ISSN: 2639-0108

Review Article

Advances in Theoretical & Computational Physics

How to execute repeated integral to generate hyper-exponential functions.

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Submitted: 29 July 2020; **Accepted**: 10 Aug 2020; **Published**: 20 Aug 2020

The following is a definition of hyper-exponential functions of n-order.

$$k_0(x; ., n, j) = seed(x; j) = \frac{x^j}{j!} \qquad (j = 0,1,2,3 \ldots n-1)$$

$$k_{i+1}(x; f, n, j) = \int_0^x \int_0^x \ldots \int_0^x f(x)\, k_i(x) dx^n$$

$$Exph_j^n(x; f(x)) = \sum_{i=0}^{\infty} k_i(x)$$

As mentioned above, we need repeated integral to generate hyper-exponential functions of n-order.

By the way, what is the way how to execute repeated integral by using a computer?

A good idea about repeated integral.

I have a good idea of only using differential operator and executing repeated integral without using integral operator.For example, integral twice is as follows.

$$\int_0^x \int_0^x f(x) dx^2 = \sum_{i=0}^{\infty} -1^{i+2} \frac{(i+1)x^{i+2}}{(2+i)!} f^{(i)}(x)$$

With integration by parts, the principle is as the next.

A function *f(x)* is an integrable and bounded function on an interval D containing *x*=0.

$$f: D \to R,\, x \in R,\, i \in N_0,$$

$$\int_0^x f(x)\, dx = x\, f(x) - \int_0^x x\, f'(x)\, dx = x\, f(x) - \frac{x^2}{2} f'(x) + \int_0^x \frac{x^2}{2} f''(x)\, dx$$

$$= x\, f(x) - \frac{x^2}{2} f'(x) + \frac{x^3}{3!} f''(x) - \frac{x^4}{4!} f^{(3)} + \int_0^x \frac{x^4}{4!} f^{(4)}(x)\, dx$$

$$= \sum_{i=0}^{\infty} (-1)^{i+2} \frac{x^{i+1}}{(i+1)!} f^{(i)}(x) = y_1$$

In practice, except some terms in the front of the series, the values of almost all the terms of the series are 0, which are calculated by a computer, by the effect of ! of the denominator.

By setting as follows,

$$\int_0^x \int_0^x \cdots \int_0^x f(x)\, dx^n = y_n$$

$$\int_0^x \int_0^x \int_0^x \cdots \int_0^x f(x)\, dx^{n+1} = y_{n+1}$$

The formula of repeated integral is as follows.

$$\left.\begin{aligned}
& n \in N, i \in N_0, j \in N_0, \\
& y_n = \sum_{i=0}^{\infty} (-1)^{i+2} \frac{a_i x^{n+i}}{(n+i)!} f^{(i)}(x) \\
& y_{n+1} = \sum_{j=0}^{\infty} (-1)^{j+2} \frac{b_j x^{n+1+j}}{(n+1+j)!} f^{(j)}(x) \\
& b_j = \sum_{i=0}^{j} a_i \\
& a_0 = 1\,;\; If\; n = 1, \qquad a_0, a_1, a_2, \cdots, a_i = 1.
\end{aligned}\right\} \quad ①$$

$$y_{n+1} = \frac{a_0 x^{n+1}}{(n+1)!} f(x) - \frac{(a_0+a_1)x^{n+2}}{(n+2)!} f'(x) + \frac{(a_0+a_1+a_2)x^{n+3}}{(n+3)!} f^{(2)}(x)$$
$$- \frac{(a_0+a_1+a_2+a_3)x^{n+4}}{(n+4)!} f^{(3)}(x) + \frac{(a_0+a_1+a_2+a_3+a_4)x^{n+5}}{(n+5)!} f^{(4)}(x)$$
$$- \frac{(a_0+a_1+a_2+a_3+a_4+a_5)x^{n+6}}{(n+6)!} f^{(5)}(x) + \cdots$$

$$y'_{n+1} = \frac{a_0 x^n}{n!} f(x) + \frac{a_0 x^{n+1}}{(n+1)!} f(x)' - \frac{(a_0+a_1)x^{n+1}}{(n+1)!} f'(x) - \frac{(a_0+a_1)x^{n+2}}{(n+2)!} f^{(2)}(x)$$
$$+ \frac{(a_0+a_1+a_2)x^{n+2}}{(n+2)!} f^{(2)}(x) + \frac{(a_0+a_1+a_2)x^{n+3}}{(n+3)!} f^{(3)}(x)$$
$$- \frac{(a_0+a_1+a_2+a_3)x^{n+3}}{(n+3)!} f^{(3)}(x) - \frac{(a_0+a_1+a_2+a_3)x^{n+4}}{(n+4)!} f^{(4)}(x)$$
$$+ \frac{(a_0+a_1+a_2+a_3+a_4)x^{n+4}}{(n+4)!} f^{(4)}(x) + \frac{(a_0+a_1+a_2+a_3+a_4)x^{n+5}}{(n+5)!} f^{(5)}(x) - \cdots$$
$$= y_n$$

Let's make y_2 from y_1 according to the above ①.

$$y_2 = \frac{x^2}{2!} f(x) - \frac{2x^3}{3!} f'(x) + \frac{3x^4}{4!} f^{(2)}(x) - \frac{4x^5}{5!} f^{(3)}(x) + \frac{5x^6}{6!} f^{(4)}(x) - \frac{6x^7}{7!} f^{(5)}(x) + \cdots$$

Let's differentiate y_2

$$y'_2 = x f(x) + \frac{x^2}{2!} f'(x) - \frac{2x^2}{2!} f'(x) - \frac{2x^3}{3!} f^{(2)}(x) + \frac{3x^3}{3!} f^{(2)}(x) + \frac{3x^4}{4!} f^{(3)}(x) - \cdots$$
$$= x f(x) - \frac{x^2}{2!} f'(x) + \frac{x^3}{3!} f^{(2)}(x) - \frac{x^4}{4!} f^{(3)}(x) + \frac{x^5}{5!} f^{(4)}(x) - \cdots$$
$$= y_1$$

The result is y_1. So, in terms of mathematical induction,① is right. We generate a hyper-exponential function of n-order by using y_n.

The advantages of the way of ① are as followings.

1. We can integral a function without using integral operator.
2. We can integral a function by only using differential operator.
3. The structure of algorithm for integral can be simplified.
4. The program on integral by any programming language can be easier than before.

As a function *f(x)*, it is assumed that it is a function that becomes zero after a few differentiations , or a function that can be differentiated as many times as we would like.

As conclusion,

I insist that we had better use the way of ① for programming about repeated integral to generate hyper-exponential functions.

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