ISSN: 2639-0108

Review Article

Applied Optics and Laser Technology

Hopf Bifurcation and Optimal Control of Thermo-Hydraulic Instabilities in a Microchannel

Lakshmi. N. Sridhar*

Chemical Engineering Department, University of Puerto Rico, Mayagüez, PR 00681, United States

*Corresponding Author
Lakshmi. N. Sridhar, Chemical Engineering Department, University of Puerto Rico, Mayagüez, PR 00681, United States

Submitted: 2026, Aug 19; **Accepted:** 2026, Sep 18; **Published:** 2026, Sep 25

Citation: Sridhar, N. L. (2026). Hopf Bifurcation and Optimal Control of Thermo-Hydraulic Instabilities in a Microchannel. *Appl Opt Laser Tech, 2*(1), 01-17.

Abstract

The growing demand for high-performance thermal management systems in data centers, power electronics, artificial intelligence hardware, and advanced manufacturing has increased industrial interest in two-phase microchannel cooling technologies. Companies such as 3M Corporation have played an important role in developing advanced cooling solutions based on phase-change heat transfer, where thermo-hydraulic instabilities can significantly affect performance and reliability. In this work, a reduced-order nonlinear dynamic model of a boiling microchannel system is developed and analyzed to investigate the onset of oscillatory behavior and its implications for system operation. The model consists of four ordinary differential equations that describe the evolution of the void fraction, pressure drop, wall temperature, and mass flow rate. Bifurcation analysis using MATCONT reveals the existence of a Hopf bifurcation, indicating a transition from stable steady-state operation to self-sustained limit-cycle oscillations. To facilitate real-time stability assessment, a neural-network surrogate model is trained using bifurcation-generated data and embedded within a dynamic optimization framework. An optimal control problem is subsequently formulated in PYOMO.DAE with the heat-input parameter as the manipulated variable. Results demonstrate that incorporating bifurcation-awareness into the optimization process enables the controller to avoid unstable operating regions while maintaining desirable thermal-fluid performance.

Keywords: Hopf Bifurcation, Microchannel Flow Boiling, Optimal Control, Thermo-Hydraulic Instability, Neural Network Surrogate Modeling

1. Introduction

The continuous increase in heat generation rates in modern engineering systems has created a growing demand for advanced thermal management technologies capable of removing large heat loads while maintaining compact size, high efficiency, and operational reliability. Applications such as high-performance computing, artificial intelligence hardware, power electronics, electric vehicles, aerospace platforms, and data centers require cooling solutions capable of dissipating increasingly large thermal loads within limited physical footprints. Among the various thermal management approaches proposed, two-phase cooling and microchannel boiling technologies have emerged as attractive solutions due to their exceptionally high heat-transfer coefficients and compact design.

The effectiveness of microchannel boiling systems stems from the latent heat of phase change, which enables the removal of large quantities of thermal energy with relatively small coolant flow rates. Consequently, considerable research has been devoted to understanding the fluid mechanics and heat-transfer characteristics of boiling flows in confined geometries. Despite their advantages, these systems exhibit highly nonlinear behavior resulting from the complex interactions among vapor generation, pressure fluctuations, wall temperature variations, and mass-flow dynamics. Such interactions can lead to thermo-hydraulic instabilities including density-wave oscillations, pressure-drop oscillations, flow reversals, and self-sustained periodic behavior. These phenomena can adversely affect thermal performance, reduce operating efficiency, increase mechanical stresses, and ultimately compromise system reliability.

The importance of understanding and controlling such instabilities has grown substantially with the increasing adoption of advanced liquid-cooling technologies by industry. For example, 3M Corporation has been a major contributor to the development of dielectric-fluid-based cooling technologies for electronics and data-center applications. The use of engineered fluids and two-phase heat-transfer processes enables substantial improvements in cooling capacity; however, stable operation remains a critical requirement for ensuring long-term performance and reliability. As cooling systems become increasingly sophisticated, the ability to predict and avoid instability-induced operating conditions becomes a key design objective.

Nonlinear dynamical systems theory provides a powerful framework for analyzing such behavior. In particular, Hopf bifurcation theory offers a rigorous mathematical mechanism for describing the transition
from stable steady-state operation to self-sustained oscillatory motion. When a Hopf bifurcation occurs, a stable equilibrium loses stability as a pair of complex conjugate eigenvalues crosses the imaginary axis, leading to the emergence of a limit cycle. Identifying these bifurcation points is important because they define boundaries between desirable and undesirable operating regimes.

While numerous investigations have examined boiling instabilities experimentally and numerically, comparatively fewer studies have integrated bifurcation analysis directly with optimal control methodologies to actively avoid unstable operating regions.

In recent years, artificial intelligence and machine learning techniques have begun to play an increasingly important role in process monitoring, optimization, and control. Data-driven surrogate models can provide rapid approximations of complex system behavior while significantly reducing computational requirements. This capability is particularly valuable in thermal-fluid applications where repeated stability calculations can become computationally expensive. The combination of machine learning with nonlinear dynamics therefore offers an opportunity to develop intelligent controllers capable of incorporating stability information into real-time decision-making processes.

Motivated by these challenges, the present work investigates the nonlinear dynamics and optimal control of a reduced-order microchannel boiling model exhibiting thermo-hydraulic instabilities. The study focuses on the interaction among vapor fraction, pressure drop, wall temperature, and mass flow rate, which collectively govern the dynamic behavior of the system. Bifurcation analysis is performed using MATCONT to identify critical operating conditions and characterize the onset of oscillatory behavior through Hopf bifurcation. The resulting stability information is subsequently used to construct a neural-network-based stability predictor that serves as a surrogate model for the local bifurcation characteristics of the system.

The primary contribution of this work is the development of a bifurcation-aware optimal control framework that integrates nonlinear dynamics, machine learning, and dynamic optimization. The optimal control problem is formulated using PYOMO.DAE and solved with IPOPT, with the heat-input parameter selected as the control variable.

Unlike conventional optimal control formulations that focus solely on tracking or performance objectives, the proposed approach explicitly incorporates stability information derived from bifurcation analysis. As a result, the controller seeks not only to improve operating performance but also to avoid trajectories that approach Hopf bifurcation regions and the associated limit-cycle behavior.

The significance of this research extends beyond the specific model considered in this study. The methodology provides a general framework for the design of intelligent, stability-aware thermal management systems applicable to advanced industrial cooling technologies. Such capabilities are particularly relevant to organizations such as 3M corporation and other developers of next-generation two-phase cooling systems, where maintaining stable operation is as important as maximizing heat-transfer performance. By combining bifurcation theory, artificial intelligence, and optimal control within a single computational framework, this work contributes toward the development of robust thermal-fluid systems capable of meeting the demanding requirements of future electronic, energy, and aerospace applications.

Finally, the integration of reduced-order modeling, Hopf bifurcation analysis, neural-network-based stability prediction, and optimal control represents a step toward the realization of digital-twin-assisted thermal management platforms. Such platforms could enable companies including 3M Corporation to deploy real-time stability monitoring and predictive control strategies for advanced cooling systems, thereby improving reliability, efficiency, and operational safety in a wide range of industrial applications.

2. Literature Survey

Research on thermo-hydraulic instabilities in boiling channels and microchannel systems has evolved significantly over the past three decades, progressing from the study of density-wave oscillations in conventional boiling systems to detailed investigations of flow boiling instabilities in microchannels and advanced cooling technologies.

Early work by Furuya et al. (1995) investigated density-wave oscillations in boiling natural circulation loops and demonstrated how flashing-induced mechanisms can generate self-sustained oscillations in two-phase flow systems [1]. Their study highlighted the importance of nonlinear feedback between flow rate and vapor generation in determining system stability.

Kennedy et al. (2000) examined the onset of flow instability in uniformly heated horizontal microchannels and showed that pressure-drop fluctuations and phase-change dynamics can trigger unstable operating conditions [2]. Their work was among the earliest studies to identify instability mechanisms in micro-scale boiling systems.

Brutin and Tadrist (2002) investigated flow boiling stability in microchannels and emphasized the influence of channel dimensions and operating conditions on oscillatory behavior [3]. Subsequently, Qu and Mudawar (2003a, 2003b) conducted comprehensive experimental and theoretical investigations of two-phase flow boiling in microchannel heat sinks. Their studies provided important insights into pressure-drop behavior, flow patterns, and heat-transfer mechanisms in micro-scale boiling systems [4,5].

Brutin and Tadrist (2004) further expanded their earlier work by examining the stability characteristics of microchannel boiling flows and discussing mechanisms responsible for the onset of oscillations [6]. Lee and Mudawar (2005) investigated two-phase flow in high-heat-flux microchannel heat sinks and demonstrated the strong coupling between boiling heat transfer and pressure-drop fluctuations. During the same year, Kandlikar (2005) provided a roadmap for high-flux heat removal using microchannels and identified flow instabilities as one of the major challenges limiting widespread implementation [7,8]. Hetsroni et al. (2005) studied flow patterns in parallel microchannels and reported significant flow maldistribution effects associated with two-phase flow behavior [9].

A major contribution to the theoretical understanding of boiling instabilities was made by Achard et al. (2006), who analyzed nonlinear density-wave oscillations in boiling channels and demonstrated the importance of nonlinear dynamic interactions in determining system behavior [10]. Revellin et al. (2006) experimentally characterized two-phase flow instabilities in microchannels and identified several instability regimes associated with pressure-drop oscillations and flow reversal phenomena [11].

Kuo and Peles (2007) investigated flow boiling instabilities in microchannels and showed that inlet compressibility can significantly affect the onset and severity of oscillations. Peles et al. (2008) subsequently examined forced convective flow boiling in microchannel heat sinks and identified operating conditions associated with stable and unstable flow regimes [12,13].

In the same year, Harirchian and Garimella (2008) developed flow-regime-based models for predicting heat transfer and pressure drop during microchannel flow boiling [14]. Harirchian and Garimella (2009) extended their earlier work by analyzing multiple flow regimes and their influence on pressure-drop characteristics and thermal performance [15]. Phan et al. (2010) investigated boiling in nanocoated microchannels and demonstrated the influence of surface modifications on boiling behavior and system stability [16].

Lee and Garimella (2011) studied saturated flow boiling heat transfer and pressure drop in microchannels and provided extensive experimental data for model validation [17]. During the same year, Bertsch et al. (2011) published a comprehensive review of saturated flow boiling in small channels and summarized major advances in heat transfer, pressure-drop prediction, and instability analysis [18].

Tibiriçá and Ribatski (2013) presented an extensive review of flow boiling in micro-scale channels, highlighting unresolved issues related to instability mechanisms and predictive modeling. Guo et al. (2014) reviewed computational approaches for modeling flow boiling in microchannels and emphasized the need for reduced-order models capable of capturing dynamic instability phenomena [19,20].

Li et al. (2016) introduced the concept of oscillate boiling and demonstrated the existence of self-sustained periodic bubble dynamics under localized heating conditions. Their work further reinforced the importance of nonlinear oscillatory behavior in boiling systems [21]. Prajapati and Bhandari (2017) reviewed flow boiling instabilities in microchannels and summarized both passive and active methods for instability suppression [22]. Siddiqui and Zubair (2018) investigated flow distribution and instability issues in parallel microchannel systems, while Özdemir and Sözbir (2018) reviewed instability mechanisms in parallel microchannel heat sinks and identified key factors affecting stable operation [23,24].

Wang et al. (2019) analyzed flow boiling instability characteristics in microchannel heat sinks and demonstrated the influence of operating parameters on instability thresholds. Xu et al. (2020) examined the dynamic characteristics of two-phase flow instabilities in microchannel evaporators and reported complex oscillatory behaviors under varying operating conditions [25,26].

Zhang and Kandlikar (2021) investigated pressure-drop oscillations and density-wave instabilities in microchannels and provided detailed explanations of the physical mechanisms responsible for instability development [27]. Li et al. (2022) subsequently published a comprehensive review of dynamic instabilities in microchannel flow boiling and summarized recent advances in experimental, numerical, and theoretical investigations [28].

More recent studies continue to focus on instability characterization under realistic operating conditions. Hu et al. (2025) experimentally investigated flow boiling heat transfer and instabilities in microchannels under high liquid subcooling conditions and provided valuable data regarding instability thresholds and thermal performance [29]. Similarly, Kriz and Moghaddam (2025) studied liquid-film instabilities during microchannel flow boiling and demonstrated how film dynamics influence boiling stability and heat-transfer behavior [30].

Overall, the literature demonstrates that thermo-hydraulic instabilities remain one of the principal challenges in microchannel boiling systems. Although significant progress has been made in understanding flow boiling behavior, pressure-drop oscillations, and density-wave instabilities, relatively few studies have integrated nonlinear bifurcation analysis, machine learning, and optimal control into a unified framework. The present work addresses this gap by combining Hopf bifurcation analysis, neural-network-based stability prediction, and bifurcation-aware optimal control to develop a stability-conscious operating strategy for advanced microchannel boiling systems relevant to modern industrial cooling technologies.

3. Main Objectives of the Present Work

The primary objective of this work is to investigate the onset of thermo-hydraulic instabilities in a nonlinear microchannel boiling system and to develop an optimal control framework capable of maintaining stable operation in the vicinity of instability boundaries. Modern two-phase cooling systems offer exceptional heat-transfer performance; however, they are also susceptible to complex nonlinear phenomena such as flow oscillations, density-wave instabilities, and limit-cycle behavior. Understanding and controlling these phenomena is essential for the reliable operation of advanced thermal management technologies.

The first objective is to develop and analyze a reduced-order nonlinear dynamic model that captures the essential interactions among vapor generation, pressure drop, wall temperature, and mass flow rate within a microchannel boiling system. The model is intended to preserve the dominant thermo-hydraulic feedback mechanisms responsible for instability while remaining sufficiently compact for bifurcation and control studies.

The second objective is to identify and characterize Hopf bifurcations within the proposed model. Continuation techniques are employed to determine critical operating conditions at which a stable equilibrium loses stability and transitions into periodic oscillations. The associated Jacobian matrix and eigenvalue spectrum are analyzed to validate the bifurcation behavior and to establish the dynamical mechanisms responsible for the emergence of limit cycles.

The third objective is to investigate the nonlinear oscillatory behavior that develops beyond the Hopf bifurcation point. Particular attention is given to the formation and characteristics of stable limit cycles, which represent sustained thermo-hydraulic oscillations that can adversely affect system performance and reliability.

The fourth objective is to formulate and solve an optimal control problem in which the heat-input parameter serves as the control variable. The goal is to minimize deviations of key state variables from their desired operating conditions while simultaneously maintaining stable system behavior. This objective reflects the practical requirement of achieving high thermal performance without operating in unstable regions.

The fifth objective is to integrate machine learning with nonlinear dynamics by constructing a neural-network-based approximation of the stability boundary using data generated from bifurcation analysis. This stability model is incorporated into the optimal control framework to create a bifurcation-aware optimization strategy capable of discouraging operation near Hopf bifurcation points.

Finally, the work seeks to demonstrate the industrial relevance of bifurcation-aware control methodologies for advanced two-phase cooling systems used in high-performance electronics, artificial intelligence hardware, aerospace platforms, and thermal management technologies developed by companies such as 3M. By combining bifurcation theory, machine learning, and optimal control, the study aims to provide a foundation for the design of intelligent thermal-fluid systems that are both high-performing and dynamically stable.

4. Novelty of the Present Work

The novelty of the present work lies in the integration of nonlinear bifurcation analysis, machine learning, and optimal control within a unified framework for the stabilization of thermo-hydraulic instabilities in microchannel boiling systems. While previous studies have investigated boiling instabilities, flow oscillations, and two-phase thermal management systems, relatively few have combined rigorous bifurcation analysis with data-driven stability prediction and optimal control to actively avoid instability-inducing operating conditions.

A key contribution of this work is the development and analysis of a reduced-order nonlinear model that captures the essential feedback mechanisms among vapor fraction, pressure drop, wall temperature, and mass flow rate responsible for thermo-hydraulic oscillations. The model exhibits a Hopf bifurcation and the subsequent emergence of a stable limit cycle, thereby providing a computationally efficient platform for investigating nonlinear dynamic behavior that would otherwise require significantly more expensive computational fluid dynamics simulations.

A second novel aspect is the explicit identification and characterization of a Hopf bifurcation through continuation analysis using MATCONT. The study not only determines the critical operating conditions at which instability occurs but also validates the bifurcation using analytical and numerical Jacobian matrices, eigenvalue analysis, and limit-cycle continuation. This provides a rigorous nonlinear dynamics foundation for the subsequent control design.

The most significant novelty of the work is the incorporation of bifurcation information directly into the optimal control formulation. Traditionally, bifurcation analysis is performed offline as a diagnostic or design tool, whereas optimal control is implemented independently without explicit knowledge of stability boundaries. In the present study, data generated from bifurcation analysis are used to train a neural network that approximates the local stability characteristics of the system. The neural-network-based stability indicator is then embedded within a PYOMO.DAE optimal control framework, allowing the optimizer to account for the proximity of Hopf bifurcation regions during operation.

This bifurcation-aware optimization strategy represents a new approach to thermal-fluid system control. Instead of merely minimizing deviations from desired operating conditions, the controller actively discourages trajectories that approach dynamically unstable regions of the state space. Consequently, stability considerations become an integral part of the optimization process rather than a separate post-analysis step. Another novel contribution is the application of machine-learning-assisted bifurcation avoidance to a boiling microchannel system. The use of a neural-network surrogate significantly reduces the computational burden associated with repeatedly evaluating stability information during optimization. This creates a practical pathway toward real-time implementation in industrial applications where rapid decision-making is required.

The methodology is also highly relevant to emerging industrial cooling technologies, including two-phase dielectric-fluid systems used in high-performance computing, artificial intelligence hardware, power electronics, and advanced thermal management platforms developed by companies such as 3M.
To the best of the authors' knowledge, the combination of Hopf bifurcation analysis, neural-network-based stability prediction, and bifurcation-aware optimal control for a reduced-order microchannel boiling model has not been previously reported in the literature.

Therefore, the present work introduces a comprehensive framework that bridges nonlinear dynamics, artificial intelligence, and optimal control, providing a new strategy for the design and operation of dynamically stable thermal-fluid systems. This integration extends the traditional role of bifurcation theory from a descriptive analysis tool to an active component of advanced process optimization and control. The rest of this paper is organized as follows. First, the model equations are presented, followed by a description of the numerical procedures used. The results discussion and conclusions are then presented.

5. Model Equations

The mathematical model developed in this work represents the dominant thermo-hydraulic interactions in a boiling microchannel using a low-dimensional nonlinear dynamical system. The formulation captures the essential coupling between phase change, fluid transport, and thermal effects while avoiding the complexity of full spatially distributed two-phase flow equations. The void fraction equation describes the rate of vapor generation due to heat input and its depletion through convective transport with the flowing mixture. This introduces a nonlinear coupling between heat input, phase fraction, and mass flow rate, which is a key mechanism driving instability in boiling systems.

The pressure-drop dynamics represent the balance between vapor-induced flow expansion and dissipative effects arising from friction and flow resistance. The resulting equation captures how variations in vapor content and flow rate directly influence the hydraulic response of the channel. The wall temperature equation reflects the competition between external heating and thermal dissipation through boiling heat transfer. It also incorporates feedback from the phase distribution in the channel, linking thermal behavior with hydrodynamic state variables. The mass flow rate equation models the dynamic response of the channel flow to pump forcing, while accounting for pressure-driven acceleration, frictional damping, vapor-induced feedback, and thermal coupling effects. This equation provides the main inertial mechanism in the system and plays a central role in the emergence of oscillatory dynamics.

The microchannel boiling system is governed by the fundamental conservation laws of mass, momentum, and energy for a one-dimensional two-phase flow. These distributed equations are first considered in their full partial-differential form and then reduced via spatial averaging to obtain a tractable nonlinear dynamical system. The starting point is the mixture continuity equation, which describes conservation of mass in the two-phase flow field:

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \left(\rho_m u \right)}{\partial z} = 0 \tag{1}$$

where p_{m} is the mixture density and is the average mixture velocity along the channel.

To construct a reduced-order representation, the vapor content in the flow is represented using the spatially averaged void fraction defined as:

$$\alpha\left(t\right) = \frac{1}{L}\int_0^L \alpha\left(z,t\right) dz \tag{2}$$

This averaging assumes that the dominant dynamics can be captured by lumped temporal evolution of the phase fraction rather than full spatial resolution.

The evolution of void fraction is governed by competition between vapor generation due to heat input and vapor removal due to convective transport. This leads to the lumped balance:

$$\frac{d\alpha}{dt} = k_1 Q - k_2 \dot{m} \alpha - k_3 \alpha \tag{3}$$

where Q represents the imposed heat input and $m^{\cdot}$ is the mass flow rate. Introducing the state definitions:

$$x_v = \alpha, w_v = \dot{m} \tag{4}$$

the void fraction dynamics become:

$$\frac{dx_v}{dt} = A_1 Q - A_2 w_v x_v - A_3 x_v \tag{5}$$

This equation captures the nonlinear feedback between boiling (through Q), flow rate, and vapor accumulation. The pressure dynamics are obtained from the one-dimensional momentum equation:

$$\frac{\partial \left(\rho_m u \right)}{\partial t} + \frac{\partial \left(\rho_m u^2 \right)}{\partial z} = -\frac{\partial P}{\partial z} - \frac{f}{2D_h} \rho_m u |\, u| \tag{6}$$

where P is pressure, , f is the friction factor, and D_h is the hydraulic diameter of the channel.
The pressure drop across the channel is defined as:

$$\Delta P = P_{in} - P_{out} \tag{7}$$

and is used as a lumped state variable . After spatial integration and closure approximations, the pressure-drop dynamics reduce to:

$$\frac{d\left(\Delta P\right)}{dt} = k_4 \alpha - k_5 \Delta P - k_6 \dot{m} \tag{8}$$

which becomes in state form:

$$\frac{dy_v}{dt} = A_4 x_v - A_5 y_v - A_6 w_v \tag{9}$$

This equation represents the competition between vapor-induced expansion and frictional dissipation in the channel. The thermal dynamics are obtained from the energy conservation equation:

$$\frac{\partial(\rho_m h)}{\partial t} + \frac{\partial(\rho_m u h)}{\partial z} = \frac{q'' P_w}{A_c} \tag{10}$$

where h is the mixture enthalpy, q'' is the wall heat flux, P_w is the wetted perimeter, and A_c is the cross-sectional area. The wall thermal response is modeled using a lumped energy balance:

$$M_w C_{pw} \frac{dT_w}{dt} = Q - h_b A_s (T_w - T_{sat}) \tag{11}$$

which after averaging leads to:

$$\frac{dT_w}{dt} = k_7 Q - k_8 T_w - k_9 \alpha \tag{12}$$

Introducing the state variable $z_c = T_w$ gives the wall temperature dynamics:

$$\frac{dz_c}{dt} = A_7 Q - A_8 z_c - A_9 x_v \tag{13}$$

This equation reflects heating from external input, cooling due to convection, and coupling with phase change through void fraction.

Finally, the mass flow dynamics are derived from a lumped momentum balance across the channel. The flow is driven by the pump pressure and opposed by frictional and thermo-hydraulic feedback effects:

$$\frac{d\dot{m}}{dt} = k_{10}(P_p - \Delta P) - k_{11}\dot{m} + k_{12}\alpha - k_{13} T_w \tag{14}$$

Using the state substitutions $w_v = \dot{m}, y_v = \Delta P, x_v = \alpha, z_c = T_w$ this becomes:

$$\frac{dw_v}{dt} = A_{10}(P_p - y_v) - A_{11} w_v + A_{12} x_v - A_{13} z_c \tag{15}$$

This equation completes the four-state nonlinear dynamical model. The final reduced-order system is therefore:

$$\begin{aligned}
\frac{dx_v}{dt} &= A_1 Q - A_2 w_v x_v - A_3 x_v \\
\frac{dy_v}{dt} &= A_4 x_v - A_5 y_v - A_6 w_v \\
\frac{dz_c}{dt} &= A_7 Q - A_8 z_c - A_9 x_v \\
\frac{dw_v}{dt} &= A_{10}(P_p - y_v) - A_{11} w_v + A_{12} x_v - A_{13} z_c
\end{aligned} \tag{16}$$

Tables 1 and 2 provide the details of variables and parameters.

Symbol	Parameter Description	Value	Units
A_1	Heat-to-void generation coefficient	1.0	–
A_2	Void-flow coupling coefficient	3.0	–
A_3	Void decay coefficient	0.4	–
A_4	Void-to-pressure coupling coefficient	2.0	–
A_5	Pressure relaxation coefficient	1.0	–
A_6	Flow damping coefficient	1.5	–
A_7	Heat-to-temperature coefficient	1.2	–
A_8	Thermal dissipation coefficient	0.2	–
A_9	Void-to-temperature coupling coefficient	0.7	–
A_{10}	Pump forcing coefficient	1.5	–
A_{11}	Flow damping coefficient	0.5	–
A_{12}	Void influence on flow	2.0	–
A_{13}	Thermal feedback coefficient	2.0	–
P_p	Pump pressure (constant)	1.0	Pa

Table 1: Model Parameters, Symbols, and Values

Symbol	Variable Name	Units
x_v	Void fraction (vapor volume fraction)	–
y_v	Pressure drop across microchannel	Pa
z_c	Wall temperature	K
w_v	Mass flow rate	kg/s
Q	Heat input (control input)	W
P_p	Pump pressure	Pa
t	Time	s

Table 2: State Variables and Model Variables

6. Bifurcation analysis and Optimal Control

6.1. Bifurcation Analysis

Continuation and bifurcation computations were carried out using the MATLAB-based software MATCONT. Bifurcation analysis provides important insights into the mechanisms underlying the existence of multiple steady states and self-sustained oscillations in nonlinear dynamical systems. In particular, branch points and limit points are associated with the emergence of multiple steady-state solutions, whereas oscillatory dynamics and periodic solutions originate from Hopf bifurcations. The package, originally developed and subsequently enhanced by several researchers (Dhooge Govearts, and Kuznetsov, 2003), enables the systematic identification of limit points (LP), branch points (BP), and Hopf bifurcation points (H) in nonlinear dynamical models [31]. This program identifies Limit points(LP), branch points(BP), and Hopf bifurcation points(H) for an system of ordinary differential equations

$$\frac{dx}{dt} = f(x,\alpha) \tag{17}$$

$x \in R^n$ where the bifurcation parameter is α .

At a limit point, the $n+1^{th}$ component of the tangent vector $w_{n+1} = 0$. For a branch point, the matrix $B = \begin{bmatrix} A \\ w^T \end{bmatrix}$ must be singular and have a determinant of 0.

At a Hopf bifurcation point,

$$\det(2 f_x(x,\alpha) @ I_n) = 0 \tag{18}$$

@ indicates the bialternate product while I_n is the n-square identity matrix. More details can be found in Kuznetsov (1998)) and

Govaerts [2000] respectively [32,33].

6.2. Optimal Control

Pyomo.dae (Hart et al, 2017) is used for the Optimal Control calculations [34]. Pyomo with its Pyomo.DAE module provides an efficient framework for the formulation and solution of dynamic optimization problems involving systems of differential and algebraic equations. The package provides a symbolic modeling environment for representing differential-algebraic equation (DAE) systems in optimization applications.

In Pyomo.DAE, the continuous-time dynamic model is transformed into a nonlinear programming (NLP) problem through discretization techniques such as orthogonal collocation, allowing the resulting optimization problem to be solved using standard NLP solvers. The NLP is solved using IPOPT.

7. Formation of Stability Dataset from MATCONT Results

A stability dataset was developed from numerical continuation calculations performed in MATCONT. The stability dataset consists of rows, each representing a continuation point from an equilibrium branch. Each row contains the state variables, the bifurcation parameter, and a stability measure. The stability measure is a numerical value derived from the Jacobian matrix. The Jacobian matrix is computed numerically at each equilibrium point. The eigenvalues are then computed automatically using MATLAB. The maximum value of the real part of these eigenvalues is then computed as a scalar stability measure.

The stability measure is computed using "eig_real_max = max(real(eigvals));" in MATLAB. The stability measure is a quantitative metric in which negative values indicate locally asymptotically stable equilibria, positive values indicate instability, and a zero crossing indicates a Hopf bifurcation. The stability dataset is then saved as a CSV file. The dataset can then be used in subsequent computational calculations to perform classification or regression to identify stability boundaries or approximate bifurcations.

8. Neural Network Surrogate for Stability Prediction

Direct embedding of eigenvalue calculations into IPOPT-based optimal control is impractical for several reasons: (i) computing eigenvalues at each time step is computationally expensive, (ii) the mapping from states to the maximum eigenvalue is non-smooth near eigenvalue crossings, and (iii) symbolic differentiation of eigenvalues is challenging.

Prior to neural-network training, all input variables were standardized to improve numerical conditioning and training stability. Let x_rawdenote the vector of state variables and bifurcation parameters= obtained from the stability dataset. For each input variable j, the training mean μ_jand training standard deviation σ_jwere computed over all training samples. = The training mean for the input variable j is defined as the

arithmetic average over all training samples as $\mu_j = \frac{1}{N}\sum_{k=1}^{N} x_j^{(k)}$ and the training standard deviation for the input variable j is defined as: $\sigma_j = \frac{1}{N}(\sum_{k=1}^{N} x_j^{(k)} - \mu_j)^2$

The standardized inputs were defined as

$$x_j = \frac{x_{raw,j} - \mu_j}{\max(\sigma_j, \varepsilon_{scale})} \tag{19}$$

where ε_scaleis a small positive regularization parameter introduced to prevent numerical singularities associated with extremely small variances. This transformation ensures that all inputs remain properly scaled while avoiding excessively large neural-network activation arguments during optimization. The normalization procedure improves neural-network conditioning and enhances the robustness of gradient-based optimization. The vectors μ and σ computed during training were stored and embedded identically within the Pyomo optimal-control formulation to ensure consistency between neural-network training and deployment.

To overcome these limitations, a feedforward neural network is trained to approximate the maximum eigenvalue as a smooth function of the system state and bifurcation parameter. A typical architecture employs the hyperbolic tangent (tanh) as a smooth activation function. If the input vector is denoted by x, which represents the scaled variables, then the network is defined as:

The vectors μ,σ computed during training were stored and embedded identically within the Pyomo optimal control formulation to ensure consistency between neural network training and deployment.

To avoid repeated eigenvalue computations during optimization, a feedforward neural network is trained to approximate the maximum real eigenvalue as a smooth function of the system states and bifurcation parameter. Using hyperbolic tangent activation functions ensures smooth differentiability required by IPOPT. If the input vector is denoted by x, which are the scaled variables, the network architecture is defined as

$$\begin{aligned} z1 &= tanh\left(W1\,x + b1\right) \\ z2 &= tanh\left(W2\,z1 + b2\right) \\ \lambda\_max\_NN &= W3\,z2 + b3 \end{aligned} \tag{20}$$

where W_i and b_i denote the weight matrices and bias vectors, respectively. The hidden-layer variables z_1 and z_2 represent nonlinear transformations of the input variables and intermediate features. The final output $\hat{\lambda}_{max}$ provides a smooth approximation of the spectral abscissa. Because tanh is infinitely differentiable, the network is fully smooth, guaranteeing the availability of first and second derivatives required by IPOPT. Without biases, the network output would be constrained to pass through the origin, limiting flexibility.

The hidden-layer outputs z_1, z_2, represent nonlinear combinations of the inputs and previous-layer features, respectively. Each element of z_1 is a smoothed combination of the original inputs, while each element of z_2 encodes more abstract patterns extracted from z_1. The final output λ_max_NN provides a smooth approximation of the maximum real eigenvalue, enabling efficient and differentiable stability evaluation within the optimal control problem. The integration into optimal control is done using a soft penalty formulation where we use a smooth-max function that converts λ into a smooth, nonnegative penalty that only "activates" when the system is unstable:

$$s_{\max}(\lambda+\varepsilon_1) = smooth - \max(\lambda+\varepsilon_1) = \frac{(\lambda+\varepsilon_1)+\sqrt{(\lambda+\varepsilon_1)^2+\varepsilon_1}}{2} \tag{21}$$

λ is the neural network's predicted maximum eigenvalue at the current state and parameter, while ε is a small positive safety margin to ensure differentiability. The soft penalty formulation involves the new objective function, where the original objective function $optv(t) = \sum (xv(t)-xv^*)^2 + (wv(t)-wv^*)^2 + (zc(t)-zc^*)^2$ is modified to $\sum (optv(t) + \alpha_{hopf}.s_{\max}(\lambda+\varepsilon_1))^2$ α_{Hopf} controls how aggressively instability is penalized, and ε_1 prevents numerical issues at exactly λ = 0 and slightly shifts the stability boundary. xv*, yv*, wv*, and zc* are chosen as 1.167012; 2.225967; 0.229527; and 0.072038 respectively. The goal is to minimize the objective function value while ensuring differentiability for IPOPT and avoiding non-smooth optimization. This approach avoids repeated eigenvalue computations and provides a smooth, differentiable surrogate suitable for gradient-based optimization. Furthermore, this enables the incorporation of stability constraints into optimal control without explicitly computing eigenvalues during optimization.

To facilitate integration into optimal control, a stability dataset is constructed from MATCONT continuation results. At each equilibrium point, the Jacobian is evaluated and its eigenvalues computed. The stability metric is defined as the maximum real part of the eigenvalues (λ_{max} = max $\Re(\lambda(J))$). Negative values indicate stability, positive values indicate instability, and zero crossings correspond to Hopf bifurcations.

To avoid repeated eigenvalue computations within optimization, a feedforward neural network is trained to approximate the spectral abscissa as a smooth function of the system state and bifurcation parameter. Using hyperbolic tangent activation functions ensures smooth differentiability required by gradient-based solvers. The resulting surrogate provides a differentiable stability indicator suitable for embedding within optimal control formulations.

9. Results

The reduced-order microchannel boiling model consists of four nonlinear ordinary differential equations describing the evolution of void fraction (x_v), pressure drop (y_v), wall temperature (z_c), and mass flow rate (w_v). The parameter values for the bifurcation and optimal control studies were

$$A_1 = 1.0, A_2 = 3.0, A_3 = 0.4, \quad A_4 = 2.0, A_5 = 1.0, A_6 = 1.5, \quad A_7 = 1.2, A_8 = 0.2, A_9 = 0.7,$$
$$A_{10} = 1.5, A_{11} = 0.5, A_{12} = 2.0, \quad A_{13} = 2.0, P_p = 1.0.$$

The heat-input parameter Q was selected as the bifurcation and control parameter.

The bifurcation analysis was performed using MATCONT and a Hopf bifurcation point was found at (xv, yv,zc,wv, Q) values of (1.167012; 2.225967; 0.229527; 0.072038; 0.719012) with a first Lyapunov coefficient of 2.542607e-02. The bifurcation diagram and the limit cycle are shown in Figs. 1a and 1b.

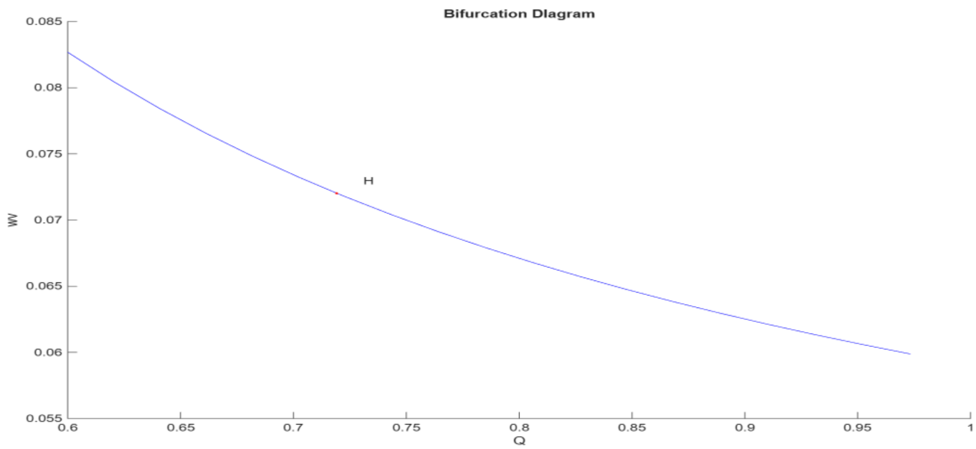

Figure 1a: Bifurcation Diagram.

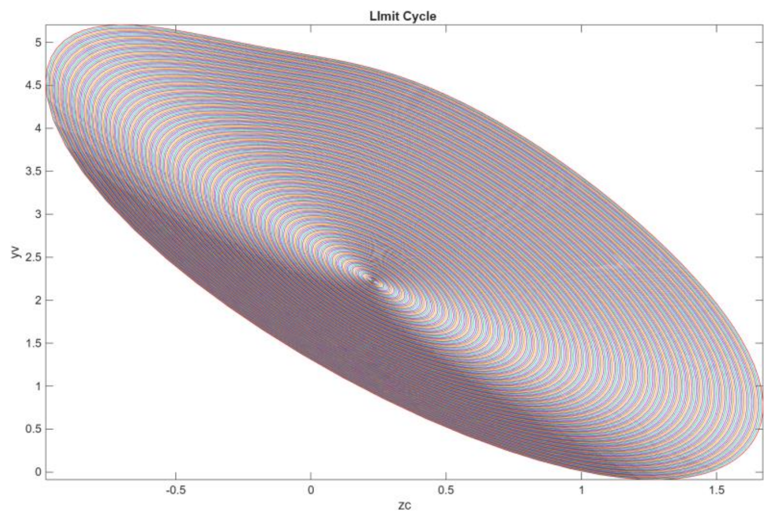

Figure 1b: Limit Cycle.

The analytical Jacobian matrix is

$$J_a = \begin{bmatrix} -\left(A_2 w_v + A_3\right) & 0 & 0 & -A_2 x_v \\ A_4 & -A_5 & 0 & -A_6 \\ -A_9 & 0 & -A_8 & 0 \\ A_{12} & -A_{10} & -A_{13} & -A_{11} \end{bmatrix} \tag{22}$$

The numerical Jacobian at the Hopf bifurcation point is

$$J_n = \begin{bmatrix} -0.616114 & 0 & 0 & -3.501036 \\ 2.0 & -1.0 & 0 & -1.5 \\ -0.7 & 0 & -0.2 & 0 \\ 2.0 & -1.5 & -2.0 & -0.5 \end{bmatrix} \tag{23}$$

The eigenvalues at this point are (−1.323, −0.412, ±1.274i). The presence of the imaginary eigenvalues that are opposite in sign confirms the existence of the Hopf bifurcation point.

For the optimal control, Q is the control parameter and xv* wv* and zc* are chosen as the Hopf co-ordinates. To investigate the effect of bifurcation-aware operation, an optimal control problem was solved both with and without a Hopf-bifurcation-avoidance constraint using PYOMO.DAE coupled with IPOPT.

To investigate the effect of bifurcation-aware operation, an optimal control problem was solved both without and with a Hopf-bifurcation-avoidance constraint. PYOMO.DAE with IPOPT was used.

The objective was to minimize ∑((*optv(t)* over the specified operating horizon. In the absence of the Hopf bifurcation constraint α_Hopf = 0, the obtained optimal objective value of $optv(t) = \sum(xv(t)-xv^*)^2 + (wv(t)-wv^*)^2 + (zc(t)-zc^*)^2$ was 72.5 When the Hopf bifurcation avoidance constraint was activated with $\alpha_{Hopf} = 142$, the optimal objective value was 90.81.

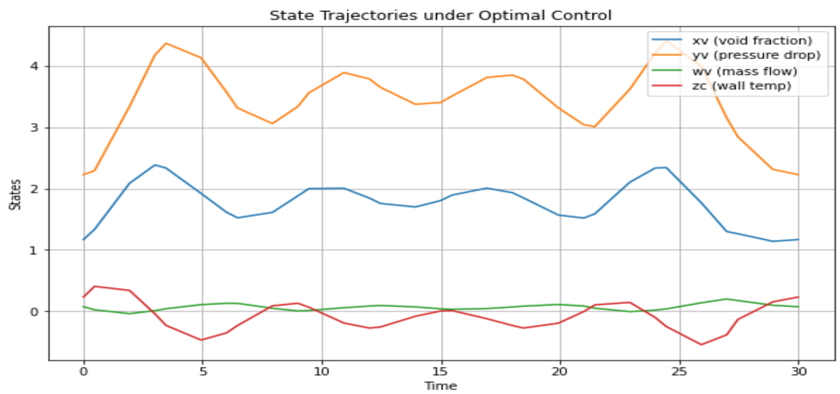

Figure 2a: Optimal Control, xv, yv, wv, zc Profiles No Hopf Constraint.

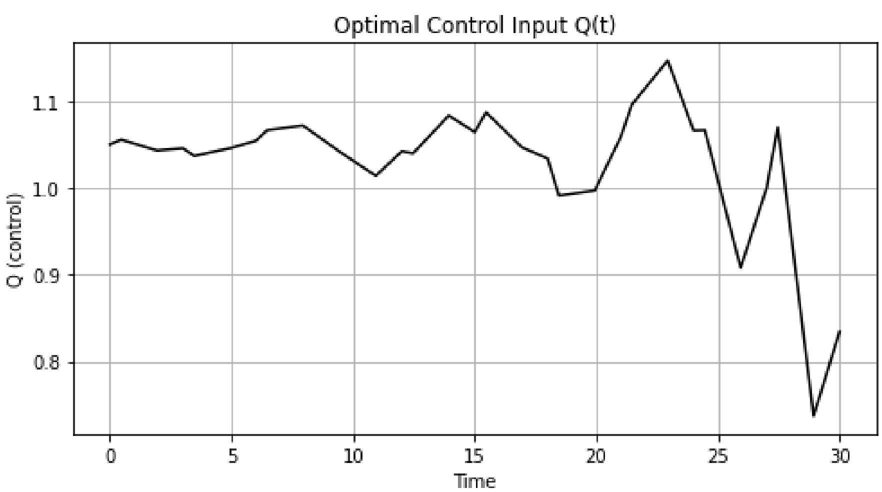

Figure 2b: Optimal Control, Q Profile No Hopf Constraint

The 5.01% decrease in the objective function value demonstrate that avoiding the Hopf bifurcation that causes limit cycles is very beneficial.

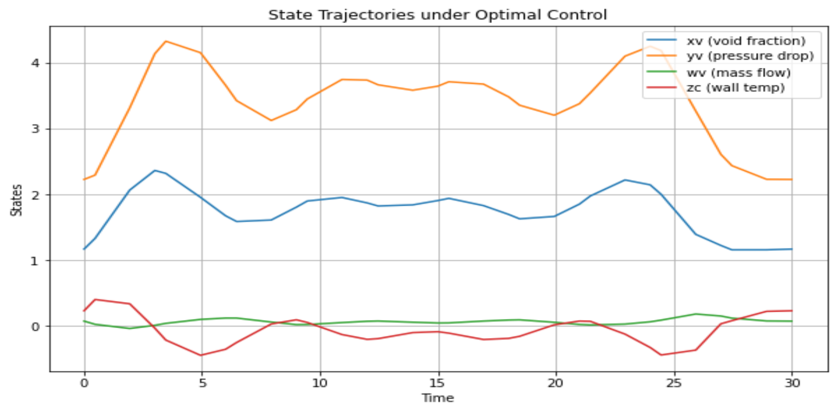

Figure 2c: Optimal Control, xv, yv, wv, zc Profiles With Hopf Constrain

Figures 2a-2b show the profiles without the Hopf Bifurcation constraint while Figures 2a-3d show the profiles with the Hopf Bifurcation constraint.

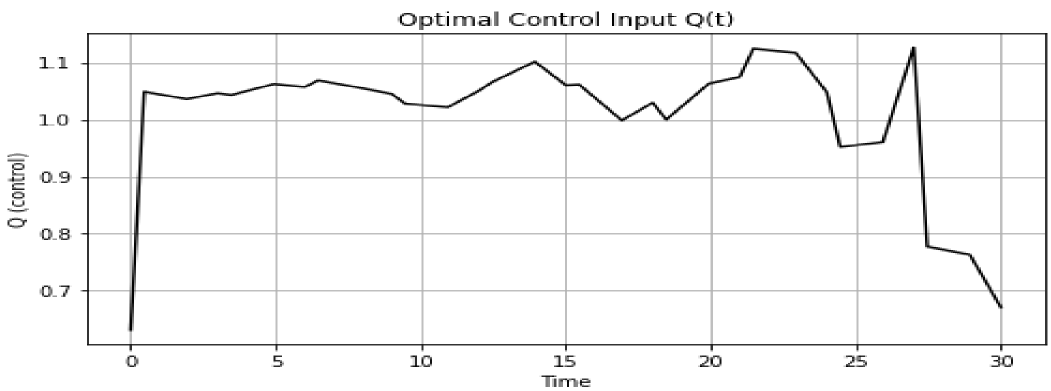

Figure 2d: Optimal Control, Q Profile with Hopf Constraint

10. Discussion of Results

The present study demonstrates how bifurcation analysis and optimal control can be combined to understand and mitigate oscillatory behavior in nonlinear two-phase flow systems. Starting from a reduced-order model derived from microchannel boiling dynamics, a Hopf bifurcation was identified using continuation techniques in MATCONT. The appearance of a Hopf point at ((x_v,y_v,z_c,w_v ,Q)=(1.167012,2.225967,0.229527,0.072038,0.719012)) confirms that the interaction between vapor generation, pressure drop, wall temperature, and mass flow rate can lead to the onset of self-sustained oscillations. Such oscillations are commonly observed in boiling systems and are associated with density-wave instabilities, thermal feedback mechanisms, and flow oscillations that degrade system performance.

The eigenvalue analysis provides further confirmation of the bifurcation structure. At the Hopf point, the Jacobian matrix possesses a pair of purely imaginary eigenvalues, (± 1.274i), while the remaining eigenvalues remain negative. This behavior is characteristic of a Hopf bifurcation, where a stable equilibrium loses stability and a periodic solution emerges. The computed first Lyapunov coefficient is positive, indicating that nonlinear effects become significant near the bifurcation point and influence the evolution of oscillatory behavior. The resulting limit cycle observed in the simulations demonstrates the existence of sustained thermo-hydraulic oscillations that persist even in the absence of external disturbances.

From an engineering perspective, the existence of such oscillatory regimes is highly significant. Modern thermal management systems increasingly rely on microchannel heat exchangers and flow-boiling technologies because of their ability to dissipate very large heat fluxes in compact volumes. However, the same mechanisms that enhance heat transfer also introduce nonlinear feedback effects that can destabilize operation. Oscillations in vapor fraction, pressure drop, wall temperature, and flow rate may lead to fluctuating thermal performance, increased mechanical stress, localized dry-out, and ultimately reduced reliability. Consequently, identifying operating regions that avoid instability is of considerable practical importance.

The industrial relevance of these findings is particularly evident in advanced cooling technologies developed for electronics, data centers, aerospace systems, and electric vehicles. Companies such as 3M have invested heavily in dielectric liquid cooling technologies that employ two-phase heat transfer for the removal of high heat loads from electronic devices. In immersion cooling and microchannel-based cooling systems, dielectric fluids undergo phase change as heat is removed from electronic components. While boiling substantially increases heat-transfer performance, it also introduces the possibility of flow instabilities similar to those captured by the present model.

For industrial applications, stable operation is often more valuable than achieving the absolute maximum heat-transfer rate. In high-performance computing facilities, for example, unexpected oscillations in coolant flow can lead to temperature fluctuations that reduce processor efficiency and shorten equipment lifetime. Similarly, in aerospace thermal management systems, oscillatory behavior can compromise reliability under varying operating conditions. Therefore, understanding the bifurcation structure of boiling systems is not merely an academic exercise but a practical requirement for robust system design.

The optimal control results further highlight this point. The control variable in the present study is the heat-input parameter (Q), which directly influences vapor generation and thermal behavior. When the optimization was performed without considering bifurcation avoidance, the controller focused exclusively on minimizing deviations from the desired operating conditions. Although this strategy produced a lower numerical objective value, it did not explicitly discourage trajectories that approached unstable operating regions. As a result, the system could operate close to the Hopf bifurcation boundary, where even small disturbances might trigger sustained oscillations.

To address this issue, a bifurcation-aware optimization framework was introduced. A neural-network-based stability model, trained using data generated from continuation analysis, was incorporated into the objective function as a Hopf-bifurcation-avoidance term. This approach effectively embeds information about system stability into the optimization problem. Instead of merely seeking the best thermodynamic performance, the controller simultaneously considers the risk of entering unstable operating regimes.

The resulting behavior is particularly relevant for industrial decision-making. In practical systems, operators rarely seek maximum performance at any cost. Rather, they aim to achieve an optimal compromise between efficiency, reliability, safety, and operating margins. The introduction of the Hopf-avoidance term produces exactly this type of compromise. Although the optimized objective value changes when stability considerations are included, the resulting operating trajectory remains farther from regions where oscillatory behavior is likely to develop. Such an approach can be viewed as a form of risk-aware optimization in which the controller actively avoids instability-inducing conditions.

For companies developing next-generation thermal management technologies, this concept has considerable value. In large-scale data centers utilizing two-phase cooling, avoiding instability can reduce maintenance costs, improve thermal uniformity, and increase equipment longevity. For manufacturers of power electronics and electric vehicle systems, stable cooling operation can enhance reliability and prevent thermal runaway events. In aerospace applications, maintaining stable heat-transfer characteristics can improve mission reliability and reduce system complexity.

An important aspect of the present work is the integration of nonlinear dynamics, machine learning, and optimal control within a unified framework. Traditionally, bifurcation analysis is performed as an offline design tool, while optimal control is implemented independently during operation. The present methodology bridges this gap by using continuation-generated stability information to influence real-time control decisions. Such an approach is particularly attractive for industrial systems because it enables the use of reduced-order models that capture essential nonlinear phenomena while remaining computationally tractable.

The methodology also aligns with current trends in industrial digitalization and artificial intelligence. Many industrial organizations are increasingly adopting digital twins and data-driven monitoring platforms to improve process performance. A neural-network approximation of the Hopf stability boundary can be viewed as a lightweight digital twin component that provides rapid stability predictions without repeatedly solving eigenvalue problems. This significantly reduces computational cost and makes stability-aware optimization feasible in real-time applications.

Another important implication concerns system design. Knowledge of the Hopf bifurcation location allows engineers to establish operating envelopes that guarantee stable behavior. Rather than relying solely on empirical safety factors, designers can use mathematically derived stability boundaries to select flow rates, heat inputs, and geometric parameters. This capability is particularly valuable during the development of new cooling technologies, where extensive experimental testing can be expensive and time-consuming.

Overall, the results demonstrate that bifurcation-aware optimal control provides a promising framework for the operation of nonlinear thermal-fluid systems. The identification of a Hopf bifurcation, validation through eigenvalue analysis, and incorporation of stability information into the optimization process collectively show how nonlinear dynamics can be transformed from a diagnostic tool into an active component of process control. For industrial organizations such as 3M and other developers of advanced two-phase cooling technologies, such methods offer the potential to improve reliability, increase operational safety, and achieve more robust thermal management in increasingly demanding applications.

As heat flux requirements continue to rise in modern electronic and energy systems, the ability to predict, avoid, and control nonlinear oscillatory phenomena will become an increasingly important component of engineering design and operation.

11. Conclusion

This work presented a nonlinear dynamics and optimal control framework for the analysis and stabilization of thermo-hydraulic instabilities in a reduced-order microchannel boiling system. Starting from a physically motivated model describing the interactions among vapor fraction, pressure drop, wall temperature, and mass flow rate, bifurcation analysis was performed using MATCONT to identify the onset of oscillatory behavior. A Hopf bifurcation point was detected at(x_v,y_v,z_c,w_v,Q)=(1.167012,2.225967,0.229527,0.072038,0.719012), indicating the loss of stability of the steady-state solution and the emergence of self-sustained periodic oscillations.

The existence of the Hopf bifurcation was confirmed through analytical and numerical Jacobian analyses, which revealed a pair of purely imaginary eigenvalues (±1.274i) at the critical operating condition. Furthermore, continuation of the periodic branch demonstrated the formation of a stable limit cycle, confirming the presence of nonlinear thermo-hydraulic oscillations in the system.

To mitigate the adverse effects of these oscillations, an optimal control formulation was developed using PYOMO.DAE and IPOPT, with the heat-input parameter Q selected as the control variable. The control objective minimized deviations of the key state variables from the Hopf equilibrium point while maintaining system stability.

A neural-network-based stability model trained from bifurcation data was incorporated into the optimization framework to create a bifurcation-aware control strategy capable of discouraging operation near unstable regions. The results demonstrated that inclusion of Hopf-bifurcation avoidance significantly altered the optimal operating trajectory and promoted operation within dynamically stable regions of the state space.

The proposed methodology has important implications for industrial thermal management systems involving boiling heat transfer and two-phase flow. In particular, companies such as 3M have developed advanced dielectric-fluid cooling technologies for data centers, power electronics, artificial intelligence hardware, aerospace systems, and high-performance computing applications. In such systems, thermo-hydraulic instabilities can reduce cooling effectiveness, increase temperature fluctuations, and compromise long-term reliability.

The ability to identify Hopf bifurcation boundaries and incorporate stability information directly into optimal control calculations provides a powerful tool for enhancing operational robustness. The integration of bifurcation theory, machine learning, and optimal control presented in this study offers a promising pathway toward developing intelligent, stability-aware thermal management systems that meet the increasing cooling demands of next-generation industrial technologies.

Data Availability Statement

All data used is presented in the paper

Conflict of interest

The author Dr. Lakshmi N Sridhar, has no conflict of interest.

Acknowledgement

Dr. Sridhar thanks Dr. Carlos Ramirez for encouraging him to write single-author papers.

References

1. Furuya, M., Inada, F., & Yasuo, A. (1995). Density Wave Oscillations of a Boiling Natural Circulation Loop Induced by Flashing. In Proceedings of the 7th International Meeting on Nuclear Reactor Thermal-Hydraulics (NURETH-7), Volume 2: Sessions 6–11 (pp. 923–932).
2. Kennedy, J. E., Roach, G. M., Dowling, M. F., Abdel-Khalik, S. I., Ghiaasiaan, S. M., &et.al. (2000). The Onset of flow Instability in Uniformly Heated Horizontal Microchannels. *Journal of Heat Transfer, 122*(1), 118–125.
3. Brutin, D., & Tadrist, L. (2002). Flow Boiling Stability in Microchannels. In 8th AIAA/ASME Joint Thermophysics and Heat Transfer Conference (p. 2781).

4. Qu, W., & Mudawar, I. (2003). Flow Boiling Heat Transfer in Two-Phase Micro-Channel Heat Sinks—I. Experimental Investigation and Assessment of Correlation Methods. *International Journal of Heat and Mass Transfer, 46*(15), 2755–2771.
5. Qu, W., & Mudawar, I. (2003). Flow Boiling Heat Transfer in Two-Phase Micro-Channel Heat Sinks-II. Annular Two-Phase Flow Model. *International Journal of Heat and Mass Transfer, 46*(15), 2773-2784.
6. Brutin, D., & Tadrist, L. (2004). Flow Boiling Stability in Microchannels. *Journal of Thermo Physics and Heat Transfer, 18*, 1–9.
7. Lee, J., & Mudawar, I. (2005). Two-Phase Flow in High-Heat-Flux Micro-Channel Heat Sink for Refrigeration Cooling Applications: Part I-Pressure Drop Characteristics. *International Journal of Heat and Mass Transfer, 48*(5), 928–940.
8. Kandlikar, S. G. (2005). High Flux Heat Removal with Microchannels—A Roadmap of Challenges and Opportunities. *Heat Transfer Engineering, 26*(8), 5–14.
9. Hetsroni, G., Mosyak, A., Segal, Z., & Pogrebnyak, E. (2003). Two-Phase Flow Patterns in Parallel Micro-Channels. *International Journal of Multiphase Flow, 29*(3), 341–360.
10. Achard, J.-L., Drew, D. A., & Lahey, R. T. (1985). The Analysis of Nonlinear Density-Wave Oscillations in Boiling Channels. *Journal of Fluid Mechanics, 155,* 213–232.
11. Revellin, R., Dupont, V., Ursenbacher, T., Thome, J. R., & Zun, I. (2006). Characterization of Two-Phase Flow Instabilities in Microchannels. *International Journal of Multiphase Flow, 32*, 755–774.
12. Kuo, C.-J., & Peles, Y. (2007). Flow Boiling Instabilities in Microchannels and Effect of Inlet Compressibility. *International Journal of Heat and Mass Transfer, 50,* 4736–4749.
13. Peles, Y., et al. (2008). Forced Convective Flow Boiling Across Microchannel Heat Sinks. *Journal of Heat Transfer, 130*, 1–12.
14. Harirchian, T., & Garimella, S. V. (2008). Flow Regime-Based Modeling of Heat Transfer and Pressure Drop in Microchannel Flow Boiling. *International Journal of Heat and Mass Transfer, 51*, 3724–3735.
15. Harirchian, T., & Garimella, S. V. (2009). Microchannel Flow Boiling Heat Transfer and Pressure Drop with Multiple Flow Regimes. *International Journal of Heat and Mass Transfer, 52*, 269–280.
16. Phan, H. T., Caney, N., Marty, P., Colasson, S., & Gavillet, J. (2010). Flow Boiling Of Water On Nanocoated Surfaces in A Microchannel. *International Journal of Thermal Sciences, 49*, 791–798.
17. Lee, P. S., & Garimella, S. V. (2011). Saturated Flow Boiling Heat Transfer and Pressure Drop in Microchannels. *International Journal of Heat and Mass Transfer, 54,* 4760–4772.
18. Bertsch, S. S., Groll, E. A., & Garimella, S. V. (2011). Review and Comparative Analysis of Studies on Saturated Flow Boiling in Small Channels. *Nanoscale and Microscale Thermophysical Engineering, 15,* 1–46.
19. Tibiriçá, C. B., & Ribatski, G. (2013). Flow Boiling in Micro-Scale Channels-a Review. *Experimental Thermal and Fluid Science, 44,* 698–729.
20. Guo, Z., Fletcher, D. F., & Haynes, B. S. (2014). A Review of Computational Modelling of Flow Boiling in Microchannels. *Journal of Computational Multiphase Flows, 6*(2), 79–112.
21. Li, F., Gonzalez-Avila, R., Nguyen, D. M., & Ohl, C.-D. (2017). Oscillate Boiling from Microheaters. *Physical Review Fluids, 2*, 014007.
22. Prajapati, Y. K., & Bhandari, P. (2017). Flow Boiling Instabilities in Microchannels and Their Promising Solutions-A Review. *Experimental Thermal and Fluid Science, 88,* 576–593.
23. Siddiqui, M. A., & Zubair, S. M. (2018). Flow Distribution and Instability Issues in Parallel Microchannel Systems. *Applied Thermal Engineering, 134,* 389–406.
24. Özdemir, M. R., & Sözbir, N. (2018). Instability Mechanisms in Parallel Microchannel Heat Sinks: A Review. *Renewable and Sustainable Energy Reviews, 81*, 2030–2047.
25. Wang, G., Cheng, P., & Bergles, A. E. (2019). Flow Boiling Instability Characteristics in Microchannel Heat Sinks. *International Journal of Heat and Mass Transfer, 131*, 641–655.
26. Xu, J., Zhang, W., & Gan, Y. (2020). Dynamic Characteristics of Two-Phase Flow Instabilities in Microchannel Evaporators. *Applied Thermal Engineering, 171*, 115107.
27. Zhang, Y., & Kandlikar, S. G. (2021). Mechanisms of Pressure-Drop Oscillations and Density-Wave Instabilities in Microchannels. *International Journal of Multiphase Flow, 139,* 103618.
28. Wang, X., Chen, Y., & Wang, H. (2022). Dynamic Instabilities of Flow Boiling in Micro-Channels: A Review. *Applied Thermal Engineering, 214,* 118773.
29. Hu, C., Ma, Z., Ma, X., Feng, Y., Xue, Y., Wei, J., & Yang, X. (2025). Experimental Investigation of Flow Boiling Heat Transfer and Instabilities in Microchannels at high liquid subcooling. *Physics of Fluids, 37*, 033303.
30. Kriz, A., & Moghaddam, S. (2025). Onset of Liquid Films Instability in Microchannel Flow Boiling. *International Journal of Heat and Mass Transfer, 244*, 126964.
31. Dhooge, A., Govaerts, W., & Kuznetsov, Y. A. (2003). MATCONT: A MATLAB Package for Numerical Bifurcation Analysis of ODEs. *ACM Transactions on Mathematical Software, 29*(2), 141–164.
32. Kuznetsov, Y. A. (1998). Elements of Applied Bifurcation Theory (2nd ed.). *Springer.*
33. Govaerts, W. J. F. (2000). Numerical Methods for Bifurcations of Dynamical Equilibria. *SIAM*.

34. Hart, W. E., Laird, C. D., Watson, J.-P., Woodruff, D. L., Hackebeil, G. A., &et.al. (2017). Pyomo-Optimization modeling in Python (2nd ed.). *Springer.*