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Review Article

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From special operators $(1-D_x)^{n+\alpha}$ to Laguerre Polynomials

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Abstract
Laguerre polynomials $L_n^\alpha(x)$ are shown to be the transforms of monomials by the special operators $(1-D_x)^{n+\alpha}$. From this their current properties such as Rodrigues formula, Lucas symbolic formula, orthogonality, generating functions, etc… are systematically obtained. This success opens the way for the study of special functions from special operators by the powerful operator calculus.

Keywords: Laguerre Polynomials, Resolution of Ode by Operator Calculus

Introduction

The Laguerre polynomials $L_n(x)$ introduced by and the generalized polynomials $L_n^\alpha(x)$ by are utilized in quantum mechanics in the resolution of the Schrödinger equation as every physicist knows [1, 2]. Although their properties are well-known in literatureand on the net such as in Wikipediaby different approaches for examples by determining them from Rodrigues formulae, from explicit expression or by lengthy resolution of its differential equation by integral factors, etc…, we would like in this work expose the studies of them firstly by resolving their differential equations by operator calculus, i.e. by utilizing the differential operator D_x side by side with the position operato $\hat{X}$ as in quantum mechanics [3,4]. By this approach, we obtain that the special operators $(1-D_x)^{n+\alpha}$ transform monomials into them. Thank to these operators, we get also easily many properties of Laguerre and generalized polynomials, specially the Rodrigues formulae, the Lucas symbolic formulae, the orthogonality, the generating functions, etc…as will be exposed in the following paragraphs.

Study of Laguerre Polynomials by Operator Calculus

Summary About Operator Calculus for Laguerre Polynomials

Consider the linear differential equation for Laguerre polynomials [1,3].

$$xy''+(1-x)y'+ny=0 \tag{1}$$

In order to resolve this equation by operator calculus let us introduce the derivative operator D_x and the Eckaert's "multiply with the argument "that we will call position operator $\hat{X}$ [5].

$$D_x f(x)=f'(x) \tag{2}$$

$$\hat{X} f(x)=x f(x) \tag{3}$$

From the property

$$(xf(x))'=xf'(x)+f(x), \forall f \text{ derivable} \tag{4}$$

we obtain the commutation relation

$$\left[D_x,\hat{X}\right]\equiv D_x\hat{X}-\hat{X}D_x\equiv\hat{I} \tag{5}$$

For generalizing the above relation we have proven the followingfundamentalidentitiesbetweenoperators [6]:

« If $A(x)$and $B(x)$ are two entire functions and $A(D_x)$, $B(\hat{X})$ two operators then

$$A(D_x)B(\hat{X})\equiv B(\hat{X})A(D_x)+\ldots+\frac{1}{k!}B^{(k)}(\hat{X})A^{(k)}(D_x)+\ldots \tag{6}$$

$$B(\hat{X})A(D_x)\equiv A(D_x)B(\hat{X})+\ldots+(-)^k\frac{1}{k!}A^{(k)}(D_x)B^{(k)}(\hat{X})+\ldots» \tag{7}$$

For example

$$D_x^{\,2}(\hat{X}-1)^2\equiv(\hat{X}-1)^2D_x^{\,2}+4(\hat{X}-1)D_x+\frac{4}{2!} \tag{8}$$

For clarity we will expose the proofs of these fundamental identities in Appendix.

Resolution of Differential Equation of Laguerre Polynomials

Consider the differential equation (1) written under the form

$$(\hat{X}D_x^{\,2} + (1-\hat{X})D_x + n)y = 0$$

i.e.

$$(\hat{X}D_x(D_x-1) + D_x + n)y = 0 \qquad (9)$$

In order to resolvethis ODE, let us apply an operator $A(D_x)$ onto both sides of it and utilizing the fundamental identity (6) we have, in writing A for $A(D_x)$ and D for D_x simplicity,

$$A\hat{X} \equiv \hat{X}A + A' \qquad (10)$$

$$A(\hat{X}D(D-1)+D+n)y = (\hat{X}D(D-1)+D+n)Ay + A'D(D-1)y \qquad (11)$$

Observing the above equation, it is natural to think of searching for an operator $A=A(D_x)$ such that $A'D(D-1)$ contains A . For this we have two choices

$$A \equiv D^{-m} \text{ and } A \equiv (D-1)^{-m} \qquad (12)$$

With the second choice we have

$$A'D(D-1) \equiv -mD(D-1)^{-m} \equiv -mDA \qquad (13)$$

and may transform the Laguerre differential equation (9) into

$$(XD^2 - (m-1)D - (XD-n))(D-1)^{-m}y = 0$$

i.e

$$((\hat{X}D_x - m + 1)D_x - (\hat{X}D_x - n))(D_x-1)^{-m}y = 0 \qquad (14)$$

Remarking that

$$(\hat{X}D_x - n)x^n = (\hat{X}D_x - n + 1)D_x x^n = 0 \qquad (15)$$

we choice $m = n$ and see that (14) leads to

$$(D_x-1)^{-n}y = c_n x^n \qquad (16)$$

For conclusion, we may state that

"The differential equation

$$(\hat{X}D_x^{\,2} + (1-\hat{X})D_x + 2n)y = 0$$

has as particular solution

$$y_n = c_n(D_x-1)^n x^n \qquad (17)$$

If we define the Laguerre polynomial as a particular solution verifying the condition

$$L_n(0) = 1 \qquad (18)$$

then we obtain a very interesting formula saying that

" A Laguerre polynomial is the transform of a monomial by the special differential operator $(D_x - 1)^n$"

i.e.

$$L_n(x) = (D_x-1)^n \frac{x^n}{n!} \qquad (19)$$

The formula (19) leads directly to many applications as we can see hereinafter.

Properties of Laguerre Polynomials

From the formula (19) we get

The Rodrigues formula

According to the fundamental identities (6)

$$D_x e^{-\hat{X}} \equiv e^{-\hat{X}}D_x - e^{-\hat{X}} \qquad (20)$$

we get the factorization relation

$$(D_x - \hat{I})^n \equiv e^{\hat{X}}D_x^{\,n}e^{-\hat{X}} \qquad (21)$$

which when applying for (19) gives immediately the Rodrigues formula

$$L_n(x) = e^x D_x^{\,n} e^{-x}x^n / n! \qquad (22)$$

Inversely from the Rodrigues formula (22) and the formula (20) we get directly (19).

The Lucas Symbolic Formula

From (19) we get

$$L_n(x) = (-)^n(1-D_x)^n\frac{x^n}{n!} = \sum_{k=0}^{n}(-)^{n-k}\binom{n}{k}D_x^{\,k}\frac{x^n}{n!}$$

$$= \sum_{k=0}^{n}(-)^{n-k}\binom{n}{k}\frac{1}{\backslash(n-k)!}x^{n-k} = \sum_{k=0}^{n}(-)^k\binom{n}{k}\frac{x^k}{k!} \qquad (23)$$

i.e. the Lucas symbolic formula for calculating explicitly $L_n(x)$

$$L_n(x) =: (L-x)^n,\ L_k = \frac{1}{k!} \qquad (24)$$

which is similar to the famous Lucas formula for Bernoulli polynomials [7].

The Derivatives of Laguerre Polynomials

From (19) we get immediately

$$L_n^{(k)}(x) = D_x^{\,k}L_n(x) = D_x^{\,k}(D_x-1)^n\frac{x^n}{n!} = (D_x-1)^n\frac{x^{n-k}}{(n-k)!}$$

$$= (D_x-1)^k L_{n-k}(x) \qquad (25)$$

The orthogonality of Laguerre Polynomials

Remarking from the definition of gamma function

$$\int_0^\infty e^{-x}x^{n-1}dx = \Gamma(n) = (n-1)! \qquad (26)$$

and from the fundamental identity (6)

$$e^{\hat{X}}D_x e^{-\hat{X}} \equiv (D_x-1) \qquad (27)$$

$$e^{\hat{X}}\int e^{-\hat{X}} \equiv (D_x-1)^{-1} \qquad (28)$$

$$(D_x-1)^{-1}\frac{\hat{X}^n}{n!} \equiv \sum_{k=0}^{n}\frac{1}{k!}\frac{\hat{X}^{n-k}}{(n-k)!}(-)^k k!(D_x-1)^{-1-k} \qquad (29)$$

we get

$$(D_x-1)^{-1}\frac{x^n}{n!}L_m(x)=\sum_{k=0}^{n}\frac{x^{n-k}}{(n-k)!}(-)^k(D_x-1)^{m-1-k}\frac{x^m}{m!} \quad (30)$$

so that for $0 < n < m$

$$\int_0^\infty e^{-x}\frac{x^n}{n!}L_m(x)dx=0-e^{-x}\sum_{k=0}^{n}\frac{x^{n-k}}{(n-k)!}(-)^k(D_x-1)^{m-1-k}\frac{x^m}{m!}\Big|_{x=0}$$

$$=-(-)^n(D_x-1)^{m-1-n}\frac{x^m}{m!}\Big|_{x=0}=0 \quad (31)$$

although for $0 \le n = m$, according to (26) and (28)

$$\int_0^\infty e^{-x}\frac{x^n}{n!}L_n(x)dx=-(-)^n(D_x-1)^{-1}\frac{x^n}{n!}\Big|_{x=0}=\frac{1}{n!}\Gamma(n+1)=1 \quad (32)$$

Because the polynomial $L_n(x)$ begins with the term $(-)^n\frac{x^n}{n!}$ we have for $0 < n < m$

$$\int_0^\infty e^{-x}L_n(x)L_m(x)dx=\delta_{n,m} \text{ QED} \quad (33)$$

Generating Functions

From (23) we have

$$x^nL_n(\frac{1}{x})=\sum_{k=0}^{n}(-)^k\binom{n}{k}\frac{x^{n-k}}{k!}={}_0F_1(-,1,-D_x)x^n \quad (34)$$

so that for $|xt|<1$

$$\sum_{n=0}^{\infty}x^nL_n(\frac{1}{x})t^n={}_0F_1(-,1,-D_x)(1-xt)^{-1}=(1-xt)^{-1}e^{-\frac{t}{1-xt}} \quad (35)$$

i.e, putting $z=\frac{1}{x}$, $xt=u$

$$\sum_{n=0}^{\infty}L_n(z)u^n=(1-u)^{-1}e^{-\frac{uz}{1-u}} \quad (36)$$

Study of Generalized Laguerre Polynomials by Operator Calculus

Resolution of Differential Equation

Consider the ODE of generalized Laguerre polynomials

$$xy''+(\alpha+1-x)y'+ny=0 \quad (37)$$

Under operator calculus it has the form

$$(\hat{X}D_x(D_x-1)+(\alpha+1)D_x+n)y=0 \quad (38)$$

Applying an operator $A(D_x)$ onto both sides and utilizing the fundamental identity (6) we have, in writing A for $A(D_x)$ and D for D_x for simplicity,

$$A(\hat{X}D(D-1)+(\alpha+1)D+n)y$$

$$=(\hat{X}D(D-1)+(\alpha+1)D+n)Ay+A'D(D-1)y=0 \quad (39)$$

Searching for an operator $A \equiv A(D_x)$ such that $A'(D_x)\,D_x\,(D_x-1)$ contains , $A(D_x)$ we have two choices

$$A(D_x)\equiv D_x^{-m-\alpha} \quad (40)$$

and

$$A(D_x)\equiv(D_x-1)^{-m-\alpha} \quad (41)$$

With the second choice we have from (39)

$$A\hat{X}\equiv\hat{X}A+A'\equiv\hat{X}A-(m+\alpha)(D-1)^{-m-\alpha-1}$$

$$\equiv\hat{X}A-(m+\alpha)(D-1)^{-m-\alpha-1} \quad (42)$$

$$A(\hat{X}D(D-1)+(\alpha+1)D+n)y$$

$$=((\hat{X}D(D-1)+(\alpha+1)D+n)-(m+\alpha)D)Ay$$

$$=((\hat{X}D(D-1)+D+n)A-mD)(D-1)^{-m-\alpha}y=0 \quad (43)$$

and consequently

$$((\hat{X}D_x-m+1)D_x-(\hat{X}D_x-n))(D_x-1)^{-m-\alpha}y=0 \quad (44)$$

Remarking that

$$(\hat{X}D_x-n)x^n=(\hat{X}D_x-n+1)D_xx^n=0 \quad (45)$$

we choice $m = n$ and see that

$$(D_x-1)^{-n-\alpha}y=c_nx^n$$

i.e. with the convention $L_n^{\alpha}(0)=(1+\alpha)_n/n!$

$$L_n^{\alpha}(x)=(-)^n(1-D_x)^{n+\alpha}\frac{x^n}{n!} \quad (46)$$

$$=(1-D_x)^{\alpha}(D_x-1)^n\frac{x^n}{n!}=(1-D_x)^{\alpha}L_n(x) \quad (47)$$

With the notation $(a_n) = a(a+1)...(a+n-1)$ we have the explicit formula

$$L_n^{\alpha}(x)=\sum_{k=0}^{n}(-)^k\frac{(1+\alpha)_n}{(1+\alpha)_k}\frac{x^k}{(n-k)!k!} \quad (48)$$

which may be put under the Lucas symbolic form

$$L_n^{\alpha}(x)=:(L^{\alpha}-x)^n \quad (49)$$

where for given undefined terms $(L^{\alpha})^k$ are to be replaced with

$$L_k^{\alpha}=\frac{1}{n!}\frac{(1+\alpha)_n}{(1+\alpha)_{n-k}} \quad (50)$$

For examples

$$L_3^{\alpha}(x)=\frac{(1+\alpha)_3}{6}(1-\frac{3}{(1+\alpha)_1}x+\frac{3}{(1+\alpha)_2}x^2-\frac{1}{(1+\alpha)_3}x^3)$$

$$L_n^{\alpha}(0)=L_n^{\alpha}=\frac{(1+\alpha)_n}{n!} \quad (51)$$

Properties of

The Representation by Hypergeometric Operator

$$x^nL_n^{\alpha}(\frac{1}{x})=(1+\alpha)_n\,{}_0F_1(-,1+\alpha,-D_x)\frac{x^n}{n!} \quad (52)$$

which represents $x^n L_n^{\alpha}(\frac{1}{x})$ under the form of a differential operator independent with respect to n

applying on the monomial $(1+\alpha)_n x^n / n!$

The Recurrence Relation Between $L_n^{\alpha}(x)$

From (45) we get

$$L_n^{\alpha}(x) = (-)^n (1-D_x)^{n+\alpha-1}(1-D_x)\frac{x^n}{n!}$$

$$= (-)^n (1-D_x)^{n+\alpha-1}(\frac{x^n}{n!} - \frac{x^{n-1}}{(n-1)!}) = L_n^{\alpha-1}(x) + L_{n-1}^{\alpha}(x) \quad (53)$$

For example

$$L_2^1(x) = \frac{x^2}{2} - 3x + 3 = L_2^0(x) + L_1^1(x) = (\frac{x^2}{2} - 2x + 1) + (2-x) \quad (54)$$

By this algorithm we may obtain many others recurrence relations.

Interrelations Between Laguerre Polynomials

Thanks to the identity deduced from the fundamental identity between operators (7) we have

$$\hat{X}^m (1-D_x)^{n+\alpha} \equiv \sum_{k=0}^{m} \frac{(-)^k}{k!}(1-D_x)^{(n+\alpha)^{(k)}} \hat{X}^{m(k)} \quad (55)$$

and may get from (45) the relation

$$(-)^n x^m (1-D_x)^{n+\alpha}\frac{x^n}{n!} = \sum_{k=0}^{m} \frac{(-)^{n-k}}{n!k!}\frac{(1+\alpha)_n}{(1+\alpha)_{n-k}}(-)^k (1-D_x)^{n+\alpha-k}\frac{m!}{(m-k)!}x^{m-k}x^n$$

$$= \frac{1}{n!}\sum_{k=0}^{m}(-)^{-m+k}(n+m-k)!\binom{m}{k}\frac{(1+\alpha)_n}{(1+\alpha)_{n-k}}(1-D_x)^{n-k+\alpha}(-)^{n+m-k}\frac{x^{n+m-k}}{(n+m-k)!}$$

i.e.

$$x^m L_n^{\alpha}(x) = \sum_{k=0}^{m}(-)^{m-k}\binom{m}{k}\frac{(n+m-k)!}{n!}\frac{(1+\alpha)_n}{(1+\alpha)_{n-k}}L_{n+m-k}^{\alpha-m}(x) \quad (56)$$

For examples

$$xL_n^{\alpha}(x) = \left[-\frac{n+1}{1}L_{n+1}^{\alpha-1}(x) + (n+\alpha)L_n^{\alpha-1}(x)\right] \quad (57)$$

$$xL_n^1(x) = \left[-\frac{n+1}{1}L_{n+1}(x) + (n+1)L_n(x)\right] \quad (58)$$

$$x(2-x) = -2L_2(x) + 2L_1(x) = -2(\frac{x^2}{2} - 2x + 1) + 2(1-x)$$

Derivatives of $L_n^{\alpha}(x)$

$$D_x^m L_n^{\alpha}(x) = (-)^n (1-D_x)^{n+\alpha}\frac{x^{n-m}}{(n-m)!} = (-)^m L_{n-m}^{\alpha+m}(x) \quad (59)$$

Addition formula of $L_n^{\alpha}(x)$

Because

$$\partial_{x+y} = \partial_x = \partial_y \quad (60)$$

we may write

$$L_n^{\alpha}(x+y) = (-)^n (1-D_x)^{n+\alpha}\frac{(x+y)^n}{n!} = (-)^n (1-D_x)^a (1-D_y)^{n+b}\frac{(x+y)^n}{n!}$$

$$= \sum_{k=0}^{n}(-)^{k+n-k}(1-D_x)^{a-k}(1-D_y)^{n-k+2k+b}\frac{x^k y^{n-k}}{k!(n-k)!}, a+b=\alpha$$

and obtain

$$L_n^{a+b}(x+y) = \sum_{k=0}^{n} L_k^{a-k}(x) L_{n-k}^{b+2k}(y) \quad (61)$$

For examples

$$L_1^1(x+y) = L_0^1(x)L_1^0(y) + L_1^0(x)L_0^1(y) = 2-x-y = (1-x)+(1-y)$$

$$L_1^2(x+y) = L_0^1(x)L_1^1(y) + L_1^0(x)L_0^3(y) = 3-(x+y) = (2-y)+(1-x) \quad (62)$$

Generating functions of $L_n^{\alpha}(x)$

Many generating functions of $L_n^{\alpha}(x)$ may be obtained from (45) and (51). They are

$$\sum_{n=0}^{\infty}\frac{1}{(1+\alpha)_n}x^n L_n^{\alpha}(\frac{1}{x})t^n = {}_0F_1(-,1+\alpha,-D_x)\sum_{n=0}^{\infty}\frac{x^n t^n}{n!} = {}_0F_1(-,1+\alpha,-t)e^{xt} \quad (63)$$

$$\sum_{n=0}^{\infty}x^n L_n^{\alpha}(\frac{1}{x})t^n = {}_0F_1(-,1+\alpha,-D_x)\sum_{n=0}^{\infty}(1+\alpha)_n\frac{x^n t^n}{n!}, |xt|<1$$

$$= {}_0F_1(-,1+\alpha,-D)(1-xt)^{-1-\alpha} = (1-xt)^{-1-\alpha}e^{-\frac{t}{1-xt}} \quad (64)$$

$$\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}x^n L_n^{\alpha}(\frac{1}{x})t^n = {}_0F_1(-,1+\alpha,-D_x)(1-xt)^{-1}, |xt|<1$$

$$= {}_1F_1(1,1+\alpha,-\frac{t}{1-xt})(1-xt)^{-1} \quad (65)$$

Putting $xt=u$, $x=1/z$ we get generating functions of $L_n^{\alpha}(x)$

$$\sum_{n=0}^{\infty}\frac{1}{(1+\alpha)_n}L_n^{\alpha}(z)u^n = {}_1F_1(-,1+\alpha,-uz)e^{ut} \quad (66)$$

$$\sum_{n=0}^{\infty}L_n^{\alpha}(z)u^n = (1-u)^{-1-\alpha}e^{-\frac{uz}{1-u}} \quad (67)$$

$$\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}L_n^{\alpha}(z)u^n = {}_1F_1(1,1+\alpha,-\frac{uz}{1-u})(1-u)^{-1} \quad (68)$$

Relations between generalized Laguerre and Hermite polynomials

From (47) we get

$$L_n^{\alpha}(x^2) = (-)^n (1-(\hat{X}^{-1}D_x)^{n+\alpha}\frac{x^{2n}}{n!} \quad (69)$$

Remarking that

$$(\hat{X}^{-1}D_x))^k \frac{x^{2n}}{(2n)!} = \frac{2^{k-1}(n-1)!}{(n-k)!(2n-1)!}x^{2(n-k)} \quad (70)$$

$$D_x^{2k}\frac{x^{2n}}{(2n)!} = \frac{x^{2n-2k}}{(2n-2k)!} \quad (71)$$

we get

$$L_n^{-\frac{1}{2}}(x^2)=(-)^n\sum_{k=0}^{n}(-)^k\frac{2^{-2n}(2n)!}{n!}\frac{(n-k)!}{2^{2k-2n}(2n-2k)!}\frac{x^{2n-2k}}{(n-k)!k!}$$

$$=\frac{(-)^n}{n!}\sum_{k=0}^{n}\frac{(-)^k}{k!}\frac{D_x^{2k}}{2^{2k}}x^{2n}=\frac{(-)^n}{n!}e^{-\frac{D_x^2}{4}}x^{2n} \quad (72)$$

But from [8]

$$H_n(x)=e^{-\frac{D_x^2}{4}}(2x)^n \quad (73)$$

$$\frac{(-)^n}{n!}e^{-\frac{D_x^2}{4}}x^{2n}=\frac{(-)^n}{n!2^{2n}}H_{2n}(x)$$

so that finally

$$L_n^{-\frac{1}{2}}(x^2)=\frac{(-)^n}{n!2^{2n}}H_{2n}(x) \quad (74)$$

By similar calculation we get also

$$xL_n^{1/2}(x^2)=\frac{(-)^n}{n!2^{2n+1}}H_{2n+1}(x) \quad (75)$$

Remarks and Conclusion

This work firstly proposes the use of a special operator, say $(D_x-1)^{n+\alpha}$, for resolving ordinary differential equations, say those of Laguerre and generalized Laguerre polynomials. This leads to the result that Laguerre polynomials and generalized one's are the transforms of monomials by the said special operator.

Secondly we obtain quasi all the well-known properties of Laguerre polynomials by utilizing this special operator together with the couple of operators $(D_x,\hat{X})$, exactly as in quantum mechanics we utilize simultaneously the momentum and position operators respecting the Dirac permutation relation.

Last but not least the aim of this work is to interest readers in utilizing operator calculus, not only differential calculus, in mathematics and physics. For that we cite in references some works of some authors in this domain [9-11]. For facilitate readers aiming to follow this direction we give as example in Appendix the proofs of fundamental identities in Operator calculus [12].

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Appendix

The fundamental identity in operator calculus

Proof of the identity $\boldsymbol{f(D)X^m\equiv\sum_{k=0}^{m}\binom{m}{k}X^{m-k}f^{(k)}(D)}$

From the identity

$$\left[D_x,\hat{X}\right]\equiv I \quad (1)$$

we may deduce successively, in writing D^m for D_x^m and X^m for $\hat{X}^m$, that

$$\left[D^m,X\right]\equiv mD^{m-1}\equiv(D^m)' \quad (2)$$

$$f(D)X-Xf(D)\equiv f'(D) \quad (3)$$

$$f(D)X^2\equiv(Xf(D)+f'(D))X\equiv X^2f(D)+2Xf'(D)+f''(D) \quad (4)$$

$$f(D)X^3\equiv X^3f(D)+3X^2f'(D)+3Xf''(D)+f'''(D) \quad (5)$$

and so on.

Thank to this remark we suppose that

$$f(D)X^m\equiv\sum_{k=0}^{m}\binom{m}{k}X^{m-k}f^{(k)}(D) \quad (6)$$

In order to prove (6) by recurrence we utilize (1) to proceed

$$f(D)X^{m+1}\equiv\sum_{k=0}^{m}\binom{m}{k}X^{m-k}f^{(k)}(D)X\equiv\sum_{k=0}^{m}\binom{m}{k}X^{m-k}(Xf^{(k)}(D)+f^{(k+1)}(D))$$

$$\equiv\sum_{k=0}^{m}\binom{m}{k}(X^{m+1-k}f^{(k)}(D)+\sum_{k=1}^{m+1}\binom{m}{k-1}X^{m-k+1}f^{(k)}(D))$$

$$\equiv X^{m+1}f(D)+\sum_{k=1}^{m}(\binom{m}{k}+\binom{m}{k-1})X^{m-k+1}f^{(k)}(D))+f^{(m+1)}(D) \quad (7)$$

$$f(D)X^{m+1}\equiv\sum_{k=0}^{m+1}\binom{m+1}{k}X^{m+1-k}f^{(k)}(D)\quad \text{QED} \quad (8)$$

Combining the above result and the fact that

$$f(D)X^0\equiv X^0f(D) \quad (9)$$

we conclude that (6) is correct.

Transforming *F(D) g(X)* into sum of operators in which *X* precedes *D.*

Under the form (6) we can't proceed further because the mixed coefficient $(m-k)!$ doesn't permit summations with respect to m.

In order to bypass this obstacle, we make use of the relation

$$\frac{m!}{(m-k)!}x^{m-k}=m(m-1)...(m-k+1)x^{m-k}\equiv x^{m(k)} \quad (10)$$

where $x^{m(k)}$ is the k^{th} derivative of x^m and obtain

$$f(D_x)\hat{X}^m \equiv \sum_{k=0}^{m} \binom{m}{k} \hat{X}^{m-k} f^{(k)}(D_x) \equiv \sum_{k=0}^{m} \frac{1}{k!} \hat{X}^{m(k)} f^{(k)}(D_x)$$

$$\equiv \sum_{k=0}^{\infty} \frac{1}{k!} \hat{X}^{m(k)} f^{(k)}(D_x) \tag{11}$$

where $\hat{X}^{m(k)}$ is obtained by replacing $x^{m(k)}$ in the function with the operator $\hat{X}$.

Finally we may conclude that for and expandable into Taylor series

$$f(D_x)g(\hat{X}) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(\hat{X}) f^{(k)}(D_x) \tag{12}$$

$$g(\hat{X})f(D_x) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!} f^{(k)}(D_x) g^{(k)}(\hat{X}) \tag{13}$$

The formulae (12) and (13) form the fundamentalidentities of operatorcaculus.

Apply the fundamental identity (12) on a function *h(x)* we find again the formula

$$f(D_x)g(x)h(x) = \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(x) f^{(k)}(D_x)h(x) \tag{14}$$

that Forsyth had found in 1888 by generalizing the Leibnitz formula but did not give details of calculations [9].

Invariance of the fundamental identity

Inspecting the way we obtain the fundamental identity in operator calculus we see that it is the consequence of one and only one condition which is that the dual couple of operators $(D_x, \hat{X})$ must respect the canonical identity $[D_x, \hat{X}] \equiv I$ We thus obtain an extremely important corollary saying that "The fundamental identity is invariant under replacing the couple of operators $(D_x, \hat{X})$ with any other couple respecting the condition "$[A,B] \equiv I$.

For example, by remarking that

$$[-X, D] \equiv -XD + DX \equiv I \tag{15}$$

we get

$$f(-X)g(D) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(D) f^{(k)}(-X)$$

Putting $h(x) = f(-x)$, we have $h'(x) = -f'(-x)$ and

$$h(X)g(D) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!} g^{(k)}(D) h^{(k)}(X)$$

i.e., because $h(x)$ is also arbitrary as is $f(x)$

$$f(X)g(D) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!} g^{(k)}(D) f^{(k)}(X) \text{ QED}$$

By replacing *(D, X)* with $(\alpha a^+ + \beta a, \gamma a^+ + \delta a)$ where a^+, a are creation and annihilation operators in quantum mechanics and $\alpha\delta - \beta\gamma = 1$ in (12) we obtain many other identities for operator calculus.

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