Free Ribbon Lemma for Surface-Link

Akio Kawauchi*

Osaka Central Advanced Mathematical Institute,
Osaka Metropolitan University Sugimoto, Japan

***Corresponding Author**
Akio Kawauchi, Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University Sugimoto, Japan

Abstract
A free surface-link is a surface-link whose fundamental group is a free group not necessarily meridian-based. Free ribbon lemma says that every free sphere-link in the 4-sphere is a ribbon sphere-link. Four different proofs of Free ribbon lemma are explained. The first proof is done in an earlier paper. The second proof is done by showing that there is an O2-handle basis of a ribbon surface- link. The third proof is done by removing the commuter relations from a Wirt Inger presentation of a free group, which a paper on another proof of Free ribbon lemma complements. The fourth proof is given by the special case of the proof of the result that every free surface-link is a ribbon surface-link which is a stabilization of a free ribbon sphere-link. As a consequence, it is shown that a surface-link is a sublink of a free surface-link if and only if it is a stabilization of a ribbon sphere-link.

Keywords: Free Ribbon Lemma, Free Surface-Link, Ribbon Sphere-Link, Stabilization.

1. Introduction

A *surface link* is a closed, possibly disconnected, oriented surface F smoothly embedded in the 4-sphere S^4, and it is called a *surface knot* if F is connected. If F consists of 2-spheres F_i ($i = 1, 2, \ldots, r$), then F is called a *sphere-link* (or an S^2-*link*) of r components. A surface-link F is a trivial surface-link (that is, surface-link bounding disjoint handlebodies in S^4) if the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is a meridian- based free group, [1-3]. A surface-link F is *ribbon* if F is obtained from a trivial S^2-link O in S^4 by surgery along a smoothly embedded disjoint 1-handle system h^O on O, [4-7]. A surface-link F in the 4-sphere S^4 is *free* if the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is a (not necessarily meridian-based) free group. In this paper, four different proofs of the following *Free ribbon lemma* and its generalization to a general free surface-link are explained.

1.1. Free Ribbon Lemma
Every free S^2-link in S^4 is a ribbon S^2-link.

Free ribbon lemma leads to the following conjectures: Poincare conjecture, [8-11]. J. H. C. Whitehead asphericity conjecture for aspherical 2-complex, [12-15]. Kervaire conjecture on group weight, [16-20]. The first proof is given [13]. For convenience, an outline of the first proof is explained here.

2. First Proof of Free Ribbon Lemma
Let L_i ($i = 1, 2, \ldots, r$) be the components of a free S^2-link L in S^4. By a base change of the free fundamental group $\pi_1(S^4 \setminus L, x_0)$, take a basis x_i ($i = 1, 2, \ldots, r$) of $\pi_1(S^4 \setminus L, x_0)$ inducing a meridian basis of L in $H_1(S^4 \setminus L; Z)$, [19]. Let Y be the 4-manifold obtained from S^4 by surgery along L, which is diffeomorphic to the connected sum of r copies $S^1 \times S_i^3$ ($i = 1, 2, \ldots, r$) of $S^1 \times S_i^3$, [13,21]. Under a canonical isomorphism $\pi_1(S^4 \setminus L, x_0) \rightarrow \pi_1(Y, x_0)$, the factors $S^1 \times p_i$ ($i = 1, 2, \ldots, r$) of $S^1 \times S_i^3$ ($i = 1, 2, \ldots, r$) with suitable paths to the base point x_0 represent the basis x_i ($i = 1, 2, \ldots, r$). Let k_i ($i = 1, 2, \ldots, r$) be the loop system in Y produced from the components L_i ($i = 1, 2, \ldots, r$) by the surgery. By using the fact that any homotopy deformations of k_i ($i = 1, 2, \ldots, r$) in Y do not change the link type of the S^2-link L in S^4, the loop system k_i ($i = 1, 2, \ldots, r$) is homotopically deformed in Y so that the S^2-link L in S^4 obtained from the deformed loop system k_i ($i = 1, 2, \ldots, r$) by back surgery is a ribbon S^2-link in S^4, completing the proof of Free ribbon lemma.

3. Second and third proof of Free ribbon lemma

To explain the second and third proofs of Free ribbon lemma, the notion of an O2-handle basis of a surface-link is needed, [1, 22]. An O2-handle pair on a surface-link F in S^4 is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I, D' \times I$ on F in S^4 which intersect orthogonally only with the attaching parts $(\partial D) \times I, (\partial D') \times I$ to F, so that the intersection $Q = (\partial D) \times I \cap (\partial D') \times I$ is a square. Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F. Let $F(D \times I)$ and $F(D' \times I)$ be the surface- links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union $\delta \cup F_\delta^c$ of the plumbed disk

$$\delta = \delta_{D \times I', D' \times I} = D \times \partial I \cup Q \cup D' \times \partial I$$

and the surface $F_\delta^c = \mathrm{cl}(F \setminus (\partial D \times I \cup \partial D' \times I))$. The surface-links $F(D \times I)$, $F(D' \times I)$ and $F(D \times I, D' \times I)$ are equivalent surface-links, [1]. An O2-handle basis of a surface-link F is a disjoint system of O2-handle pairs $(D_i \times I, D_i' \times I)$ $(i = 1, 2, \ldots, r)$ on F in S^4 such that the boundary loop pair system $(\partial D_i, \partial D_i')$ $(i = 1, 2, \ldots, r)$ of the core disk system (D_i, D_i') $(i = 1, 2, \ldots, r)$ of $(D_i \times I, D_i' \times I)$ $(i = 1, 2, \ldots, r)$ is a spin loop basis for F in S^4, which is a system of a spin loop basis of every component F_i of F. Note that there is a spin loop basis for every surface-knot in F, [3]. For simplicity, an O2-handle basis $(D_i \times I, D_i' \times I)$ $(i = 1, 2, \ldots, r)$ for F is denoted by $(D \times I, D' \times I)$. The surgery surface-link of F by $(D_i \times I, D_i' \times I)$ $(i = 1, 2, \ldots, r)$ is denoted by $F(D \times I, D' \times I)$. The following theorem is shown for the second and third proofs of Free ribbon lemma.

3.1. Theorem 1.1

For every free ribbon surface-link F in S^4, there is an O2-handle basis $(D \times I, D' \times I)$ on F in S^4 such that $D \times I$ belongs to the 1-handle system of the ribbon surface-link F.

The second proof of Free ribbon lemma is explained as follows.

3.2. Second Proof of Free Ribbon Lemma

Let L be a free S^2-link such that the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is isomorphic to the free fundamental group $\pi_1(S^4 \setminus L, x_0)$ by a meridian-preserving isomorphism, [24]. By Theorem 1.1, the surgery surface-link $L' = F(D \times I, D' \times I)$ is a ribbon S^2-link, [1,22]. Then there is a meridian-preserving isomorphism $\pi_1(S^4 \setminus L', x_0) \to \pi_1(S^4 \setminus L, x_0)$ on free groups, which implies that L' is equivalent to L, [13,24]. Thus, L is a ribbon S^2-link, completing the proof of Free ribbon lemma.

The third proof of Free ribbon lemma is related to a Wirtinger presentation of a free group. A finite group presentation $(x_1, x_2, \ldots, x_n| R_1, R_2, \ldots, R_m)$ is a Wirtinger presentation if $R_j = W_j x_{s_j} W_j^{-1} x_{t_j}^{-1}$ for some word W_j in the free group $(x_1, x_2, \ldots, x_n)$ and some indexes s_j, t_j in $\{1, 2, \ldots, n\}$ for every j $(j = 1, 2, \ldots, m)$. The relator R_j is a *commutator relation* if $x_{s_j} = x_{t_j}$. It is well-known that a Wirtinger presentation of a finitely presented group G with $H_1(G; Z) \cong Z^r$ is always equivalent (without changing the generating set) to a Wirtinger presentation P such that the Wirtinger presentation P' obtained by removing all the commutator relations from P has

deficiency r. Such a Wirtinger presentation P is called a *normal Wirtinger presentation*. The following corollary is obtained from Theorem 1.1.

3.3. Corollary 1.2

If a free group G of rank r has a normal Wirtinger presentation P, then G has the Wirtinger presentation P' of deficiency r obtained from P by removing all the commutator relations.

3.3.1. Proof of Corollary 1.2 assuming Theorem 1.1

For a free group G of rank r, let $P = (x_1, x_2, \ldots, x_n| R_1, R_2, \ldots, R_m)$ be a normal Wirtinger presentation of G such that the relators R_j $(n - r + 1 \le j \le m)$ are the commutator relations. Let O be a trivial S^2-link of n components in S^4 such that the meridian basis of the free fundamental group $\pi_1(S^4 \setminus O, x_0)$ are identified with x_i $(i = 1, 2, \ldots, n)$. Let h_j $(1 \le j \le m)$ be the 1-handles on O indicated by the relators R_j $(1 \le j \le m)$. By the van Kampen theorem, the ribbon surface-link F in S^4 obtained by surgery along h_j $(1 \le j \le m)$ has the normal Wirtinger presentation P of the fundamental group $\pi_1(S^4 \setminus F, x_0)$ with the meridian generators set $\{x_1, x_2, \ldots, x_n\}$, [25, 26]. Let L be the ribbon surface-link obtained from O by surgery along the 1-handles h_j $(1 \le j \le n-r)$, which is a ribbon S^2-link of r components. The fundamental group $\pi_1(S^4 \setminus L, x_0)$ has the Wirtinger presentation P' of deficiency r obtained from P by removing all the commutator relations. By Theorem 1.1, the 1-handles h_j $(n - r + 1 \le j \le m)$ on L are trivial 1-handles, so that $\pi_1(S^4 \setminus L, x_0)$ is isomorphic to $\pi_1(S^4 \setminus F, x_0)$ by a meridian-preserving isomorphism. This completes the proof of Corollary 1.2 assuming Theorem 1.1.

The author has published a paper on another proof of Free ribbon lemma, which this paper complements, [24]. The third proof of Free ribbon lemma is nothing but the proof of the paper except for adding to it the assertion of Corollary 1.2 which was missing from it. For convenience, an outline of the third proof is explained here.

3.4 Third Proof of Free Ribbon Lemma

Let L be a free S^2-link of r components. Since the fundamental group $G = \pi_1(S^4 \setminus L, x_0)$ is a free group with $H_1(G; Z) = Z^r$ and $H_2(G; Z) = 0$, there is a normal Wirtinger presentation P of G whose generator set comes from meridians of L in S^4, [24,27]. Note that there is also another method to find such a normal Wirtinger presentation P using a normal form of L in S^4, [25,26,28, 29]. Let L' be a ribbon S^2-link given by the Wirtinger presentation P' obtained from P by removing all the commutators. By Corollary 1.2, there is a meridian- preserving isomorphism $\pi_1(S^4 \setminus L', x_0) \to \pi_1(S^4 \setminus L, x_0)$, so that L' is equivalent to L. Thus, L is a ribbon S^2-link, completing the proof of Free ribbon lemma.

4. Fourth proof of Free ribbon lemma

The fourth proof of Free ribbon lemma is a direct consequence of the following theorem, restricting F to every free S^2-link.

4.1. Theorem 1.3

Every free surface-link F in S^4 is a ribbon surface-link in S^4.

Thus, after the proofs of Theorems 1.1 and 1.3, there are four

different proofs of Free ribbon lemma. To generalize the free ribbon lemma to a free surface-link, the notion of a stabilization of a surface-link is needed, [1,22]. A *stabilization* of a surface-link L is a connected sum $F = L\#_{k=1}^s T_k$ of L and a system of trivial torus-knots T_k ($k = 1, 2, \ldots, s$). By granting $s = 0$, a surface-link L itself is regarded as a stabilization of L. Free ribbon lemma is generalized to a general free surface-link as follows.

4.2. Corollary 1.4
Every free surface-link F in S^4 is a stabilization of a free ribbon S^2-link L in S^4.

4.2.1. Proof of Corollary 1.4 Assuming Theorems 1.1 and 1.3
Theorems 1.1 and 1.3 imply that every free surface-link F is a ribbon surface-link and a stabilization of a free S^2-link L, [1]. By Free ribbon lemma, the free S^2-link L is a ribbon S^2-link. This result also follows directly, [22]. This completes the proof of Corollary 1.4 assuming Theorems 1.1 and 1.3.

It is shown that an S^2-link L is a sublink of a free S^2-link if and only if L is a ribbon S^2-link, [13]. The following corollary generalizes this property to a general surface-link.

4.3. Corollary 1.5
A surface-link L in S^4 is a sublink of a free surface-link F in S^4 if and only if L is a stabilization of a ribbon S^2-link in S^4.

4.3.1. Proof of Corollary 1.5 Assuming Corollary 1.4
If L is a sublink of a free surface-link F, then L is a stabilization of a ribbon S^2-link since every free surface-link is a stabilization of a free ribbon S^2-link by Corollary 1.4. Conversely, if L is a stabilization of a ribbon S^2-link, then L is a sublink of a stabilization of a free ribbon S^2-link which is a free surface-link F since every ribbon S^2-link is a sublink of a free S^2-link. This completes the proof of Corollary 1.5 assuming Corollary 1.4.

5. Proofs of Theorems 1.1 and 1.3
Let F be a free surface-link in S^4 with components F_i ($i = 1, 2, \ldots, r$). Let $N(F) = \cup_{i=1}^r N(F_i)$ be a tubular neighborhood of $F = \cup_{i=1}^r F_i$ in S^4 which is a trivial normal disk bundle $F \times D^2$ over F, where D^2 denotes the unit disk of complex numbers of norm $\leqq 1$. Let $E = E(F) = \text{cl}(S^4 \setminus N(F))$ be the exterior of F in S^4. The boundary $\partial E = \partial N(F) = \cup_{i=1}^r \partial N(F_i)$ of the exterior E is a trivial normal circle bundle over $F = \cup_{i=1}^r F_i$. Identify $\partial N(F_i) = F_i \times S^1$ for $S^1 = \partial D^2$ such that the composite inclusion

$$F_i \times 1 \to \partial N(F_i) \to \text{cl}(S^4 \setminus N(F_i))$$

induces the zero-map in the integral first homology. The following lemma uses the assumption that the fundamental group $\pi_1(E, x_0)$ is a free group of rank r and the fact that the first homology group $H_1(E; Z)$ is a free abelian group of rank r with meridian basis.

5.1. Lemma 2.1
The composite inclusion $F_i \times 1 \to \partial N(F_i) \to E$ is null-homotopic for all i.

5.1.1. Proof of Lemma 2.1
Since $\partial N(F_i) = F_i \times S^1$, the fundamental group elements between the factors $F_i \times 1$ and $q_i \times S^1$ are commutive. Let a_i ($i = 1, 2, \ldots, r$) be embedded edges with common vertex x_0 in E such that $a_i \setminus \{x_0\}$ ($i = 1, 2, \ldots, r$) are mutually disjoint and $a_i \cap (\cup_{j=1}^r F_j \times 1) = p_i \times 1$ for a point p_i of $F_i \times 1$. The surface $F_i \times 1$ in $\partial N(F_i) = F_i \times S^1$ is chosen so that the inclusion $F_i \times 1 \to \text{cl}(S^4 \setminus N(F_i))$ induces the zero-map in the integral first homology. Since $H_1(E; Z)$ is a free abelian group of rank r with meridian basis and $\pi_1(E, x_0)$ is a free group of rank r, the image of the homomorphism $\pi_1(a_i \cup F_i \times S^1, x_0) \to \pi_1(E, x_0)$ is an infinite cyclic group generated by the homotopy class $[a_i \cup p_i \times S^1]$. This implies that the inclusion $F_i \times 1 \to E$ is null-homotopic. This completes the proof of Lemma 2.1.

By using the free group $\pi_1(E, x_0)$ of rank r, let $\Gamma = \cup_{i=1}^r a_i \cup c_i$ be a connected graph in the interior $\text{Int}(E)$ of E consisting of embedded edges a_i ($i = 1, 2, \ldots, r$) with the common base point x_0 and disjoint embedded circles C_i ($i = 1, 2, \ldots, r$) such that
(1) the half-open edges $a_i \setminus \{x_0\}$ ($i = 1, 2, \ldots, r$) are mutually disjoint and $a_i \cap (\cup_{j=1}^r c_i) = v_i$, a point in C_i for every i,
(2) the inclusion $i : (\Gamma, x_0) \to (E, x_0)$ induces an isomorphism $i_\# : \pi_1(\Gamma, x_0) \to \pi_1(E, x_0)$, and
(3) the homology class $[p_i \times S^1] = [C_i]$ in $H_1(E; Z)$ for all i.

In fact, by (2), the homotropy classes $[a_i \cup C_i]$ ($i = 1, 2, \ldots, r$) form a basis of the free group $\pi_1(E, q_0)$. (3) is obtained by a base change of the free group $\pi_1(E, x_0)$, [19]. Since Γ is a $K(\pi, 1)$-space, there is a piecewise-linear map $f : (E, x_0) \to (\Gamma, x_0)$ inducing the inverse isomorphism $f_\# = (i_\#)^{-1} : \pi_1(E, q_0) \to \pi_1(\Gamma, q_0)$ of $i_\#$, and by the homotropy extension property, the restriction of f to Γ is the identity map, [30]. By Lemma 2.1, the restriction of f to ∂E is homotopic to the composite map

$$g : \partial E = F \times S^1 \to \cup_{i=1}^r q_i \times S^1 \to \Gamma$$

such that the first map $F \times S^1 \to \cup_{i=1}^r q_i \times S^1$ is induced from the constant map $F \to \cup_{i=1}^r \{q_i\}$ and the second map $\cup_{i=1}^r q_i \times S^1 \to \Gamma$ is defined in the map f. By using a a boundary collar of ∂E in E, assume that the piecewise-linear map $f : (E, x_0) \to (\Gamma, x_0)$ defines the map $g : \partial E \to \Gamma$. For a non-vertex point p_i of C_i, the preimage $V_i = (f)^{-1}(p_i)$ is a bi-collarable compact oriented proper piecewise-linear 3-manifold in E. Take the compact 4-manifold E' obtained from E by splitting along $V = \cup_{i=1}^r V_i$ to be connected. Then join the components in each V_i with 1-handles in E'. By these modifications, V_i is assumed to be connected for all i within a homotopic deformation of f. The boundary ∂V_i is the disjoint union $P_i(F)$ of m_{ij} parallel copies $m_{ij} F_j$ of $F_j \times 1$ for all j ($j = 1, 2, \ldots, r$) in S^4, where m_{ii} is an odd integer and m_{ij} for distinct i, j is an even integer. An *anti-parallel surface-link* of F_j in S^4 is the boundary $\partial(F_j \times J)$ of a normal line bundle $F_j \times J$ of F_j in S^4 such that the natural homology homomorphism $H_1(F_j \times 1; Z) \to H_1(S^4 \setminus F_j \times 0; Z)$ is the zero-map, where $J = [0,1]$. Note that the boundary $\partial(F_j^{(0)} \times J)$ for a compact once-punctured surface $F_j^{(0)}$ of F_j is a trivial surface-knot in S^4 since $F_j^{(0)} \times J$ is a handlebody. Let $n_{ii} = (m_{ii} -1)/2$ and $n_{ij} = m_{ij}/2$ for distinct i, j. The surface-link $P_i(F) = \partial V_i$

consists of a component identified with F_i and some anti-parallel surface links of F_j in $P_i(F)$, denoted by $P_{ijk}(F)$ (j= 1, 2, ..., r; k= 1, 2, ..., n_{ij}). Let $P(F) = \cup_{i=1}^r P_i(F)$ be the surface-link in S^4. The proof of Theorem 1.1 is done as follows.

5.2. Proof of Theorem 1.1

A ribbon surface-link F is obtained from a trivial S^2-link O in S^4 by surgery along a disjoint 1-handle system h^O on O, so that the surface-link $P(F)$ of a free ribbon surface-link F is a ribbon surface-link obtained from a trivial S^2-link $P(O)$, a parallel S^2-link of O, in S^4 by surgery along a disjoint 1-handle system $P(h^O)$, a parallel 1-handle system of h^O, on $P(O)$. Let $VP(F)$ be a multi punctured handlebody system with $\partial VP(F) = P(F) \cup P(O)$ constructed from the collar $P(O) \times [0, 1]$ of O in S^4 by attaching the 1-handle system $P(h^O)$ to $P(O) \times 0 = P(O)$ which is called a SUPH system for the ribbon surface-link $P(F)$ in S^4, [23]. Let (ℓ, ℓ') be a spin loop basis for $P(F)$ consisting of a spin loop basis of every component of $P(F)$, where the spin loop system ℓ is the boundary loop system of a transverse disk system D of the 1-handle system $P(h^O)$ and the spin loop system ℓ' is the union of an arc system parallel to the core arc system of $P(h_O)$ and an arc system in $P(F)$. Because of the isomorphism $f_{\#}$, the compact 4-manifold E' is simply connected, so that every loop of ℓ' is null-homotopic in E' and hence bounds an immersed disk system in E'. Let (ℓ_1, ℓ'_1) be a loop pair in (ℓ, ℓ') meeting at a boundary point of a 1-handle h^O_1 in $P(h^O)$, and δ'_1 an immersed disk in E' with an immersed disk in E' with $\partial \delta'_1 = \ell'_1$. By thickening the arc $\ell'_1 \cap P(h^O)$ to the 1-handle h^O_1, the loop pair (ℓ_1, ℓ'_1) bounds an immersed O2-handle pair $(D_1 \times I, \delta'_1 \times I)$ on $P(F)$ and hence there is an O2-handle pair $(D_1 \times I, D'_1 \times I)$ on $P(F)$ such that the boundary loop pair $(\partial D_1, \partial D'_1)$ of the core disk pair (D_1, D'_1) is the loop pair (ℓ_1, ℓ'_1) by Recovery Lemma, [2]. By using the 3-ball $D_1 \times I \cup D'_1 \times I$, every loop of $\ell' \setminus \ell'_1$ bounds an immersed disk system in E' not meeting $D'_1 \times I$. Thus, by continuing this procedure, the spin simple loop basis (ℓ, ℓ') bounds an O2-handle basis $(D \times I, D' \times I)$ on $P(F)$ such that the 2-handle system $D \times I$ belongs to $P(h^O)$, whose subsystem on F gives a desired O2-handle basis on F in S^4. This completes the proof of Theorem 1.1.

Let h_i be a disjoint 1-handle system on $P_i(F)$ embedded in V_i such that the surface $P_i(F; h_i)$ obtained from $P_i(F)$ by surgery along h_i is connected and the genus of $P^i(F; h_i)$ is equal to the total genus of $P_i(F)$. Assume that one copy of the parallel $m_u F_i$ of F_i is identified with F_i and just one 1-handle h_i^F of h_i attaches to F_i. Let $P(F; h) = \cup_{i=1}^r P_i(F; h_i)$ be a surface-link in S^4. By further taking a disjoint 1-handle system h'_i on $P_i(F; h_i)$ embedded in V_i, the closed surface $P_i(F; h_i, h'_i)$ obtained from $P_i(F; h_i)$ by surgery along h'_i bounds a handlebody V'_i in V_i, so that the surface-link $P(F; h, h') = \cup_{i=1}^r P_i(F; h_i, h'_i)$ is a trivial surface-link in S^4. Because of the isomorphism $f_{\#}$, the compact 4-manifold E' is simply connected, so that the 1-handle system $h' = \cup_{i=1}^r h'_i$ is a trivial 1-handle system on the surface-link $P(F; h)$ in S^4, [3,31]. Thus, the surface-link $P(F; h)$ is a trivial surface-link in S^4, [1,2].
The proof of Theorem 1.3 is done as follows.

5.3. Proof of Theorem 1.3.

Since E' is simply connected, the 1-handle system h_i can be chosen

so that a trivial surface-knot F_{ijk} is obtained from the anti-parallel surface-link $P_{ijk}(F)$ in $P_i(F)$ by surgery along a unique 1-handle h_{ijk} in h_i. Let $h(0)$ be the system of the 1-handles h_{ijk} (i, j = 1,2, ..., r; k = 1, 2, ..., n_{ij}) in h, and $h(1)$ the complementary system of $h(0)$ in h. Let $P(F; h(0))$ be the surface-link obtained from $P(F)$ by surgery along $h(0)$, which consists of the surface-link F and the trivial surface-knots F_{ijk} (i, j = 1, 2, ..., r; k = 1, 2, ..., n_{ij}). The trivial surface-link $P(F; h)$ is obtained from the surface-link $P(F; h(0))$ by surgery along the 1-handle system $h(1)$. As stated just before the proof of Theorem 1.1, the surface-link $P(F; h, h')$ bounds a handlebody system V' in V. Note that h' is a trivial 1-handle system on $P(F; h)$. Then there is a disjoint handlebody system U in S^4 with $\partial U = P(F; h)$ extending the handlebody system V' by adding a 2-handle system $e \times I$ which makes an O2-handle system together with a thickened transverse disk system $m(h')$ of h', [1, 2]. Let d be a transverse disk system of the 1-handle system $h(1)$. In general, the disk system d meets the core disk system e of the 2-handle system $e \times I$ transversely in finite points in S^4, but the handlebody system U is isotopically deformed so to have $d \cap U = \partial d$ by an isotopic deformation of U in a neighborhood of the 3-disk system $e \times I \cup m(h') \times I$ in S^4, [2]. For $h(1) = d \times I$, the union $U \cup h(1)$ is a compact oriented 3-manifold with boundary $P(F; h(0))$ obtained from U by adding the 2-handle system $h(1)$ since $h(1) \cap U = h(1) \cap \partial U = (\partial d) \times I$. Let (ℓ, ℓ') be a spin loop basis of $P(F; h)$ given for $P(F; h(0))$ such that when restricted to every trivial surface-knot F_{ijk}, it becomes a standard spin loop basis. Since $P(F; h)$ is a trivial surface-link, there is an orientation-preserving diffeomorphism w of S^4 sending $P(F; h)$ to the boundary ∂W of a standard handlebody system W in S^4 such that the spin loop basis $(w(\ell), w(\ell'))$ of ∂W is a meridian-longitude pair system of W, [1, 33]. Let $U(W) = w^{-1}(W)$ be the handlebody system in S^4 with $\partial U(W) = P(F; h)$. The spin loop basis $(w(\ell), w(\ell'))$ of ∂W bounds a core disk-pair system (δ, δ') of an O2-handle basis $(\delta \times I, \delta' \times I)$ of the trivial surface-link ∂W in S^4, where δ denotes a meridian disk system of W. Thus, the spin loop basis (ℓ, ℓ') of $P(F; h)$ bounds the core disk pair system (D, D') of the O2-handle basis $(D \times I, D' \times I) = (w^{-1}(\delta) \times I, w^{-1}(\delta') \times I)$ on $P(F; h)$ in S^4 with D in $U(W)$, so that $d \cap D = h(1) \cap D = \emptyset$. The handlebody system $U(W)$ is isotopically deformed to be $U(W) = U$ in S^4. Consider the 3-manifold $U' \cup h(1)$ obtained from the 3-manifold $U \cup h(1)$ by splitting along the disk system D, which is a multi-punctured 3-sphere system. Since $\partial(U \cup h(1)) = P(F; h(0))$, the boundary $\partial(U' \cup h(1))$ is obtained from $P(F; h(0))$ by surgery along the 2-handle system $D \times I$ and consists of the S^2-link S (obtained from F) and the S^2-knots S_{ijk} (obtained from F_{ijk}) for all i, j, k. The S^2-knots S_{ijk} for all i, j, k are shown to form a trivial S^2-link in S^4. To see this, consider a collar $V \times J$ of V in S^4 to move (via a boundary collar of V) the anti-parallel surface $P_{ijk}(F)$ and the 1-handle h_{ijk} into a disjoint subcollar $V \times J_{ijk}$ of $V \times J$ for each i, j, k. Then by forgetting the 1-handle system $h(1)$, the trivial surface-knots F_{ijk} for all i, j, k (not containing F) in the boundary of $U \cup h(1)$ are isotopic to the boundaries of disjoint handlebodies U_{ijk} for all i, j, k in S^4. For the transverse disk system d of the 1-handle system $h(1)$ on $P(F; h(0))$, there is a proper disk system d_U in U with $\partial d_U = \partial d$ which splits U into disjoint handlebodies U^0 and U^0_{ijk} for all i, j, k such that the boundaries ∂U^0 and ∂U^0_{ijk} contain the multi-punctured surfaces of F and F_{ijk}, respectively and U^0_{ijk} contains the subsystem D_{ijk} for F_{ijk} of the 2-handle core disk system D on $P(F; h(0))$. Deform

the handlebody U^0_{ijk} into the handlebody U_{ijk} for all $i, j, k,$ so that the S^2-knots S_{ijk} for all i, j, k are isotopic to the boundaries of the disjoint 3-balls U'_{ijk} for all i, j, k obtained from the disjoint handlebodies U_{ijk} for all i, j, k by splitting along the disk systems D_{ijk} for all $i, j, k.$ Thus, the system of the S^2-knots S_{ijk} for all i, j, k forms a trivial S^2-link in S^4, as desired. Then the multi-punctured 3-sphere system $U' \cup h(1)$ means that the S^2-link S is a ribbon S^2-link in S^4, [13]. Because F is obtained from the ribbon S^2-link S by surgery along a 1-handle system, the surface-link F is a ribbon surface-link in S^4. This completes the proof of Theorem 1.3.

References
1. Kawauchi, A. (2021). Ribbonness of a stable- ribbon surface-link, I. A stably trivial surface-link, *Topology and its Applications,* 301, 107522 (16 pages).
2. Kawauchi, A. (2023). Uniqueness of an orthogonal 2-handle pair on a surface-link, *Contemporary Mathematics (UWP)*, 4, 182-188.
3. Kawauchi, A. (2023). Triviality of a surface-link with meridian-based free fundamental group, *Transnational Journal of Mathematical Analysis and Applications, 11,* 19-27.
4. Kawauchi, A., Shibuya, T. and Suzuki, S. (1983). Descriptions on surfaces in four-space II: Singularities and cross-sectional links, *Math Seminar Notes Kobe Univ,* 11, 31-69.
5. Yajima T. (1962). On the fundamental groups of knotted 2-manifolds in the 4-space, *J Math Osaka City Univ* 13, 63-71.
6. Yajima, T. (1964). On simply knotted spheres in R⁴, *Osaka J Math,* 1, 133-152.
7. Yanagawa, T. (1969). On ribbon 2- knots; the 3-manifold bounded by the 2-knot, *Osaka J Math*, 6, 447-164.
8. Kawauchi, A. (2024). Classical Poincaré conjecture via 4D topology, *J Math Techniques Comput Math, 3*(4), 1-7.
9. Perelman, G. (2003) Ricci flow with surgery on three-manifolds. arXiv: math DG/0303109 v1.
10. Poincaré, H. (1900). Second complément à l'analysis situs, *Proc London Math Soc*, 32, 277-308.
11. Poincaré, H. (1904). Cinquième complément à l'analysis situs, *Rendiconti del Circolo Matematico di Palermo* (1884 1940), 18, 45-110.
12. Howie. J. (1983). Some remarks on a problem of J. H. C. Whitehead, *Topology*, 22, 475-485.
13. Kawauchi, A. (2024). Ribbonness of Kervaire's sphere-link in homotopy 4- sphere and its consequences to 2-complexes, *Techniques Comput Techniques Comput Math*, 3(4), 01-08.
14. Kawauchi, A. (2024). Whitehead aspherical conjecture via ribbon sphere-link, *J Math Techniques Comput Math*, 3(5), 01-10.
15. Whitehead, J. H. C. (1941). On adding relations to homotopy groups, *Ann Math*, 42, 409-428.
16. González Acuña, F. and Ramírez, A. (2006). A knot-theoretic equivalent of the Kervaire conjecture, *J Knot Theory Ramifications,* 15, 471–478.
17. Kawauchi, A. (2024). Kervaire conjecture on weight of group via fundamental group of Ribbon sphere-link, *J Math Techniques Comput Math,* 3(4), 1-3.
18. Kervaire, M. A. (1965). On higher dimensional knots, *Differential and combinatorial topology, 27,* 105-119.
19. Magnus, W., Karrass A. and Solitar, D. (2004). *Combinatorial group theory: Presentations of groups in termsof generators and relations,* Courier Corporation.
20. Klyachko, Ant. A. (1993). A funny property of a sphere and equations over groups, *Comm. Algebra*, 21, 2555-2575.
21. Kawauchi, A. (2023) Smooth homotopy 4-sphere, *WSEAS Transactions on Mathematics,* 22, 690-701.
22. Kawauchi, A. (2025). Ribbonness of a stable-ribbon surface-link, II. General case, *(MDPI) Mathematics*, 13 (3), 402, 1-11.
23. Hillman, J. A. and Kawauchi, A. (1995). Unknotting orientable surfaces in the 4-sphere, *J Knot Theory Ramifications*, 4, 213-224.
24. Kawauchi, A. (2024). Another proof of free ribbon lemma, *J Math Techniques Comput Math*, 3(9), 01-03.
25. Kawauchi, A. (2007). On the surface-Link groups, *Intelligence of low dimensional topology 2006,* 40, 157-164.
26. Kawauchi, A. (2008). The first Alexander Z[Z]-modules of surface-links and of virtual links, *Geometry and Topology Monographs*, 14, 353-371.
27. Yajima, T. (1970). Wirtinger presentations of knot groups, *Proc Japan Acad,* 46, 997-1000.
28. Kamada, S. (2001). Wirtinger presentations for higher dimensional manifold knots obtained from diagrams, *Fund Math,* 168, 105-112.
29. A. Kawauchi, A., Shibuya, T. and S. Suzuki, S. (1982). Descriptions on surfaces in four-space I : Normal forms, *Mathematics Seminar Notes Kobe Univ*,10, 75-125.
30. Spanier, E. H. (1966). *Algebraic topology*, Springer Science+Business Media New York.
31. Hosokawa, F. and Kawauchi, A. (1979). Proposals for unknotted surfaces in four-space, *Osaka J Math* 16, 233-248.
32. Kawauchi, A. (2018). Faithful equivalence of equivalent ribbon surface-links, *J Knot Theory Ramifications*, 27, 1843003 (23 pages).
33. Hirose, S.(2002). On diffeomorphisms over surfaces trivially embedded in the 4-sphere, *Algebraic and Geometric Topology Topology 2*, 791-824.