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Fourier Transform of a Single-Frequency Octonion

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Abstract
A method for Fourier transforms in 8D space of a single-frequency octonion for 8 different pulses is presented. The octonion Fourier transform is calculated using the fundamental matrix of the octonion by integrating the product of the signal vector with this matrix. The inverse Fourier transform uses the transposed fundamental matrix multiplied by the spectrum vector. Spectra of different pulses were obtained. It is shown that the pulse spectra are formed using the theorem of shifting spectra along the frequency axis in accordance with their shift in time. In this case, the shift of the signal vector elements is performed using a direct and quadrature shift matrix. As is known, the octonion is used to form a MIMO system with 8 inputs and 8 outputs. Therefore, the spectra of each pulse are located on each axis of spatial coordinates of 8D space. In this case, we obtain the sum of the spectra of different pulses as elements of the output vector. Since the sum spectra are formed from the spectra of individual pulses shifted along the frequency axis in accordance with their time shift, the sum spectra have a wider frequency band and are more resistant to interference. The spatial coordinate axes are orthogonal, since the octonion-based fundamental matrix is orthogonal, therefore, the total spectra on the axes will be orthogonal and the pulses are separated during the inverse transformation.

Keywords: Octonion, Quaternion, Hypercomplex Signals, Spectrum, Fourier Transform, MIMO

1. Introduction

The Fourier transform appeared in 1807 as a way to represent a function as a sum of harmonic oscillations of different frequencies. It is noteworthy that the Fourier transform used a complex function on the complex plane in the form of an exponential of the circular frequency $\omega = 2\pi f$ and time t: $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ As can be seen from the given expression, the function consists of the sum of the real part and the imaginary part. If an exponential complex function has continuous derivatives of any order, it is called analytic or harmonic. It is known that the first derivatives of a harmonic function satisfy the Cauchy-Riemann conditions (CRC), and the second derivatives satisfy the Laplace equation. The CRC requires that as the amplitude of real oscillations increases, the amplitude of imaginary oscillations decreases. In this case, in accordance with the Laplace equation, the total power of the oscillations should remain unchanged. These two conditions are of fundamental importance for the analysis of physical processes, since, according to the law of conservation of energy, in physical space, with any transformation of energy, it must remain the same in magnitude. For example, if the real part of a complex function is defined as potential energy, and the imaginary part as kinetic energy, then the sum of the energies must be constant at any moment in time. It follows that the harmonic function describes a circle with constant radius on the complex plane. However, the dimension of physical space is greater than the dimension of the complex plane and is equal to 3D – length, width, height. Therefore, to analyze physical processes in real space, it is necessary to use harmonic functions of higher dimension than complex ones on the plane. It is clear that these functions must be analytical and harmonic, i.e. satisfy the CRC and the Laplace equation. Complex numbers of higher dimensions are known as hypercomplex. In 1843, Hamilton discovered quaternions. Later, using the doubling procedure, octonions, sedenions, and other hypercomplex numbers of higher dimension were defined. According to the doubling procedure, the dimension of hypercomplex numbers N describing hypercomplex spaces is a multiple of 2, i.e. $N = 2^n$, where n = 1,2,3, ... Hence, the dimension of a quaternion is 4, an octonion is 8, a sedenion is 16, etc. Hypercomplex functions constructed on the basis of hypercomplex numbers also satisfy the CRC and the Laplace equation. Therefore, an analytical physical space constructed using hypercomplex functions can have a large dimension, a multiple of 2, and be analytical and harmonic. Therefore, to obtain the Fourier transform in a space with higher dimensions, it is necessary to use hypercomplex functions based on hypercomplex numbers. It is important to note that, despite the fact that physical space has a 3D dimension, it is possible to form spaces of very high dimensions in it using hypercomplex numbers. Methods of quaternion single-frequency and three-frequency Fourier transform and discrete quaternion Fourier transform for calculating the spectra of a sequence of different 4 pulses are known [1-3]. The aim of the article is to present a technique for octonion single-frequency Fourier transform in 8D space for 8 different pulses.

2. Materials and Methods for Solving the Problem

Let us write the octonion as a hypercomplex number with real values $s, x, y, z, s_1, x_1, y_1$, z_1 on the corresponding coordinate axes of the 8D space with one real axis e and seven mutually orthogonal imaginary spatial axes $i, j, k, e_1, i_1, j_1, k_1$:

$$o = se + xi + yj + zk + s_1e_1 + x_1i_1 + y_1j_1 + z_1k_1 . \qquad (1)$$

Let us represent the octonion (1) as a matrix [4]:

$$\mathbf{O} = \begin{bmatrix} s & x & y & z & s_1 & x_1 & y_1 & z_1 \\ -x & s & -z & y & -x_1 & s_1 & z_1 & -y_1 \\ -y & z & s & -x & -y_1 & -z_1 & s_1 & x_1 \\ -z & -y & x & s & -z_1 & y_1 & -x_1 & s_1 \\ -s_1 & x_1 & y_1 & z_1 & s & -x & -y & -z \\ -x_1 & -s_1 & z_1 & -y_1 & x & s & z & -y \\ -y_1 & -z_1 & -s_1 & x_1 & y & -z & s & x \\ -z_1 & y_1 & -x_1 & -s_1 & z & y & -x & s \end{bmatrix} . \qquad (2)$$

Matrix (2) can be decomposed into basis matrices and written as (2) as a sum:

$$\mathbf{O} = s\mathbf{E} + x\mathbf{I} + y\mathbf{J} + z\mathbf{K} + s_1\mathbf{E}_1 + x_1\mathbf{I}_1 + y_1\mathbf{J}_1 + z_1\mathbf{K}_1 .$$

We will represent the model of change of the harmonic function in the form of a dynamic equation in the state space.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) , \qquad (3)$$

$$\mathbf{A} = \begin{bmatrix} 0 & \omega_i & \omega_j & \omega_k & \omega_{e_1} & \omega_{i_1} & \omega_{j_1} & \omega_{k_1} \\ -\omega_i & 0 & -\omega_k & \omega_j & -\omega_{i_1} & \omega_{e_1} & \omega_{k_1} & -\omega_{j_1} \\ -\omega_j & \omega_k & 0 & -\omega_i & -\omega_{j_1} & -\omega_{k_1} & \omega_{e_1} & \omega_{i_1} \\ -\omega_k & -\omega_j & \omega_i & 0 & -\omega_{k_1} & \omega_{j_1} & -\omega_{i_1} & \omega_{e_1} \\ -\omega_{e_1} & \omega_{i_1} & \omega_{j_1} & \omega_{k_1} & 0 & -\omega_i & -\omega_j & -\omega_k \\ -\omega_{i_1} & -\omega_{e_1} & \omega_{k_1} & -\omega_{j_1} & \omega_i & 0 & \omega_k & -\omega_j \\ -\omega_{j_1} & -\omega_{k_1} & -\omega_{e_1} & \omega_{i_1} & \omega_j & -\omega_k & 0 & \omega_i \\ -\omega_{k_1} & \omega_{j_1} & -\omega_{i_1} & -\omega_{e_1} & \omega_k & \omega_j & -\omega_i & 0 \end{bmatrix} ,$$

where matrix **A** corresponds to the state transition matrix for the 7-frequency octonion [4].
The solution to equation (3) when angular frequencies change over time will have the form

$$\mathbf{\Phi}(\omega_i, \omega_j, \omega_k, \omega_{e_1}, \omega_{i_1}, \omega_{j_1}, \omega_{k_1}, t) = e^{\mathbf{A}t} = \exp\left\{(\omega_i\mathbf{I} + \omega_j\mathbf{J} + \omega_k\mathbf{K} + \omega_{e_1}\mathbf{E}_1 + \omega_{i_1}\mathbf{I}_1 + \omega_{j_1}\mathbf{J}_1 + \omega_{k_1}\mathbf{K}_1)t\right\} .$$

For a single-frequency octonion, all frequencies will be the same, so we write this solution as:

$$\mathbf{\Phi}(\omega,t)=\mathrm{e}^{\mathbf{A}t}=\mathrm{e}^{\hat{\mathbf{I}}t}=\cos(\omega t)\mathbf{E}+\sin(\omega t)\hat{\mathbf{I}}\,, \tag{4}$$

where $\hat{\mathbf{I}}=\mathbf{I}+\mathbf{J}+\mathbf{K}+\mathbf{E}_1+\mathbf{I}_1+\mathbf{J}_1+\mathbf{K}_1$ - imaginary unit of the single-frequency octonion, and $\mathbf{A}=\omega\hat{\mathbf{I}}$.

In matrix form, the imaginary numbers of the octonion correspond to basis matrices that, when superimposed on each other, do not intersect, i.e., they are spatially separated and, accordingly, orthogonal. As can be seen from (4), the fundamental matrix has a real part and an imaginary part with a matrix imaginary unit $\hat{\mathbf{I}}$. For clarity, we write the fundamental matrix of the single-frequency octonion (4) in expanded form as

$$\mathbf{\Phi}(\omega,t)=\mathrm{e}^{\mathbf{A}t}=\mathrm{e}^{\hat{\mathbf{I}}t}=\frac{1}{\sqrt{7}}\times \tag{5}$$

$$\begin{bmatrix}
\sqrt{7}\cos(\omega t) & \sin(\omega t) & \sin(\omega t) & \sin(\omega t) & \sin(\omega t) & \sin(\omega t) & \sin(\omega t) & \sin(\omega t)\\
-\sin(\omega t) & \sqrt{7}\cos(\omega t) & -\sin(\omega t) & \sin(\omega t) & -\sin(\omega t) & \sin(\omega t) & \sin(\omega t) & -\sin(\omega t)\\
-\sin(\omega t) & \sin(\omega t) & \sqrt{7}\cos(\omega t) & -\sin(\omega t) & -\sin(\omega t) & -\sin(\omega t) & \sin(\omega t) & \sin(\omega t)\\
-\sin(\omega t) & -\sin(\omega t) & \sin(\omega t) & \sqrt{7}\cos(\omega t) & -\sin(\omega t) & \sin(\omega t) & -\sin(\omega t) & \sin(\omega t)\\
-\sin(\omega t) & \sin(\omega t) & \sin(\omega t) & \sin(\omega t) & \sqrt{7}\cos(\omega t) & -\sin(\omega t) & -\sin(\omega t) & -\sin(\omega t)\\
-\sin(\omega t) & -\sin(\omega t) & \sin(\omega t) & -\sin(\omega t) & \sin(\omega t) & \sqrt{7}\cos(\omega t) & \sin(\omega t) & -\sin(\omega t)\\
-\sin(\omega t) & -\sin(\omega t) & -\sin(\omega t) & \sin(\omega t) & \sin(\omega t) & -\sin(\omega t) & \sqrt{7}\cos(\omega t) & \sin(\omega t)\\
-\sin(\omega t) & \sin(\omega t) & -\sin(\omega t) & -\sin(\omega t) & \sin(\omega t) & \sin(\omega t) & -\sin(\omega t) & \sqrt{7}\cos(\omega t)
\end{bmatrix}.$$

The fundamental matrix of the octonion is orthogonal, since $\mathbf{\Phi}(\omega,t)\mathbf{\Phi}^{\mathrm{T}}(\omega,t)=\mathbf{\Phi}^{\mathrm{T}}(\omega,t)\mathbf{\Phi}(\omega,t)=\mathbf{E}$.

3. Method for Obtaining the Single-Frequency Octonion Fourier Transform

In general, the octonion Fourier transform (OFT) is written as the integral over time of an 8-dimensional vector s(t) changing over time in 8D space:

$$\mathbf{S}(\omega)=\int_{\mathbf{O}}\mathrm{e}^{-\hat{\mathbf{I}}\omega t}\,\mathbf{s}(t)d^8t=\mathrm{OFT}\ \mathbf{s}(t)\ , \tag{6}$$

The inverse Fourier transform for the octonion is written as:

$$\mathbf{s}(t)=\frac{1}{(2\pi)^8}\int_{\mathbf{O}}\mathrm{e}^{\hat{\mathbf{I}}\omega t}\,\mathbf{S}(\omega)d^8\omega=\mathrm{IOFT}\ \mathbf{S}(\omega)\ . \tag{7}$$

For a single-frequency octonion, we will use the representation of the matrix exponential in the form of a fundamental matrix (4) or in expanded form (5). Since the fundamental matrix of the octonion is orthogonal, the result of integrating the product of signal vectors $\mathbf{s}(t)$ or spectra $\mathbf{S}(\omega)$ by fundamental matrices along one row does not depend on the values of the integration of vectors along other rows. Then, expressions (6) and (7), by analogy with the Fourier transform for a quaternion, can be represented in the form.

$$\mathbf{S}(\omega)=\int_{-\infty}^{\infty}\mathbf{\Phi}^{\mathrm{T}}(\omega,t)\mathbf{s}(t)dt=\mathrm{OFT}\ \mathbf{s}(t)\ , \tag{8}$$

$$\mathbf{s}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Phi}(\omega,t)\mathbf{S}(\omega)d\omega=\mathrm{IOFT}\ \mathbf{S}(\omega)\ . \tag{9}$$

In this case, integration can be performed separately on the duration of the analyzed pulses in signal vectors or in spectrum vectors. Let us represent the signal vector as 8 elements of some octonion functions:

$$\mathbf{s}(t)=[p(t)\quad u(t)\quad v(t)\quad w(t)\quad p_1(t)\quad u_1(t)\quad v_1(t)\quad w_1(t)]^{\mathrm{T}} \tag{10}$$

We will define the amplitude and sign of the pulses by the initial state vector:

$$\mathbf{x}(0)=[x_0\quad x_1\quad x_2\quad x_3\quad x_4\quad x_5\quad x_6\quad x_7]^{\mathrm{T}}. \tag{11}$$

The corresponding vector of octonion spectra for each orthogonal frequency axis will be:

$$\mathbf{S}(\omega)=[p(\omega)\quad u(\omega)\quad v(\omega)\quad w(\omega)\quad p_1(\omega)\quad u_1(\omega)\quad v_1(\omega)\quad w_1(\omega)]^{\mathrm{T}}. \tag{12}$$

The pulses in vector (10) can have different time shifts t_m relative to the zero time of the spatial coordinate axes. By analogy with the quaternion shift matrix, various shifts of the elements of the pulses vector (10) can be determined using the shift matrix for the octonion and the pulse time lag theorem [1]:

$$\mathrm{OFT}\{\mathbf{s}(t-t_m)\}\Leftrightarrow\mathbf{\Phi}^{\mathrm{T}}(\omega,t_m)\mathbf{S}(\omega)=e^{-\hat{\mathbf{I}}\omega t_m}\mathbf{S}(\omega), \tag{13}$$

where matrix (5) for different shifts of elements takes the form:

$$\mathbf{D}(\omega,t_m)=\mathbf{\Phi}^{\mathrm{T}}(\omega,t_m)=\frac{1}{\sqrt{7}}\times \tag{14}$$

$$\begin{bmatrix}
\sqrt{7}\cos(\omega t_0) & -\sin(\omega t_1) & -\sin(\omega t_2) & -\sin(\omega t_3) & -\sin(\omega t_4) & -\sin(\omega t_5) & -\sin(\omega t_6) & -\sin(\omega t_7)\\
\sin(\omega t_0) & \sqrt{7}\cos(\omega t_1) & \sin(\omega t_2) & -\sin(\omega t_3) & \sin(\omega t_4) & -\sin(\omega t_5) & -\sin(\omega t_6) & \sin(\omega t_7)\\
\sin(\omega t_0) & -\sin(\omega t_1) & \sqrt{7}\cos(\omega t_2) & \sin(\omega t_3) & \sin(\omega t_4) & \sin(\omega t_5) & -\sin(\omega t_6) & -\sin(\omega t_7)\\
\sin(\omega t_0) & \sin(\omega t_1) & -\sin(\omega t_2) & \sqrt{7}\cos(\omega t_3) & \sin(\omega t_4) & -\sin(\omega t_5) & \sin(\omega t_6) & -\sin(\omega t_7)\\
\sin(\omega t_0) & -\sin(\omega t_1) & -\sin(\omega t_2) & -\sin(\omega t_3) & \sqrt{7}\cos(\omega t_4) & \sin(\omega t_5) & \sin(\omega t_6) & \sin(\omega t_7)\\
\sin(\omega t_0) & \sin(\omega t_1) & -\sin(\omega t_2) & \sin(\omega t_3) & -\sin(\omega t_4) & \sqrt{7}\cos(\omega t_5) & -\sin(\omega t_6) & \sin(\omega t_7)\\
\sin(\omega t_0) & \sin(\omega t_1) & \sin(\omega t_2) & -\sin(\omega t_3) & -\sin(\omega t_4) & \sin(\omega t) & \sqrt{7}\cos(\omega t_6) & -\sin(\omega t_7)\\
\sin(\omega t_0) & -\sin(\omega t_1) & \sin(\omega t_2) & \sin(\omega t_3) & -\sin(\omega t_4) & -\sin(\omega t_5) & \sin(\omega t_6) & \sqrt{7}\cos(\omega t_7)
\end{bmatrix}.$$

To avoid confusing the shift matrix with the fundamental matrix, we introduce a special notation for it $\mathbf{D}(\omega, t_m)$.The delay occurs for a time t_m, m = 0.1…7, corresponding to the number of the element in the signal vector (10). According to (4), the fundamental matrix is equal to the sum of the real and imaginary parts of the harmonic function. Therefore, the signal spectrum will also have real and imaginary components.

When calculating the spectra of the signal vector elements that are asymmetric relative to zero time, a quadrature spectrum appears, which is determined by the quadrature shift matrix [1]:

$$\bar{\mathbf{D}}(\omega,t_m)=\hat{\mathbf{I}}^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}(\omega,t_m)=\frac{1}{\sqrt{7}}\times \tag{15}$$

$$\begin{bmatrix}
-\sqrt{7}\sin(\omega t_0) & -\cos(\omega t_1) & -\cos(\omega t_2) & -\cos(\omega t_3) & -\cos(\omega t_4) & -\cos(\omega t_5) & -\cos(\omega t_6) & -\cos(\omega t_7)\\
\cos(\omega t_0) & -\sqrt{7}\sin(\omega t_1) & \cos(\omega t_2) & -\cos(\omega t_3) & \cos(\omega t_4) & -\cos(\omega t_5) & -\cos(\omega t_6) & \cos(\omega t_7)\\
\cos(\omega t_0) & -\cos(\omega t_1) & -\sqrt{7}\sin(\omega t_2) & \cos(\omega t_3) & \cos(\omega t_4) & \cos(\omega t_5) & -\cos(\omega t_6) & -\cos(\omega t_7)\\
\cos(\omega t_0) & \cos(\omega t_1) & -\cos(\omega t_2) & -\sqrt{7}\sin(\omega t_3) & \cos(\omega t_4) & -\cos(\omega t_5) & \cos(\omega t_6) & -\cos(\omega t_7)\\
\cos(\omega t_0) & -\cos(\omega t_1) & -\cos(\omega t_2) & -\cos(\omega t_3) & -\sqrt{7}\sin(\omega t_4) & \cos(\omega t_5) & \cos(\omega t_6) & \cos(\omega t_7)\\
\cos(\omega t_0) & \cos(\omega t_1) & -\cos(\omega t_2) & \cos(\omega t_3) & -\cos(\omega t_4) & -\sqrt{7}\sin(\omega t_5) & -\cos(\omega t_6) & \cos(\omega t_7)\\
\cos(\omega t_0) & \cos(\omega t_1) & \cos(\omega t_2) & -\cos(\omega t_3) & -\cos(\omega t_4) & \cos(\omega t) & -\sqrt{7}\sin(\omega t_6) & -\cos(\omega t_7)\\
\cos(\omega t_0) & -\cos(\omega t_1) & \cos(\omega t_2) & \cos(\omega t_3) & -\cos(\omega t_4) & -\cos(\omega t_5) & \cos(\omega t_6) & -\sqrt{7}\sin(\omega t_7)
\end{bmatrix}.$$

As can be seen from (15), the quadrature spectrum also has real and imaginary parts. In our case, the term "quadrature matrix" means a rotation of the shift matrix (14) using the imaginary matrix unit $\hat{\mathbf{I}}^{\mathrm{T}}$ by 90°. In this case, a coordinate system appears in 8D space, orthogonal to the original 8D coordinate system. In radio engineering, a quadrature signal is considered as a phase shift of one signal relative to another by 90°. However, when using complex or hypercomplex signals, it is necessary to apply the concept of duality, which means the difference between the real and imaginary parts of the signals in their properties. For example, potential and kinetic energy, electric and magnetic field, mass and acceleration.

The corresponding theorem for the spectrum when the pulses are advanced in time will have the form [1]:

$$\mathrm{OFT}\left\{\mathbf{s}(t+t_m)\right\} \Leftrightarrow \mathbf{\Phi}(\omega,t_m)\mathbf{S}(\omega) = e^{\hat{\mathbf{I}}\omega t_m}\mathbf{S}(\omega)\,.$$

Since the pulse shift matrix (14) and the quadrature shift matrix (15) contain elements, whose value is determined by the product of the angular frequency ω and the shift time t_m, then when the shift time changes, the value of these elements will change. However, as already mentioned, in accordance with the CRC, frequencies must change in such a way that the energy of the system is preserved. In our case, it simply shifts the spectrum along the frequency axis. As is known, the rotation frequency corresponds to kinetic energy, which increases with increasing frequency. Therefore, according to the laws of physics, as kinetic energy increases, potential energy must decrease. Similarly, changing the pulse amplitude with time affects the value of the angular frequency.

Thus, using the octonion to find the Fourier transform increases the dimension of space to 8 and represents it as 8 orthogonal imaginary spatial axes. In this case, the octonion Fourier transform is reduced to multiplying the signal vector by the fundamental matrix, followed by element-by-element integration of the multiplication results and their summation. In the inverse Fourier transform, the pulse spectra are separated.

4. Examples of obtaining the octonion Fourier transform

Let's find OFT and IOFT for the most common pulses. In this case, we will consider the successive shift of pulses in time, a multiple of the pulse length T:

$$t_0 = T/2,\ t_1 = 3T/2,\ t_2 = 5T/2,\ t_3 = 7T/2,\ t_4 = 9T/2,\ t_5 = 11T/2,\ t_6 = 13T/2,\ t_7 = 15T/2. \qquad (16).$$

It is assumed that the middle of the first pulse is located at the zero point of the time axis. With such a shift, each pulse of the signal vector (10) will be multiplied by the corresponding column of the fundamental matrix (5) and the pulse time shift matrices (14) and (15).

4.1. OFT and IOFT Rectangular Pulses

In accordance with formula (8), for the vector of rectangular pulses (10), the integral for a duration from 0 to $8T$ can be written as separate integrals for the duration of individual pulses T, i.e., in the form of a matrix of the spectrum of a rectangular pulse symmetrical to zero $r(\omega) = T\,\mathrm{sinc}\left(T\omega/2\right)$ with a corresponding frequency shift. In accordance with the shift theorem (13), when multiplying $r(\omega)$ by the matrix (14) with the values of the pulse shifts in time (16), we obtain a matrix of spectra of rectangular pulses with a unit pulse amplitude:

$$\mathbf{R}(\omega,t_m) = r(\omega)\mathbf{D}(\omega,t_m) = T\,\mathrm{sinc}\left(\frac{T\omega}{2}\right)\mathbf{D}(\omega,t_m)\,. \qquad (17)$$

As can be seen from (17), when calculating the spectrum of rectangular pulses with a sequential shift, the quadrature shift matrix (15) is not used. This is explained by the fact that the initial pulse is taken to be a pulse that is symmetrical relative to zero time with a constant amplitude. In other words, the mirror image of a rectangular pulse has the same shape as the original pulse. By adding up the spectra of all pulses of each row with the corresponding signs and amplitudes, i.e. by multiplying the matrix (17) by the vector of initial states (11), we obtain the vector of spectra:

$$\mathbf{S}(\omega,t_m) = \mathbf{R}(\omega,t_m)\mathbf{x}(0)\,. \qquad (18)$$

Figure 1 shows the spectra of the elements of vector (18) for x(0) = [1-1 1-1-1-1 1 1] ^{T}. According to the delay theorem (13), the spectrum of a rectangular pulse will be shifted along the angular frequency axis relative to the zero value according to the pulse delay values. As can be seen from Figure 1, the spectra are formed by summing the spectra at different shifts and, accordingly, the spectrum is expanded. The figure also shows the spectrum module. Calculations show that as the considered spectrum width increases, the power of each element of the spectrum vector tends to 1, i.e. to the value of the power of each element of the initial vector x(0).

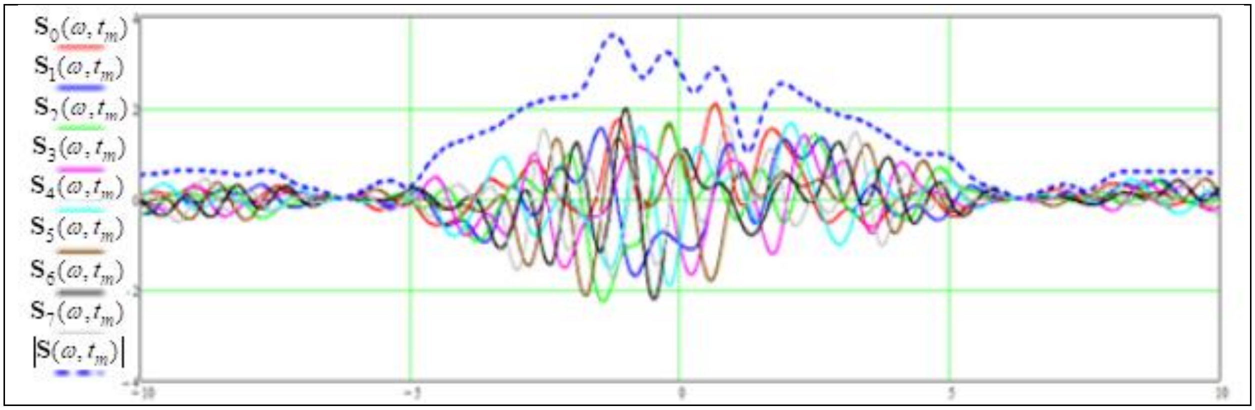

Figure 1: Spectra of Rectangular Pulse Vector Elements

Note that the spectra are located on 7 orthogonal imaginary coordinate axes and one scalar axis with the values of angular frequencies ω. Therefore, with IOFT (9) the pulses can be separated and each pulse can be calculated separately by summing its spectra on orthogonal axes. Figure 2 shows the initial vector impulses obtained using IOFT (9).

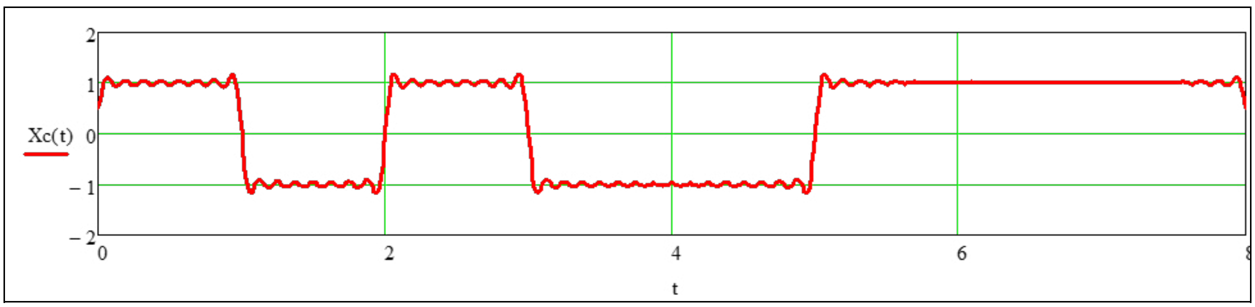

Figure 2: Initial Vector Impulses X(0) = [1-1 1-1-1 1 1 1] Obtained Using IOFT

4.2. OFT and IOFT of Meander pulses

A meander is a signal consisting of two rectangular pulses of different signs. With a rectangular pulse width of $T/2$, the meander pulse duration is T. The spectrum of a rectangular pulse has the shape (17). We will represent the meander spectrum as the spectrum of a rectangular positive pulse of duration $T/2$ and a negative pulse shifted in time by $T/2$ relative to the first pulse:

$$\mathbf{M}(\omega,t_m)=\mathbf{R}(\omega,t_{m,1})-\mathbf{R}(\omega,t_{m,2})\,. \qquad (19)$$

The first time index m shows the time shifts of the meander pulses, and the second index 1 and 2 shows the time shift of the first and second rectangular pulses relative to the time shift of the 8 meander pulses.

Let us calculate the spectra of the meander pulses for the initial state of the first meander
pulses x1(0)=$[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T$. The second pulses will be $x_2(0) = -x_1(0)$. The delay time of the elements of the first pulse vector will be equal to равно t_0,1= $T/2$, $t_{1,1}= 5T/2$, $t_{2,1}= 9T/2$, $t_{3,1}= 13T/2$, $t_{4,1}= 17T/2$, $t_{5,1}= 21T/2$, $t_{6,1}= 25T/2$, $t_{7.1}= 29T/2$. The delay time of the elements of the second pulse vector is equal to $t_{0,2\_}= 3T/2$, $t_{1,2}= 7T/2$, $t_{2,2}= 11T/2$, $t_{3,2}= 15T/2$, $t_{4,2}= 19T/2$, $t_{5,2}= 23T/2$, $t_{6,2}= 27T/2$, $t_{7.2}= 31T/2$. Figure 3 shows the spectra of meander pulses with the indicated shifts for each orthogonal coordinate axis, i.e. for each row of the fundamental matrix. The rows of the fundamental matrix are orthogonal, therefore the spectra will also be orthogonal. Since the quadrature part of the spectra is not used when using rectangular pulses, it is also absent when calculating the meander spectra. The spectrum of a meander is wider than the spectrum of a rectangular pulse with the same length T, since there are 2 rectangular pulses located at the duration T.

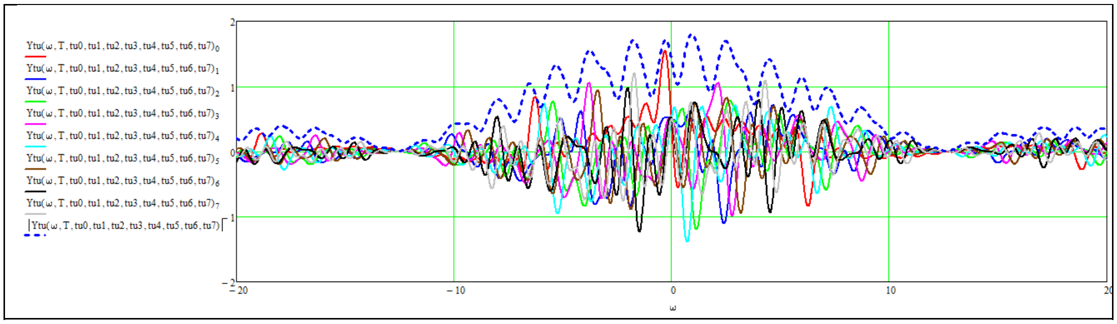

Figure 3: Spectrum of a Vector With Sequentially Arranged Meander Pulses

The IOFT is calculated using formula (9). Integrationcan also be performed $\mathbf{S}(\omega)$ separately for each row of the fundamental matrix $\mathbf{\Phi}(\omega,t)$ and for each meander pulse. Figure 4 shows the IOFT results for meanders.

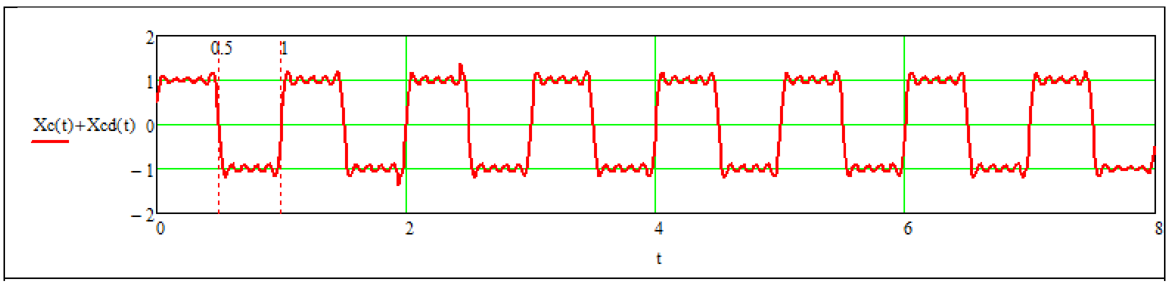

Figure 4: IOFT for a Sequence of Meanders

4.3. OFT and IOFT Sawtooth Pulses

Sawtooth pulses can be increasing or decreasing in amplitude. Let's find OFT and IOFT of increasing sawtooth pulses. The middle of the first sawtooth pulse is located at the 0 time coordinate, so the sawtooth pulse is not symmetrical. Therefore, the spectrum of the sawtooth pulse has a quadrature component. When calculating spectra, it is also necessary to use the quadrature shift matrix (15). As a result of calculations we obtain:

$$\mathbf{P}_{r,u}(\omega,T,t_m)=\frac{T}{\omega}\cdot\sin\left(\frac{T\omega}{2}\right)\mathbf{D}(\omega,t_m)\text{ - direct spectrum matrix,} \quad (20)$$

$$\mathbf{P}_{i,u}(\omega,T,t_m)=\frac{T}{\omega}\cdot\left(\operatorname{sinc}\left(\frac{T\omega}{2}\right)-\cos\left(\frac{T\omega}{2}\right)\right)\bar{\mathbf{D}}(\omega,t_m)\text{ , - quadrature spectrum.} \quad (21)$$

The corresponding resulting spectrum will look like:

$$\mathbf{P}_{u}(\omega,T,t_m)=\mathbf{P}_{r,u}(\omega,T,t_m)+\mathbf{P}_{i,u}(\omega,T,t_m)\,. \quad (22)$$

Figure 5 shows the spectrum (22) for the vector $\mathbf{x}(0)$-[1 -1 1 -1 -1 1 1 1]^T.

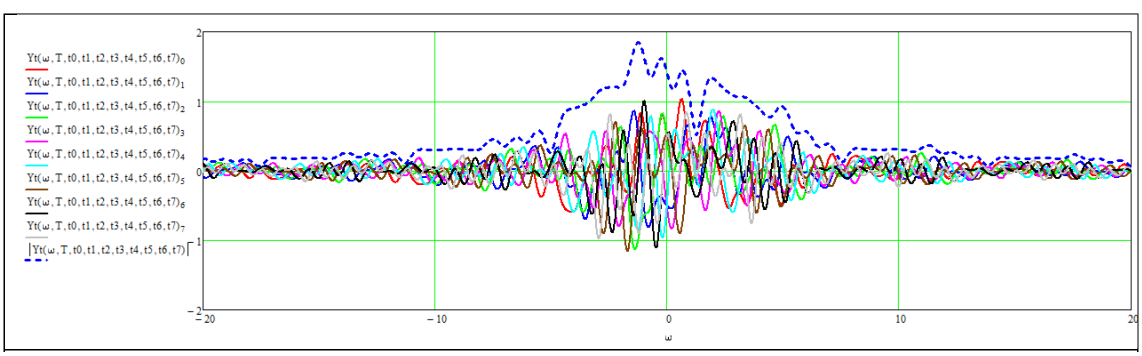

Figure 5: Spectrum of a Sequence Of Increasing Sawtooth Pulses

Figure 6 shows the IOFT of increasing sawtooth pulses.

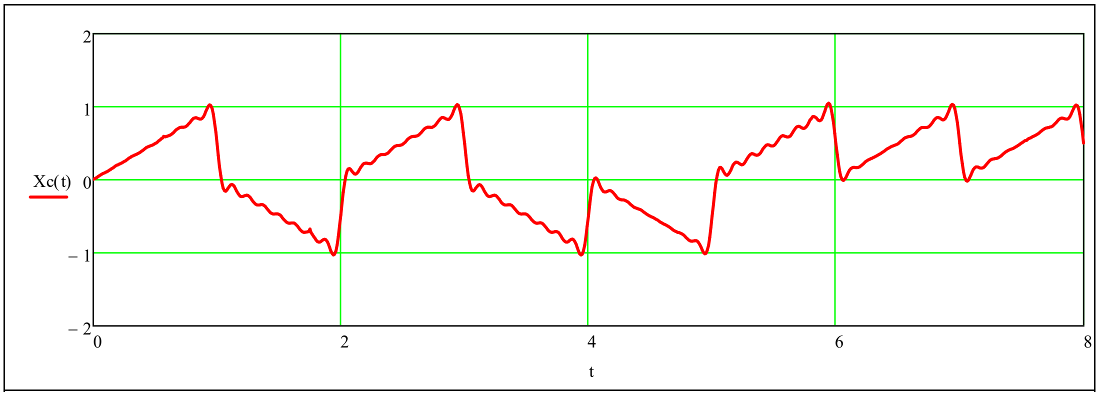

Figure 6: IOFT of Increasing Sawtooth Pulses

Let us calculate the spectra of decreasing sawtooth pulses. The matrix of spectra of decreasing sawtooth pulses has the form:

$$\mathbf{P}_{r,d}(\omega,T,t_m)=\frac{T}{\omega}\sin\left(\frac{T\omega}{2}\right)\mathbf{D}(\omega,t_m) \text{ - direct spectrum matrix,} \tag{23}$$

$$\mathbf{P}_{i,d}(\omega,T,t_m)=\frac{T}{\omega}\left(\cos\left(\frac{T\omega}{2}\right)-\mathrm{sinc}\left(\frac{T\omega}{2}\right)\right)\bar{\mathbf{D}}(\omega,t_m) \text{ - quadrature spectrum.} \tag{24}$$

The resulting spectrum:

$$\mathbf{P}_d(\omega,T,t_m)=\mathbf{P}_{r,d}(\omega,T,t_m)+\mathbf{P}_{i,d}(\omega,T,t_m)\,. \tag{25}$$

Figure 7 shows the spectra of a sequence of identical decreasing sawtooth pulses.

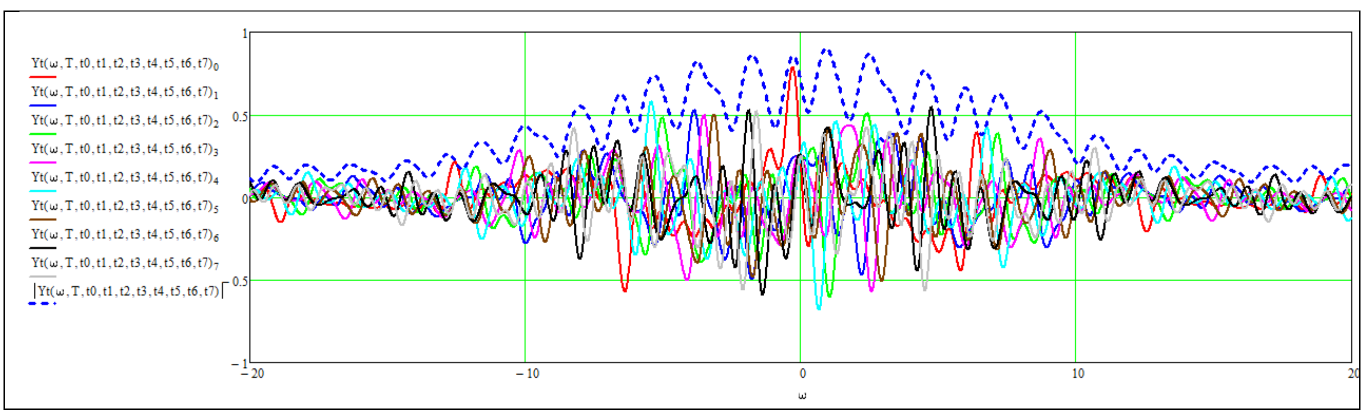

Figure 7: Spectrum of a Sequence of Decreasing Sawtooth Pulses

Figure 8 shows the IOFT of a sequence of identical decreasing sawtooth pulses.

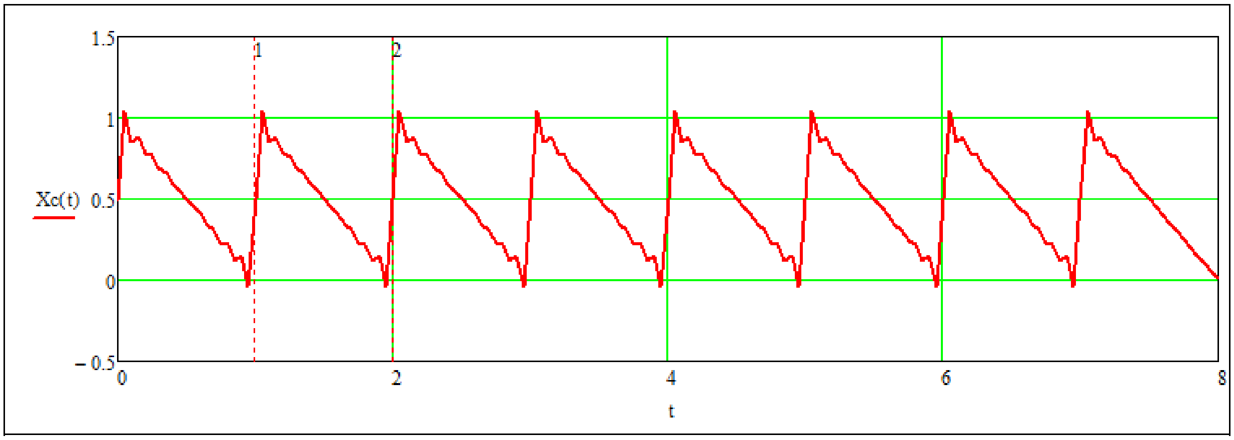

Figure 8: IOFT of Decreasing Sawtooth Pulses

4.4. OFT and IOFT Triangular Pulses

The triangular pulse is usually written as a function $\Lambda(t) = 1 - |t|/\tau$. In this case, time is calculated through its modulus. When calculating the spectrum of a triangular pulse sequence, this representation causes some difficulties in modeling. Therefore, we represent a triangular pulse through the sum of time-shifted spectra of increasing and decreasing sawtooth pulses with a duration of $T/2$.

The matrix of spectra of increasing sawtooth pulses is written as a sum (22). Since, when representing triangular pulses, the increasing and decreasing pulses have different time shifts, we denote the time shifts of the increasing pulse by an additional index 1:

$$\mathbf{P}_u(\omega, T, t_{m,1}) = \mathbf{P}_{r,u}(\omega, T, t_{m,1}) + \mathbf{P}_{i,u}(\omega, T, t_{m,1}) . \quad (26)$$

We denote the time shifts of the decreasing pulse (25) by an additional index 2:

$$\mathbf{P}_d(\omega, T, t_{m,2}) = \mathbf{P}_{r,d}(\omega, T, t_{m,2}) + \mathbf{P}_{i,d}(\omega, T, t_{m,2}) . \quad (27)$$

We write the matrix of spectra of a sequence of triangular pulses in the form:

$$\mathbf{T}(\omega, T, t_m) = \mathbf{P}_u(\omega, T, t_{m,1}) + \mathbf{P}_d(\omega, T, t_{m,2}) . \quad (28)$$

Figure 9 shows the spectrum of a sequence of identical triangular pulses.

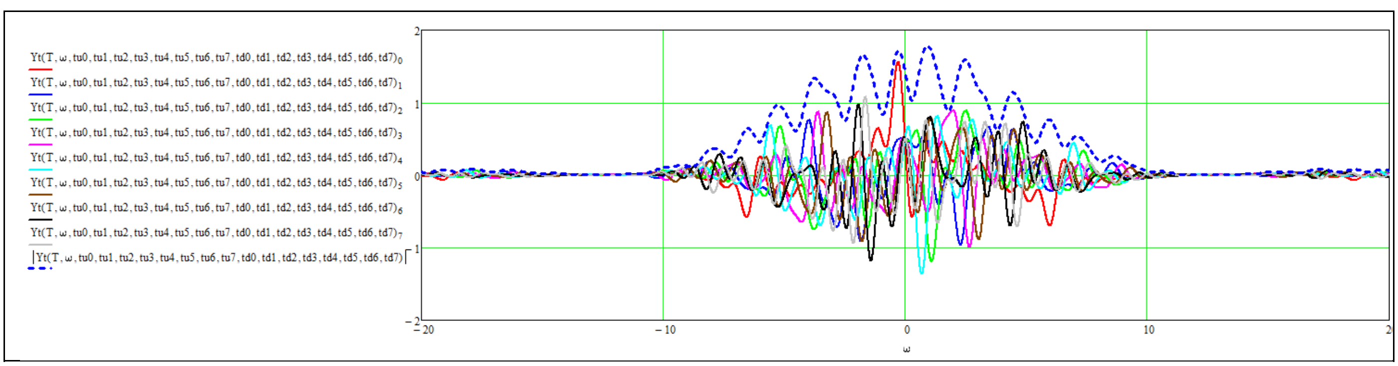

Figure 9: Spectrum of a Triangular Pulse Sequence

The sequence of triangular pulses obtained using IOFT is shown in Figure 10.

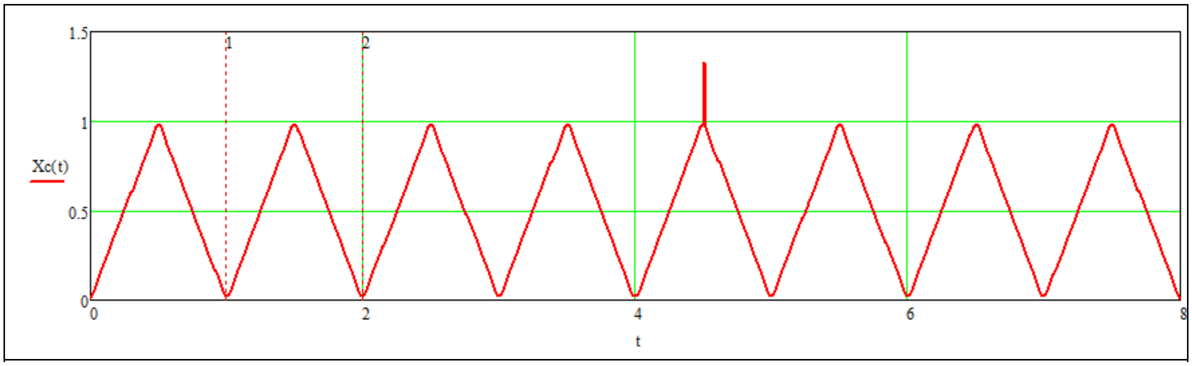

Figure 10: IOFT of Triangular Pulses

4.5. OFT and IOFT Trapezoidal Pulses

We form the matrix of spectra of an isosceles trapezoid using the matrices of spectra of an increasing sawtooth pulse (22) and a decreasing sawtooth pulse (25) on the sides and a rectangular pulse (17) in the middle. Let the pulse duration be equal to 0.33T, then, in general, the pulse length of the trapezoid will be equal to T=1. Let us write the spectrum of a trapezoid as

$$\mathbf{Tr}(\omega,T,t_m) = \mathbf{P}_u(\omega,T,t_{m,1}) + \mathbf{R}(\omega,t_{m,2}) + \mathbf{P}_d(\omega,T,t_{m,3})\ . \qquad (29)$$

In this case, we use additional indices 1, 2, 3 in the shift time designation, which show the shifts of the increasing sawtooth pulse, rectangular and decreasing pulses, respectively. The trapezoid spectrum will contain a quadrature part, since the sawtooth spectrum contains quadrature parts. The spectrum of the trapezoid pulse sequence is shown in Figure 11.

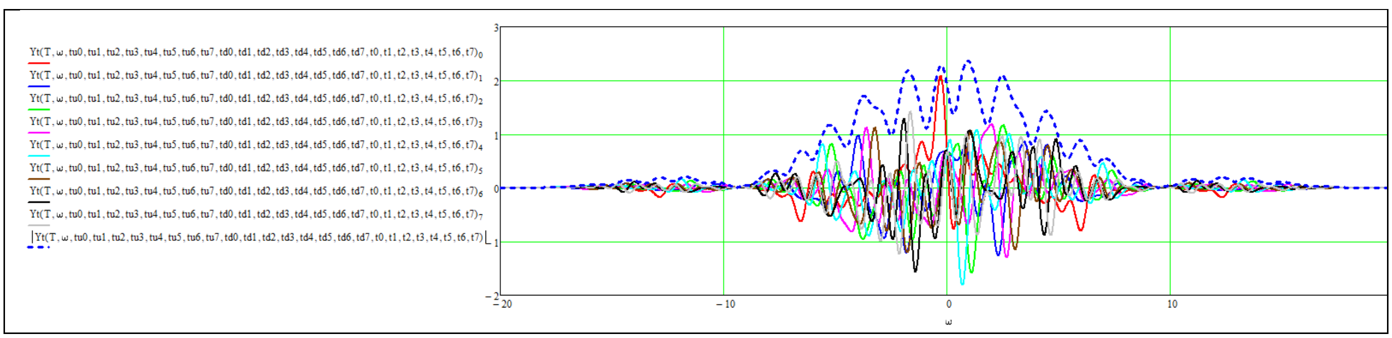

Figure 11: Spectrum of a Trapezoid Pulse Sequence

A pulse sequence of identical trapezoids obtained using IOFT is shown in Figure 12.

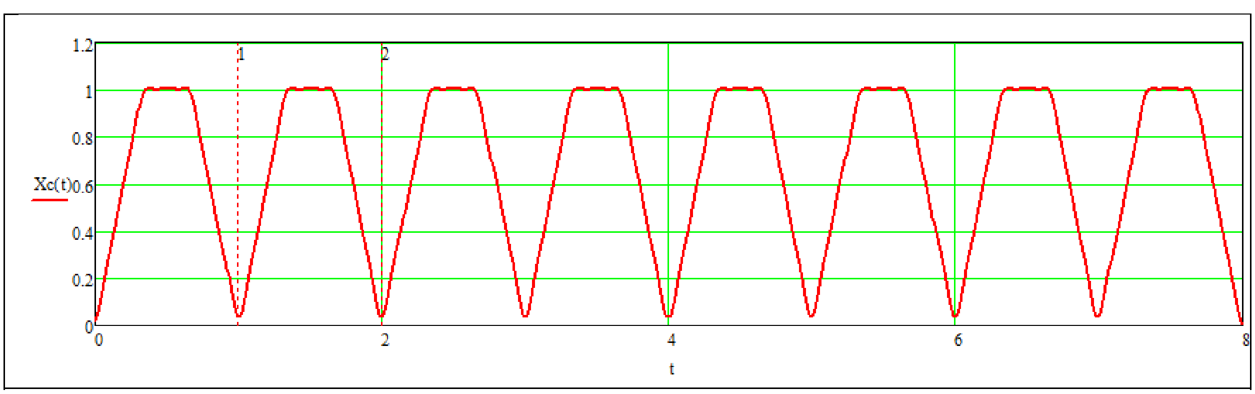

Figure 12: IOFT of Trapezoid Pulses

4.6. OFT and IOFT Sinus impulses

Let's consider the first half of a sine wave over the duration of its period as a sine pulse.
The direct spectral matrix will look like this:

$$\mathbf{S}_r(\omega,\Omega,T,t_m)=\frac{1}{\Omega^2-\omega^2}\left[\left(\Omega\cos\left(\frac{T\omega}{2}\right)(1-\cos(T\Omega))-\omega\sin(T\Omega)\sin\left(\frac{T\omega}{2}\right)\right)\right]\mathbf{D}(\omega,t_m)\,. \quad (30)$$

The quadrature matrix of spectra is equal to

$$\mathbf{S}_i(\omega,\Omega,T,t_m)=\frac{1}{\Omega^2-\omega^2}\left[\left(\omega\sin(T\Omega)\cos\left(\frac{T\omega}{2}\right)-\Omega(1+\cos(T\Omega))\sin\left(\frac{T\omega}{2}\right)\right)\right]\bar{\mathbf{D}}(\omega,t_m)\,. \quad (31)$$

Here Ω denotes the frequency of change of the sinusoidal pulse. The spectrum of a sinusoidal pulse will be equal to the sum of (30) and (31):

$$\mathbf{S}(\omega,\Omega,T,t_m)=\mathbf{S}_r(\omega,\Omega,T,t_m)+\mathbf{S}_i(\omega,\Omega,T,t_m)\,. \quad (32)$$

Figure 13 shows the spectrum of a sequence of identical sinusoidal pulses, calculated using formula (32).

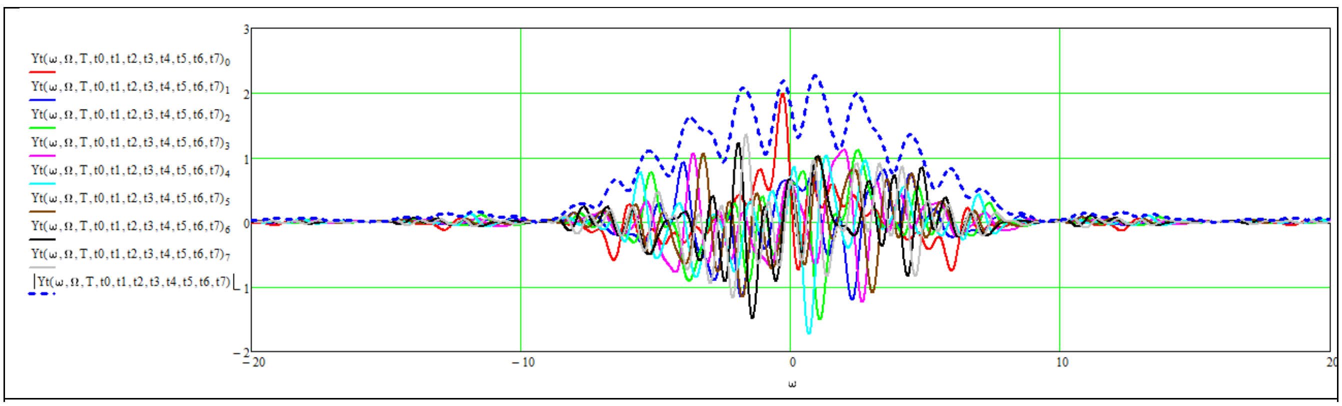

Figure 13: Spectrum of a Sine Wave Sequence

Figure 14 shows the IOFT of a sequence of identical sine pulses.

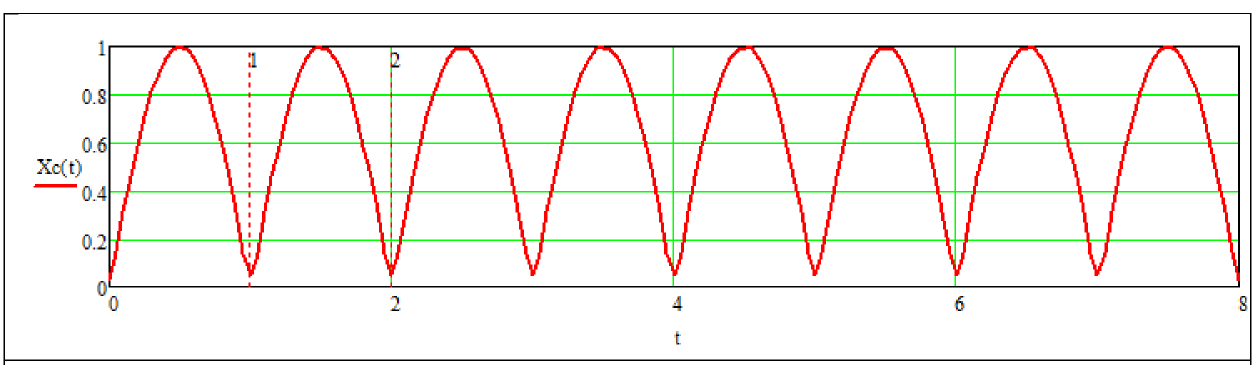

Figure 14: IOFT of Sine Pulses

4.7. OFT and IOFT Cosine Pulses

The direct part of the spectrum of cosine pulses is calculated as

$$\mathbf{C}_r(\omega,\Omega,T,t_m)=\frac{1}{\Omega^2-\omega^2}\left[\left(\Omega\sin(T\Omega)\cos\left(\frac{T\omega}{2}\right)-\omega(1+\cos(T\Omega))\sin\left(\frac{T\omega}{2}\right)\right)\right]\mathbf{D}(\omega,t_m)\,. \quad (33)$$

The quadrature part is equal to

$$\mathbf{C}_i(\omega,\Omega,T,t_m)=\frac{1}{\Omega^2-\omega^2}\left[\left(\Omega\sin(T\Omega)\sin\left(\frac{T\omega}{2}\right)-\omega(1-\cos(T\Omega))\cos\left(\frac{T\omega}{2}\right)\right)\right]\bar{\mathbf{D}}(\omega,t_m)\ . \qquad (34)$$

The spectrum of cosine pulses is equal to the sum of (33) and (34):

$$\mathbf{C}(\omega,\Omega,T)=\mathbf{C}_r(\omega,\Omega,t_m)+\mathbf{C}_i(\omega,\Omega,t_m)\ . \qquad (35)$$

Figure 15 shows the spectra of a sequence of cosine pulses calculated using formula (35).

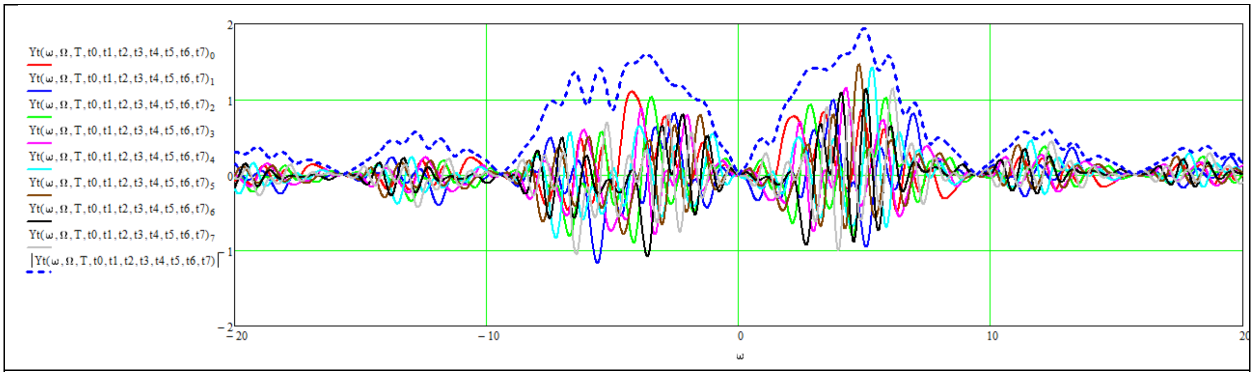

Figure 15: Spectrum of a Cosine Pulse Sequence

Figure 16 shows the IOFT of cosine pulses for x(0) = [1-1 1 1-1 1 1 1].

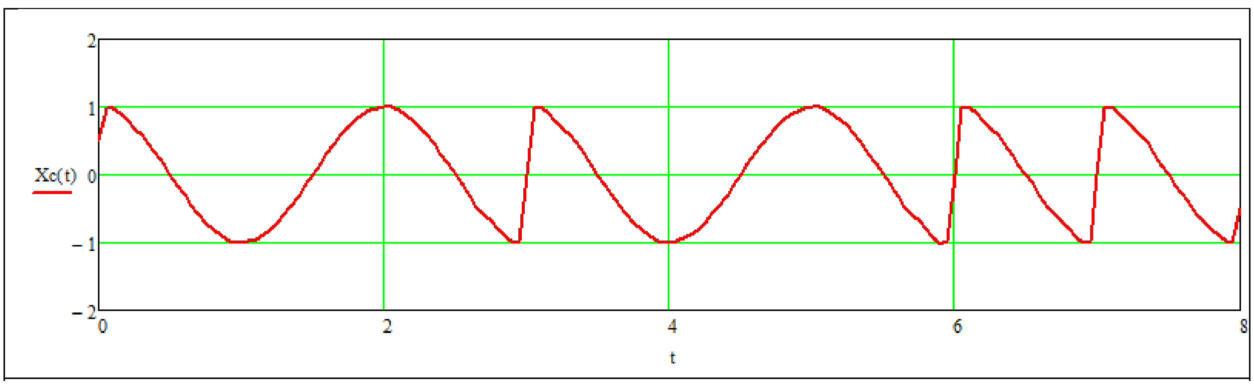

Figure 16: IFOT of Cosine Pulses for x(0)=[1-1 1 1-1 1 1 1]

4.8. OFT and OIFT of Different Pulses

Since the octonion is an 8D orthogonal space, it is used to form a MIMO system [4]. Consequently, the spectra of different pulses can be separated and OFT and IOFT can be performed not only for the same but also for different pulses. Let us consider the corresponding transformations for the 8 types of pulses presented above.

In accordance with the theorem of pulse delay in time (13), the matrix of different delays of the direct part of the pulse spectrum has the form (14). The quadrature part of the spectrum with delay is calculated using the quadrature matrix (15). Let us denote each column of the matrix (14), with the corresponding time shift t_m, as $D_0(\omega,t_0)$, $D_1(\omega,t_1)$, $D_2(\omega,t_2)$, $D_3(\omega,t_3)$, $D_4(\omega,t_4)$, $D_5(\omega,t_5)$, $D_6(\omega,t_6)$, $D_7(\omega,t_7)$. t_m – shows the pulse delay time, m=0,1,…,7. For example, the zero column will look like this:

$$D_0(\omega,t_0)=\frac{1}{\sqrt{7}}\left[\sqrt{7}\cos(\omega t_0)\quad \sin(\omega t_0)\quad \sin(\omega t_0)\quad \sin(\omega t_0)\quad \sin(\omega t_0)\quad \sin(\omega t_0)\quad \sin(\omega t_0)\quad \sin(\omega t_0)\right]^{\mathrm{T}}.$$

We denote the columns of the quadrature matrix (15) for a given pulse delay as $\bar{D}_0(\omega,t_0)$, $\bar{D}_1(\omega,t_1)$, $\bar{D}_2(\omega,t_2)$, $\bar{D}_3(\omega,t_3)$, $\bar{D}_4(\omega,t_4)$, $\bar{D}_5(\omega,t_5)$, $\bar{D}_6(\omega,t_6)$, $\bar{D}_7(\omega,t_7)$. For example, column zero (15) will look like:

$$\bar{D}_0(\omega,t_0)=\frac{1}{\sqrt{7}}\left[-\sqrt{7}\sin(\omega t_0)\quad \cos(\omega t_0)\quad \cos(\omega t_0)\quad \cos(\omega t_0)\quad \cos(\omega t_0)\quad \cos(\omega t_0)\quad \cos(\omega t_0)\quad \cos(\omega t_0)\right]^{\mathrm{T}}.$$

Using the introduced notations for the columns of the time shift matrices and the spectra of the different pulses found above, we write down the expressions for the spectral matrices for the vector of the different pulses in the following order:

0) Trapezium - $\mathbf{T}(\omega,T,t_0)=\left[\frac{T}{\omega}\sin\left(\frac{T\omega}{2}\right)D_0(\omega,t_{0,1})+\frac{T}{\omega}\left(\operatorname{sinc}\left(\frac{T\omega}{2}\right)-\cos\left(\frac{T\omega}{2}\right)\right)\bar{D}_0(\omega,t_{0,1})\right]+$
$+T\operatorname{sinc}\left(\frac{T\omega}{2}\right)D_0(\omega,t_{0,2})+\left[\frac{T}{\omega}\sin\left(\frac{T\omega}{2}\right)D_0(\omega,t_{0,3})+\frac{T}{\omega}\left(\cos\left(\frac{T\omega}{2}\right)-\operatorname{sinc}\left(\frac{T\omega}{2}\right)\right)\bar{D}_0(\omega,t_{0,3})\right]$, where T=0.3.

1) Meander - $\mathbf{M}(\omega,T,t_1)=T\operatorname{sinc}\left(\frac{T\omega}{2}\right)D_1(\omega,t_{1,1})-T\operatorname{sinc}\left(\frac{T\omega}{2}\right)D_1(\omega,t_{1,2})$, where T=0.5.

2) Triangle – $\mathbf{T}(\omega,T,t_2)=\frac{T}{\omega}\sin\left(\frac{T\omega}{2}\right)D_2(\omega,t_{2,1})+\frac{T}{\omega}\left(\operatorname{sinc}\left(\frac{T\omega}{2}\right)-\cos\left(\frac{T\omega}{2}\right)\right)\bar{D}_2(\omega,t_{2,1})+$
$+\frac{T}{\omega}\sin\left(\frac{T\omega}{2}\right)D_2(\omega,t_{2,2})+\frac{T}{\omega}\left(\cos\left(\frac{T\omega}{2}\right)-\operatorname{sinc}\left(\frac{T\omega}{2}\right)\right)\bar{D}_2(\omega,t_{2,2})$, where T=0.5.

3) Rectangle - $\mathbf{R}(\omega,t_3)=T\operatorname{sinc}\left(\frac{T\omega}{2}\right)D_3(\omega,t_3)$, where T=1.

4) Rising saw –
$\mathbf{P}_u(\omega,T,t_4)=\frac{T}{\omega}\sin\left(\frac{T\omega}{2}\right)D_4(\omega,t_4)+\frac{T}{\omega}\left(\operatorname{sinc}\left(\frac{T\omega}{2}\right)-\cos\left(\frac{T\omega}{2}\right)\right)\bar{D}_4(\omega,t_4)$, where T=1.

5) Sinus –
$\mathbf{S}(\omega,\Omega,T,t_5)=\frac{1}{\Omega^2-\omega^2}\left[\Omega\cos\left(\frac{T\omega}{2}\right)(1-\cos(T\Omega))-\omega\sin(T\Omega)\sin\left(\frac{T\omega}{2}\right)\right]D_5(\omega,t_5)+$
$+\frac{1}{\Omega^2-\omega^2}\left[\omega\sin(T\Omega)\cos\left(\frac{T\omega}{2}\right)-\Omega\sin\left(\frac{T\omega}{2}\right)(1+\cos(T\Omega))\right]\bar{D}_5(\omega,t_5)$, where T=1.

6) Decreasing saw –
$\mathbf{P}_d(\omega,T,t_6)=\frac{T}{\omega}\sin\left(\frac{T\omega}{2}\right)D_6(\omega,t_6)+\frac{T}{\omega}\left(\cos\left(\frac{T\omega}{2}\right)-\operatorname{sinc}\left(\frac{T\omega}{2}\right)\right)\bar{D}_6(\omega,t_6)$, where T=1.

7) Cosine - $\mathbf{C}(\omega,\Omega,T)=\frac{1}{\Omega^2-\omega^2}\left[\Omega\sin(T\Omega)\cos\left(\frac{T\omega}{2}\right)-\omega\sin\left(\frac{T\omega}{2}\right)(1+\cos(T\Omega))\right]D_7(\omega,t_7)+$
$+\frac{1}{\Omega^2-\omega^2}\left[\Omega\sin(T\Omega)\sin\left(\frac{T\omega}{2}\right)-\omega\cos\left(\frac{T\omega}{2}\right)(1-\cos(T\Omega))\right]\bar{D}_7(\omega,t_7)$, where T=1.

For the constituent elements that form the trapezoid, meander and triangle pulses, the relative time shifts are shown as a second index in t_m.

Figure 17 shows the spectra of different pulses for each coordinate axis of the 8D octonion space.

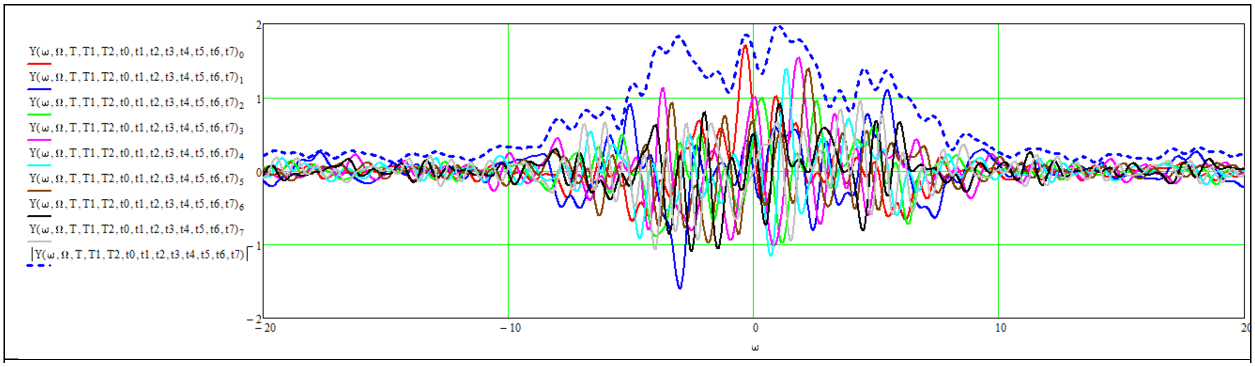

Figure 17: Spectra of a Sequence of Different Pulses

Figure 18 shows the IOFT for a sequence of different pulses.

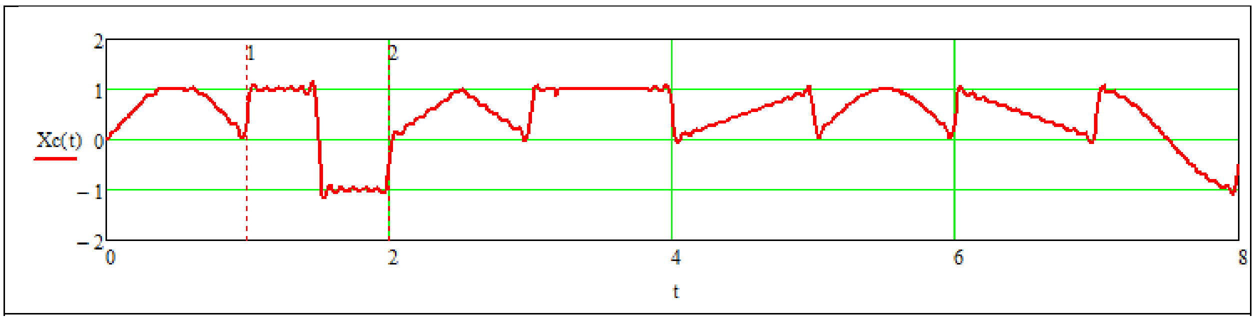

Figure 18: IOFT of Various Pulses

5. Conclusion

The octonion forms an orthogonal 8-dimensional space, so using the octonion significantly simplifies the Fourier transform of the 8-dimensional vector pulses, since integration can be performed for each pulse separately along each orthogonal coordinate axis.

The octonion is also used to form a MIMO scheme, since each pulse of the 8 at the input of the channel matrix is associated with all 8 pulses at its output. However, due to the orthogonality of space, the output pulses can be freely separated by IOFT, where identical spectra of each axis are added together to form corresponding pulses with a given time delay.

The spectra of the same 8 pulses are formed in accordance with the theorem of time shift of pulses from the spectrum of a single pulse by means of its frequency shift. In this case, the spectrum of pulses along each axis expands in accordance with the number of pulses. The expansion of the spectrum due to the joint processing of 8 pulses at once and the addition of the spectra of identical pulses along 8 orthogonal axes contributes to an increase in the noise immunity of the octonion Fourier transform.

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4. Sovetov, V. (2024). The MIMO Data Transfer Line with Seven-Frequency Octonion Carrier. *Eng OA, 2*(3), 01-23.