Existence, Stability and Instability Conditions of a Static Meniscus Appearing in Thin Rod Single Crystal Growth from Melt using EFG Method

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Abstract
This study presents necessary conditions for the existence and sufficient conditions for the stability or instability of the static meniscus (liquid bridge) appearing in the thin rod single crystal growth from the melt, of predetermined sizes, by using the edge-defined-film- fed (EFG) growth method. The cases when the contact angle and the growth angle verify the inequality $0 < \alpha_c < \frac{\pi}{2} - \alpha_g$ *or* $\frac{\pi}{2} > \alpha_c > \frac{\pi}{2} - \alpha_g$ *are treated separately. Experimentally, only static meniscus*

(liquid bridges) which verifies the necessary condition of existence and the sufficient conditions of stability can be created; static meniscus (liquid bridges) which does not verify both of these conditions, exist only in theory because in reality they collapse. The results of this study is significant for thin rod single crystal growth from melt, with prior given macroscopic dimensions, using prior given specific equipment. That is because the obtained inequalities represent limits for what can and cannot be achieved experimentally. Numerical illustrations are given for NdYAG laser thin rod, and InSb sensor laser thin rod.

Keywords: Existence, Static Stability, Meniscus, Thin Rod Growth, Edge-Defined-Film-Fed-Growth
**MSC: 34. B.15.**

1. Introduction
The free surface of the static meniscus, in single crystal growth by EFG method, in hydrostatic approximation is described by the Young-Laplace capillary equation [1]:

$$\gamma \times \left(\frac{1}{R_1} + \frac{1}{R_2}\right) = P_a - P_m \tag{1.1}$$

Here: γ is the melt surface tension; $\frac{1}{R_1}, \frac{1}{R_2}$ denote the main normal curvatures of the free surface at an arbitrary point M of the

surface ; P_a is the pressure above the free surface, equal to the pressure of the gas flow introduced in the furnace for release the heat and thereafter is denoted by p_g ($P_a = p_g$).The pressure P_m under the free surface in hydrostatic approximation is the hydrostatic pressure of the melt column equal to $-\rho \times g \times (z + H)$.

Here: ρ denotes the melt density; g is the gravity acceleration; z is the coordinate of M with respect to the Oz axis, directed vertically upwards; H denotes the melt column height between the horizontal crucible melt level and the shaper top level. H is positive when the crucible melt level is under the shaper top level and it is negative when the shaper top level is under the crucible melt level.(see Fig.1. 1).
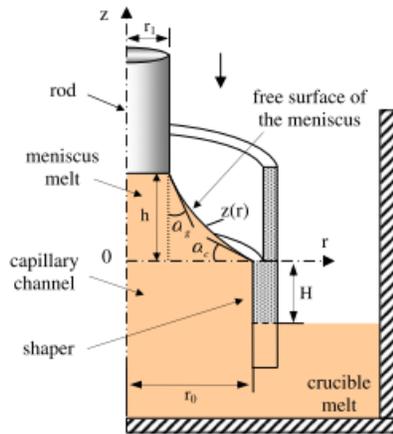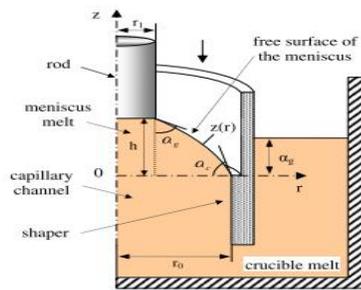
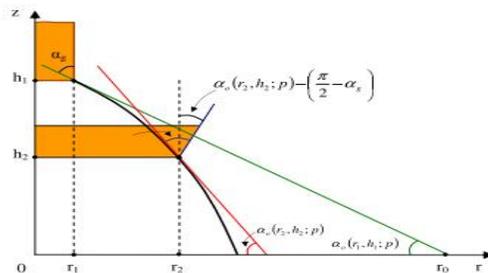a. Meniscus with convex meridian curve
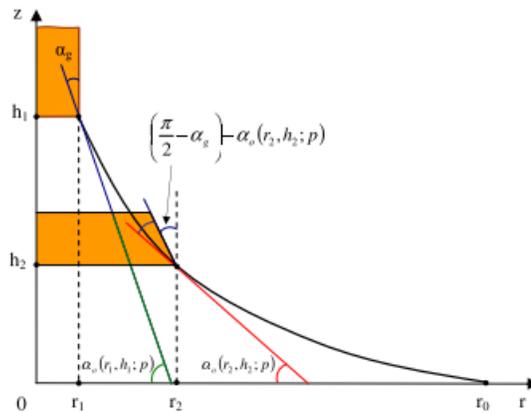
b. Deviation from the vertical for meniscus with convex meridian curve

Figure 1.1: Meniscus with Convex Meridian Curve

c. Meniscus with concave meridian curve

d. Deviation from the vertical for meniscus with concave meridian curve

Figure 1.2: Meniscus with Concave Meridian Curve

In hydrostatic approximation the Young -Laplace capillary surface equation can be written as :

$$\gamma \times \left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \rho \times g \times z - p \tag{1.2}$$

Where

$$p = -p_g - \rho \times g \times H \tag{1.3}$$

To calculate the meniscus surface shape and size in hydrostatic approximation is convenient to employ the Young–Laplace equation (1.2) in its differential form with respect to the Oxyz, Cartesian reference frame having the Oz axis oriented upward. That is:

$$\left[1 + \left(\frac{\partial z}{\partial y}\right)^2\right] \times \frac{\partial^2 z}{\partial x \partial x} - 2 \times \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} \times \frac{\partial^2 z}{\partial x \partial y} + \left[1 + \left(\frac{\partial z}{\partial x}\right)^2\right] \times \frac{\partial^2 z}{\partial y \partial y}$$

$$= \frac{\rho \times g \times z - p}{\gamma} \times \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{3}{2}} \tag{1.4}$$

With respect to an axis-symmetric reference frame (see Fig.1.1 and Fig.1.2) according to equation (1.4) become [2-4]:

$$z'' = \frac{\rho \times g \times z - p}{\gamma} \times \sqrt{[1 + z'^2]^3} - \frac{1}{r} \times [1 + z'^2] \times z' \ \ for \ 0 < r_1 \leq r \leq r_0 \tag{1.5}$$

where $r_1 > 0$ is the single crystal thin rod radius and r_0 $(r_0 > r_1)$ is the shaper radius.

In Eq. (1.5) $z = z(r)$ describes the meridian curve of the free surface as function of the radial coordinate r.

Remark that the Young-Laplace capillary equation (1.5) coincide with the Euler equation for the free energy of the melt column i.e. the first order necessary condition of minimum of functional $I(z)$ defined by:

$$I(z) = \int_{r_1}^{r_0} \{\gamma \times [1 + \sqrt{1 + (z')^2} + \frac{1}{2} \times \rho \times g \times z^2 - p \times z\} \times r dr \ \ z(r_1) = h \ , \ z(r_0) = 0 \tag{1.6}$$

The Young-Laplace equation (1.5) is obtained starting from the general Euler equation

$$\frac{d}{dr}\left(\frac{\partial F}{\partial z'}\right) - \left(\frac{\partial F}{\partial z}\right) = 0 \tag{1.7}$$

which is the first order necessary condition of minimum of the functional
$$I(z) = \int_{r_1}^{r_0} F(r, z, z') dr \ \ z(r_1) = h \ , \ z(r_0) = 0$$

taking

$$F(r, z, z') = \{\gamma \times [1 + \sqrt{1 + (z')^2} + \frac{1}{2} \times \rho \times g \times z^2 - p \times z\} \times r \tag{1.8}$$

In other worlds the Young- Laplace equation (1.5) can be obtained from (1.7),taking $F(r, z, z')$ given by (1.8).

As the Young- Laplace equation in hydrostatic approximation (1.5) is a second ordinary order differential equation formulation of boundary conditions requires assignment of two boundary conditions; one of the melt crystal interface, the second one on the melt and shaper interface. These conditions are: solution $z = z(r)$ of the Eq. (1.5) satisfy the following boundary conditions:

$$z'(r_1) = -\tan\left(\frac{\pi}{2} - \alpha_g\right), \ z'(r_0) - \tan(\alpha_c), \ z(r_0) = 0 \text{ and}$$ (1.9)

$z(r)$ is strictly decreasing on $[r_1, r_0]$

Where α_g is the growth angle, α_c is the contact angle between the meniscus free surface and the edge of the shaper top.

2. Existence, Stability or Instability of a Convex Meniscus

A meniscus is convex if $z''(r) > 0$ for $r_1 \le r \le r_0$

Remark first that in case of a convex meniscus the function $z'(r)$ is increasing. This means that the angle between the tangent line to meniscus in every point, and the Or axis $\alpha(r) = -\arctan z'(r)$ is decreasing. In particular, it follows that $\alpha(r_1) > \alpha(r_0)$. Since $\alpha(r_1) = \frac{\pi}{2} - \alpha_g$ and $\alpha(r_0) = \alpha_c$ we obtain inequality $\frac{\pi}{2} - \alpha_g > \alpha_c > 0$.(see Fig.1.1)

Using the Young-Laplace capillary equations (1.5) and conditions (1.8) in [3] the following result was established. If there exists a solution $z(r)$ of Eq. (1.5), which describes the free surface of a static meniscus having convex meridian curve on the closed interval $[r_1, r_0]$, then the following inequalities hold:

$$-\gamma \times \frac{\frac{\pi}{2} - (\alpha_c + \alpha_g)}{r_0 - r_1} \times \cos\alpha_c + \frac{\gamma}{r_0} \times \sin\alpha_c \le p \le -\gamma \times \frac{\frac{\pi}{2} - (\alpha_c + \alpha_g)}{r_0 - r_1} \times \sin\alpha_g +$$ (2.1)

$$(r_0 - r_1) \times \rho \times g \times \tan\left(\frac{\pi}{2} - \alpha_g\right) + \frac{\gamma}{r_1} \times \cos\alpha_g$$

Therefore in case of the existence of a computed convex static meniscus the values of the pressure difference $p = -p_g - \rho \times g \times H$ has to be in researched in the interval $[L_{left}, L_{right}]$ where :

$$L_{left} = -\gamma \times \frac{\frac{\pi}{2} - (\alpha_c + \alpha_g)}{r_0 - r_1} \times \cos\alpha_c + \frac{\gamma}{r_0} \times \sin\alpha_c$$ (2.2)

$$L_{right} = -\gamma \times \frac{\frac{\pi}{2} - (\alpha_c + \alpha_g)}{r_0 - r_1} \times \sin\alpha_g +$$ (2.3)

$$(r_0 - r_1) \times \rho \times g \times \tan\left(\frac{\pi}{2} - \alpha_g\right) + \frac{\gamma}{r_1} \times \cos\alpha_g$$

For $p = -p_g - \rho \times g \times H < L_{left}$ and for $p = -p_g - \rho \times g \times H > L_{right}$ (2.4)

computed convex static meniscus like in Fig.1.2 does not exit. Therefore if $p = -p_g - \rho \times g \times H$ is in the range defined by (2.4) convex static meniscus can not be obtained experimentally because it collapse.

Retain that the pressure difference $p = -p_g - \rho \times g \times H$ can be controlled by the gas pressure p_g and the parameter H.

For the existence of a convex static meniscus numerical computations were performed for NdYAG laser thin rod growth using the following numerical data: $r_0 = 5 \times 10^{-3}[m]$, $r_1 = 2.5 \times 10^{-4}[m]$; $\alpha_c = 0.523[rad]$; $\alpha_g = 0.2967[rad]$ $\rho = 3.6 \times$ $\times 10^3 [\frac{kg}{m^3}]$; $\gamma = 7.81 \times 10^{-1} [\frac{N}{m}]$; $g = 9.81[\frac{m}{s^2}]$;
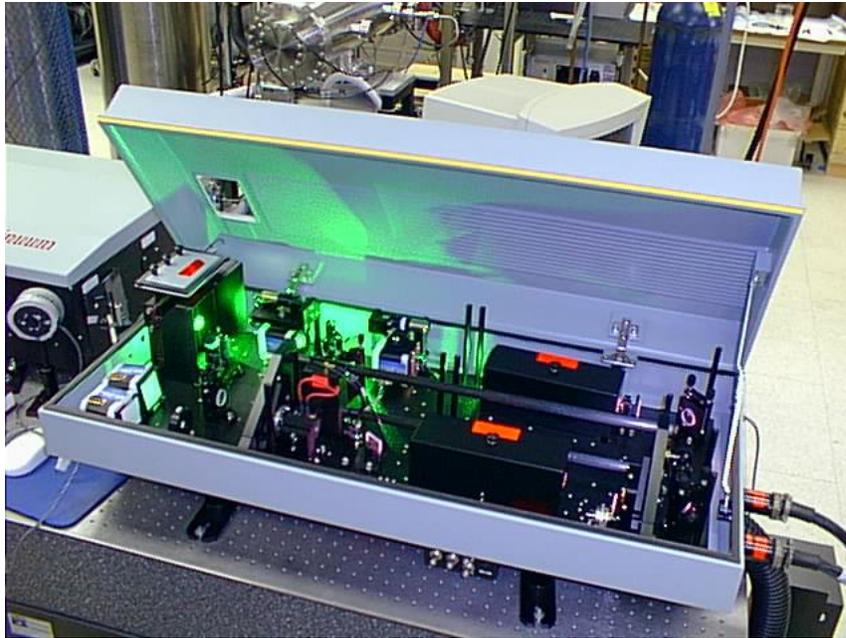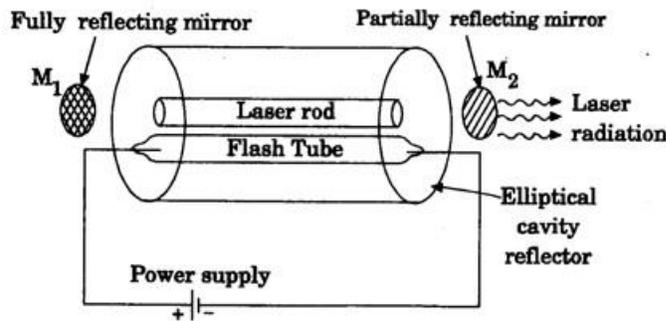
Figure 2.1: NDYAG Laser Device

Fig. 5.17 Nd - YAG Laser

Figure 2.2: Schematic of NDYAG Laser Functioning

Computation shows that for the considered numerical data, the inequality (2.1) becomes: -124.9427[Pa] $\leq p \leq 518.9855$[Pa] i.e. $L_{left} = -124.9427$[Pa] , $L_{right} = 518.9855$[Pa] .

In the following we will illustrate first the existence of convex static meniscus in case of NdYAG laser thin rod for $h = 0.0025$[m].Such meniscus can be obtained by solving the following initial value problem

$$\begin{cases} \dfrac{dz}{dr} = -\tan\alpha \\ \dfrac{d\alpha}{dr} = \dfrac{p - \rho \cdot g \cdot z}{\gamma} \cdot \dfrac{1}{\cos\alpha} - \dfrac{1}{r} \cdot \tan\alpha \end{cases} \quad z(0.0025)=0.0025,\ \alpha(0.0025)=1.274096327$$

For different values of $p \in [-124.9427, 518.9855]$ [Pa]

In this way a convex meniscus is obtained for $p = 37.7[Pa]$ .The convex static meniscus NDYAG shape with the variation of alpha is represented on the next figure
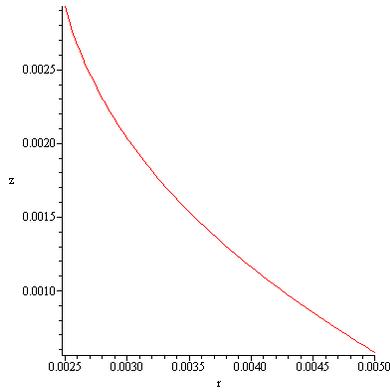
Figure 2.3: Convex NDYAG Meniscus Shape $p = 37.7[Pa]$
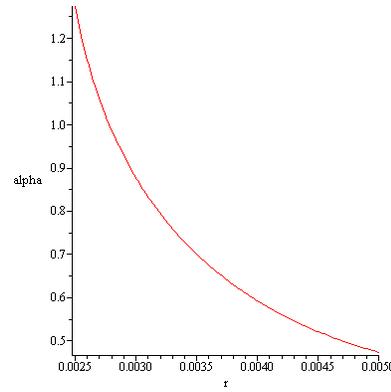

Figure 2.4: Alpha Versus r, $p = 37.7[Pa]$

For $p < -124.9427[Pa]$ and $p > 518.9855[Pa]$ convex NDYAG static meniscus ,like in Fig.1.1 does not exist.This fact is illustrated in Fig.2.5,Fig.2.6 and Fig.2.7,Fig.2.8 respectively:
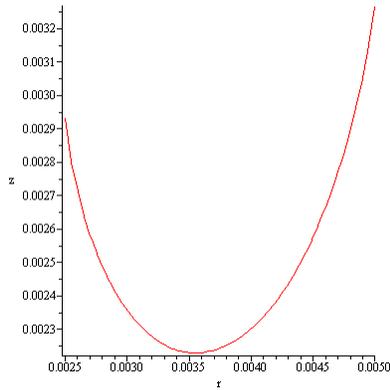
Figure 2.5: Meniscus Shape for $p = -500[Pa]$
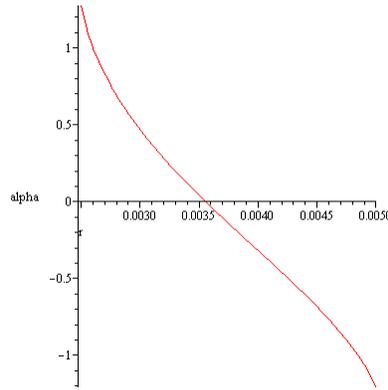
Figure 2.6: Alpha Versus r, $p = -500[Pa]$
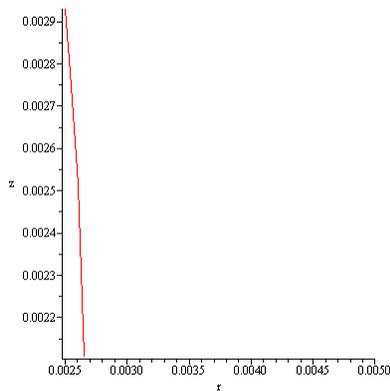
Figure 2.7: Meniscus Shape for $p = 600[Pa]$
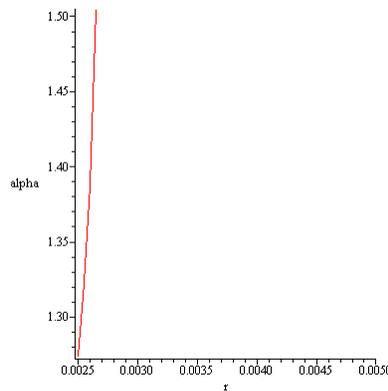
Figure 2.8: Alpha Versus r, $p = 600[Pa]$

Concerning the stability of an existing convex static meniscus it should be noted that, the convex meniscus stability (the static one) should be distinguished from the dynamic stability of the crystallization process considered [5-9].

For statically stable convex meniscus, indispensable (necessary) first order conditions and also second order sufficient conditions for the minimum of functional (1.6) should be satisfied.

The first order necessary conditions are the Euler equations which leads to the Young-Laplace capillary equation (1.5).

The second order sufficient conditions for the minimum of functional (1.6) are the Legendre condition and the Jacobi condition [10]. The Legendre condition is

$$\frac{\partial^2 F}{\partial z' \partial z'} > 0 \tag{2.5}$$

Computing $\frac{\partial^2 F}{\partial z' \partial z'}$ we find $\frac{\partial^2 F}{\partial z' \partial z'} = \frac{r \times \gamma}{(1+(z')^2)^{\frac{3}{2}}} > 0$. Therefore the Legendre condition is verified.

The Jacoby condition concern the so-called Jacoby equation:

$$\left[\frac{\partial^2 F}{\partial z \partial z} - \frac{d}{dr}\left(\frac{\partial^2 F}{\partial z \partial z'}\right)\right] \times \eta - \frac{d}{dr}\left[\frac{\partial^2 F}{\partial z' \partial z'} \times \eta'\right] = 0 \tag{2.6}$$

In case of the functional (1.6) Jacobi equation (2.6) become:

$$\frac{d}{dr}\left[\frac{r \times \gamma}{(1+(z')^2)^{\frac{3}{2}}} \times \eta'\right] - g \times \rho \times r \times \eta = 0 \tag{2.7}$$

Jacobi requirement of stability condition is that the solution of equation (2.7) which verifies the initial condition $\eta(r_0)=0$ and $\eta'(r_0) = 1$ vanishes at most ones on the interval $[r_1, r_0]$.

In order to find a sufficient condition for that, according to, for the equation (2.7) a Sturm type upper bound has to be found [10].

Remark that for the coefficients of (2.7) the following inequalities hold:

$$\frac{r \times \gamma}{(1+(z')^2)^{\frac{3}{2}}} > r_1 \times \gamma \times (sin\alpha_g)^3 \ and \ -g \times \rho \times r < -g \times \rho \times r_1 \tag{2.8}$$

Hence

$$(\theta' \times r_1 \times \gamma \times (sin\alpha_g)^3)' - g \times \rho \times r_1 = 0 \ or \ \theta'' = \frac{g \times \rho}{\gamma \times (sin\alpha_g)^3} \tag{2.9}$$

is a Sturm –type upper bound for (2.7).

Using the Sturm –type upper bound $\theta'' = \frac{g \times \rho}{\gamma \times (sin\alpha_g)^3}$, remark that an arbitrary solution of (2.9) is given by

$$\theta(r) = A \times \sin(\omega \times r + \varphi) \tag{2.10}$$

where A and φ are real constants and

$$\omega^2 = \frac{g \times \rho}{\gamma \times (sin\alpha_g)^3} \tag{2.11}$$

The half period of a non-zero solution $\theta(r)$ defined by (2.10) is given by

$$\frac{\pi}{\omega} = \pi \times \sqrt{\frac{\gamma \times (sin\alpha_g)^3}{g \times \rho}}$$

(2.12)

If the half period given by (2.12) verifies inequality

$$\frac{\pi}{\omega} = \pi \times \sqrt{\frac{\gamma \times (sin\alpha_g)^3}{g \times \rho}} > r_0 - r_1$$

(2.13)

then the function $\theta(r)$ defined by (2.10) vanishes at most ones on the interval $[r_1, r_0]$.

Hence, according to the solution of Jacobi equation (2.7) which verifies $\eta(r_0) = 0$ and $\eta'(r_0) = 1$ has only one zero on the interval $[r_1, r_0]$. Therefore the stability condition of Jacobi is verified [10].

The conclusion is:
Statement 2.1. If for a certain value of the pressure difference $p = -p_g - \rho \times g \times H$ a convex static meniscus exist and the following inequality holds

$$r_0 - r_1 < \pi \times \sqrt{\frac{\gamma \times (sin\alpha_g)^3}{g \times \rho}}$$

(2.14)

then the convex static meniscus is stable.

Inequality (2.14) is a sufficient condition of stability for a static convex meniscus which exist theoretically for a certain value of the pressure difference p. Note that inequality (2.14) is not a necessary condition of stability i.e. inequality

$$r_0 - r_1 > \pi \times \sqrt{\frac{\gamma \times (sin\alpha_g)^3}{g \times \rho}}$$

(2.15)

not imply instability.

For find sufficient condition concerning instability the Jacobi requirement of instability condition has to be proven. This is that the solution of equation (2.7) which verifies the initial condition $\eta(r_0) = 0$ and $\eta'(r_0) = 1$ vanishes at least twice on the interval $[r_1, r_0]$. For this purpose according to a Sturm-type lower bound for the Jacobi equation (2.7) has to be found [10].

For this purpose, remark that for the coefficients of (2.7) the following inequalities hold:

$$\frac{r \times \gamma}{(1 + (z')^2)^{\frac{3}{2}}} < r_0 \times \gamma \times (cos\alpha_c)^3 \text{ and } -g \times \rho \times r_0 < -g \times \rho \times r$$

(2.16)

Hence

$$(\theta' \times r_0 \times \gamma \times (cos\alpha_c)^3)' - g \times \rho \times r_0 = 0$$

(2.17)

is a Sturm-type lower bound for the Jacobi equation.

Using the Sturm –type lower bound (2.17), remark that an arbitrary solution of (2.17) is given by

$$\theta(r) = A \times \sin(\omega \times r + \varphi) \tag{2.18}$$

where A and φ are real constants and

$$\omega^2 = \frac{g \times \rho}{\gamma \times (cos\alpha_c)^3} \tag{2.19}$$

The period of a non-zero solution $\theta(r)$ defined by (2.17) is given by

$$\frac{2\pi}{\omega} = 2\pi \times \sqrt{\frac{\gamma \times (cos\alpha_c)^3}{g \times \rho}} \tag{2.20}$$

If the period of the function $\theta(r)$, verifies inequality:

$$2\pi \times \sqrt{\frac{\gamma \times (cos\alpha_c)^3}{g \times \rho}} < r_0 - r_1 \tag{2.21}$$

then the function $\theta(r)$ vanishes at least twice on the interval $[r_1, r_0]$. Therefore according to the Jacoby condition: $\eta(r)$ solution of the equation (2.7) which verifies $\eta(r_0) = 0$ and $\eta'(r_0) = 1$ vanish at least twice on the interval $[r_1, r_0]$ is verified [10]. The conclusion is:
Statement 2.2. If for a certain value of the pressure difference $p = -p_g - \rho \times g \times H$ a convex static meniscus exist and the following inequality holds

$$2\pi \times \sqrt{\frac{\gamma \times (cos\alpha_c)^3}{g \times \rho}} < r_0 - r_1 \tag{2.22}$$

then the convex static meniscus is unstable.

For an NdYAG laser thin rod the sufficient condition of stability is $r_0 - r_1 < 0.002335500254[m]$ and the sufficient condition of instability is $r_0 - r_1 > 0.02382542348[m]$

It follows that the Statements 2.1 and 2.2 cannot give an answer to the problem of stability or instability Since in the case of the NdYAG convex static meniscus, we have found by computation, for the pressure difference $p = 37.7[Pa]$ (represented on Fig.2.3,2.4) the difference $r_0 - r_1$ is $r_0 - r_1 = 0.005\text{-}0.0025=0.0025$, it follows that the Statements 2.1 and 2.2 cannot give an answer to the problem of stability or instability of that meniscus.

Despite this shortcoming, the general inequalities (2.14), (2.22), and the particular ones $r_0 - r_1 < 0.002335500254[m]$ and $r_0 - r_1 > 0.02382542348[m]$ are important. They indicate how exactly we should choose r_0 and r_1 if we want to decide on the stability or instability of the static meniscus obtained by computation.

For example choosing $r_0 = 0.004 [m]$ *insted of* $0.005[m]$ the difference $r_0 - r_1$ become $r_0 - r_1 = 0.004[m] -0.0025[m] = 0.0015[m]$ and the stability condition is verified. The problem which appear is the existence of the static meniscus. In order to solve this new problem concerning the existence the new range of the pressure difference

$$L_{left} = -\gamma \times \frac{\frac{\pi}{2} - (\alpha_c + \alpha_g)}{r_0 - r_1} \times \cos\alpha_c + \frac{\gamma}{r_0} \times \sin\alpha_c \tag{2.23}$$

$$L_{right} = -\gamma \times \frac{\frac{\pi}{2} - (\alpha_c + \alpha_g)}{r_0 - r_1} \times \sin \alpha_g + (r_0 - r_1) \times \rho \times g \qquad (2.24)$$

$$\times \tan(\frac{\pi}{2} - \alpha_g) + \frac{\gamma}{r_1} \times \cos \alpha_g$$

of the pressure difference

$$p = -p_g - \rho \times g \times H \qquad (2.25)$$

Replacing in (2.23), (2.24) the new value $r_0 = 0.004$ we found:
$L_{left} = $ -241.270552 [Pa]
$L_{right} = $ 357.6881909[Pa].

For analyze the existence of convex static meniscus in case of NdYAG laser thin rod for $h = 0.0019[m]$ the following initial value problem has to be solved for different values of $p \in$ [-241.270552,357.6881909] [Pa]

$$\begin{cases} \dfrac{dz}{dr} = -\tan \alpha \\ \dfrac{d\alpha}{dr} = \dfrac{p - \rho \cdot g \cdot z}{\gamma} \cdot \dfrac{1}{\cos \alpha} - \dfrac{1}{r} \cdot \tan \alpha \qquad z(0.0025) = 0.0019, \alpha(0.0025) = 1.274096327 \end{cases}$$

For p=37.7[Pa] is found that a convex static meniscus exist for $r \in$ [0.0025,0.004]

The convex static meniscus NDYAG shape vith the variation of alpha is represented on the next figure
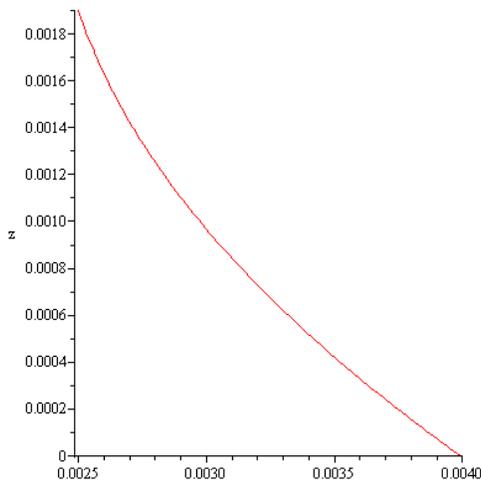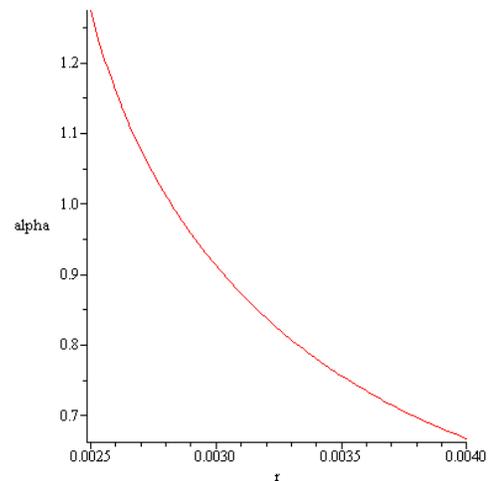

Figure 2.9: Convex NDYAG meniscus shape

Figure 2.10: Alpha versus r

$p = 37.7[Pa]$, $h = 0.0019[m]$, $r \in$ [0.0025,0.004]
According to Statement 2.1 the above NdYAG convex static meniscus is stable.

3. Existence, Stability or Instability of Concave Static Meniscus
A meniscus is concave if $z''(r) < 0$ for $r_1 \le r \le r_0$

Remark that in case of a concave meniscus the function $z'(r)$ is decreasing. Therefore, the angle between the tangent line to meniscus in every point, and the *Or* axis $\alpha(r) = -arctan z'(r)$ is increasing. In particular, it follows that $\alpha(r_1) < \alpha(r_0)$. Since $\alpha(r_1) = \frac{\pi}{2} - \alpha_g$ and $\alpha(r_0) = \alpha_c$ we obtain inequality $0 < \frac{\pi}{2} - \alpha_g < \alpha_c$. (see Fig.1.2).

Using the Young-Laplace capillary equations (1.5) and conditions (1.8) in [3] the following result was established;

If there exists a solution $z(r)$ of Eq. (1.5), which describes the free surface of a static meniscus having concave meridian curve on the closed interval $[r_1, r_0]$, then the following inequalities hold:

$$\gamma \times \frac{(\alpha_c + \alpha_g) - \frac{\pi}{2}}{r_0 - r_1} \times \cos \alpha_c$$

$$+ \frac{\gamma}{r_0} \times \cos \alpha_g \le p \le \gamma \times \frac{(\alpha_c + \alpha_g) - \frac{\pi}{2}}{r_0 - r_1} \times \sin \alpha_g + (r_0 - r_1) \times \rho \times g \quad (3.1)$$

$$\times \tan(\alpha_c) + \frac{\gamma}{r_1} \times \sin \alpha_c$$

Therefore in case of the existence of a concave static meniscus the values of the pressure difference $p = -p_g - \rho \times g \times H$ has to be researched in the interval $[L_{left}, L_{right}]$ where :

$$L_{left} = \gamma \times \frac{(\alpha_c + \alpha_g) - \frac{\pi}{2}}{r_0 - r_1} \times \cos \alpha_c + \frac{\gamma}{r_0} \times \cos \alpha_g \quad (3.2)$$

$$L_{right} = \gamma \times \frac{(\alpha_c + \alpha_g) - \frac{\pi}{2}}{r_0 - r_1} \times \sin \alpha_g + (r_0 - r_1) \times \rho \times g \times \tan(\alpha_c) + \frac{\gamma}{r_1} \times \sin \alpha_c \quad (3.3)$$

$$\text{For } p = -p_g - \rho \times g \times H < L_{left} \quad \text{and} \quad \text{for } p = -p_g - \rho \times g \times H > L_{right} \quad (3.4)$$

concave static meniscus like in Fig.1.2 does not exit. If $p = -p_g - \rho \times g \times H$ is in the range defined by (3.4) concave static meniscus cannot be obtained experimentally because it collapse.

Retain that the pressure difference $p = -p_g - \rho \times g \times H$ can be controlled by the gas pressure p_g and the parameter H.

For the existence of a convex static meniscus numerical computations were performed for InSb thin laser rod using the following numerical data: $r_0 = 7 \times 10^{-4}[m]$, $r_1 = 3.5 \times 10^{-4}[m]$; $\alpha_c = 1.1135[rad]$; $\alpha_g = 0.5044[rad]$ $\rho = 6.582 \times 10^3 [\frac{kg}{m^3}]$; $\gamma = 4.2 \times 10^{-1}$ $[\frac{N}{m}]$; $g = 9.81[\frac{m}{s^2}]$;

InSb, or indium antimonide thin rods are nanostructured materials with promising applications in various electronic and optoelectronic devices. They are thin, elongated structures made of the compound semiconductor indium antimonide. InSb is known for its high electron mobility, narrow energy bandgap, and low effective mass, making it a suitable material for infrared detectors, high-speed devices, and magnetic sensors.

Figure 3.1: Indium Antimonide Thin Rod Laser based Biometric Identification and Body Temperature Detection

Computation shows that for the considered numerical data, the inequality (3.1) becomes: 550.2356349 [Pa] $\leq p \leq$ 1149.984032[Pa] i.e. L_{left} = 550.2356349[Pa] , L_{right} = 1149.984032[Pa] .

In the following we will illustrate first the existence of concave static meniscus in case of InSb laser thin rod for h = 0.0015[m].Such meniscus can be obtained by solving the following initial value problem

$$\begin{cases} \dfrac{dz}{dr} = -\tan\alpha \\ \dfrac{d\alpha}{dr} = \dfrac{p-\rho\cdot g\cdot z}{\gamma}\cdot\dfrac{1}{\cos\alpha} - \dfrac{1}{r}\cdot\tan\alpha \quad\quad z\,(0.00035) = 0.0015 \, , \, \alpha\,(0.00035) = 1.066396327 \end{cases}$$

For different values of p ∈ [550.2356349 ,1149.984032][Pa]
In this way we find that for p = 800[Pa] a concave static InSb meniscus exist.The concave static meniscus shape is represented on the next figure
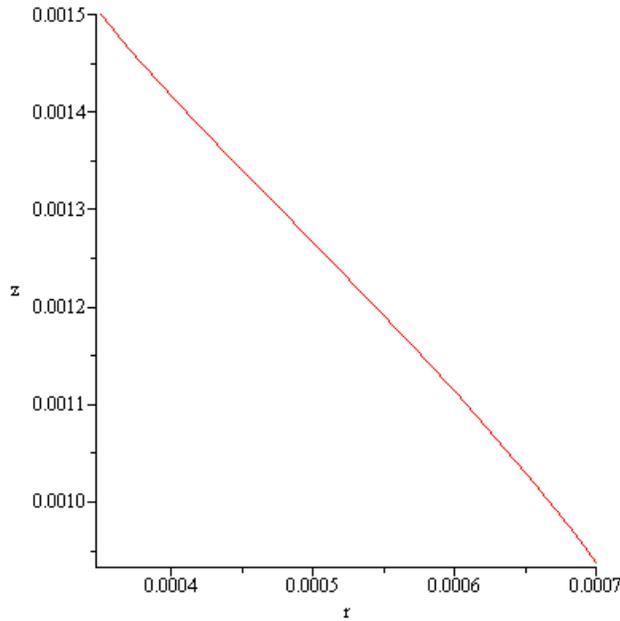
Figure 3.2: Concave InSb static meniscus for $p = 800[Pa]$, $h = 0.0015[m]$, r $\in[0.00035, 0.0007]$

Concerning the stability of an existing concave static meniscus it should be noted that, the concave meniscus stability (the static one) should be distinguished from the dynamic stability of the crystallization process considered [5-9].

For statically stable concave meniscus, indispensable (necessary) first order conditions and also second order sufficient conditions for the minimum of functional (1.6) should be satisfied.

The first order necessary conditions are the Euler equations which leads to the Young-Laplace capillary equation (1.5).

The second order sufficient conditions for the minimum of functional (1.6) are the Legendre condition and the Jacobi condition [10]. The Legendre condition is

$$\frac{\partial^2 F}{\partial z' \partial z'} > 0 \tag{3.5}$$

Computing $\frac{\partial^2 F}{\partial z' \partial z'}$ we find $\frac{\partial^2 F}{\partial z' \partial z'} = \frac{r \times \gamma}{(1+(z')^2)^{\frac{3}{2}}} > 0$.

Therefore, the Legendre condition is verified.

The Jacoby condition concern the so-called Jacoby equation:

$$\left[\frac{\partial^2 F}{\partial z \partial z} - \frac{d}{dr}\left(\frac{\partial^2 F}{\partial z \partial z'} \right) \right] \times \eta - \frac{d}{dr}\left[\frac{\partial^2 F}{\partial z' \partial z'} \times \eta' \right] = 0 \tag{3.6}$$

In case of the functional (1.6) equation (3.6) become:

$$\frac{d}{dr}\left[\frac{r \times \gamma}{(1+(z')^2)^{\frac{3}{2}}} \times \eta' \right] - g \times \rho \times r \times \eta = 0 \tag{3.7}$$

Jacobi requirement of stability condition is that the solution of equation (3.7) which verifies the initial condition $\eta(r_0) = 0$ and $\eta'(r_0) = 1$ vanishes at most ones on the interval $[r_1, r_0]$.

In order to find a sufficient condition for that, according to for the equation (3.7) a Sturm type upper bound has to be found [10].

Remark that for the coefficients of (3.7) the following inequalities hold:

$$\frac{r \times \gamma}{(1 + (z')^2)^{\frac{3}{2}}} > r_1 \times \gamma \times (cos\alpha_c)^3 \text{ and } -g \times \rho \times r < -g \times \rho \times r_1 \tag{3.8}$$

Hence

$$(\theta' \times r_1 \times \gamma \times (cos\alpha_c)^3)' - g \times \rho \times r_1 = 0 \text{ or } \theta'' = \frac{g \times \rho}{\gamma \times (cos\alpha_c)^3} \tag{3.9}$$

is a Sturm –type upper bound for (3.7).

Using the Sturm –type upper bound $\theta'' = \frac{g \times \rho}{\gamma \times (cos\alpha_c)^3}$, for stability it has to be find condition assuring that a nonzero solution of the equation (3.9) has only one zero on the interval $[r_1, r_0]$

For that remark that an arbitrary solution of (3.9) is given by

$$\theta(r) = A \times \sin(\omega \times r + \varphi) \tag{3.10}$$

where A and φ are real constants and

$$\omega^2 = \frac{g \times \rho}{\gamma \times (cos\alpha_c)^3} \tag{3.11}$$

The half period of a non-zero solution of (3.10) is given by

$$\frac{\pi}{\omega} = \pi \times \sqrt{\frac{\gamma \times (cos\alpha_c)^3}{g \times \rho}} \tag{3.12}$$

If the half period given by (3.12) verifies inequality

$$\frac{\pi}{\omega} = \pi \times \sqrt{\frac{\gamma \times (cos\alpha_c)^3}{g \times \rho}} > r_0 - r_1 \tag{3.13}$$

then the function $\theta(r)$ defined by (3.10) vanishes at most ones on the interval $[r_1, r_0]$.

Hence, according to the solution of Jacobi equation (3.7) which verifies $\eta(r_0) = 0$ and $\eta'(r_0) = 1$ vanishes at most once on the interval $[r_1, r_0]$ .Therefore the Jacobi condition of stability is verified [10].

The conclusion is:
Statement 3.1. If for a certain value of the pressure difference $p = -p_g - \rho \times g \times H$ a concave static meniscus exist and the following inequality holds

$$r_0 - r_1 < \pi \times \sqrt{\frac{\gamma \times (cos\alpha_c)^3}{g \times \rho}} \tag{3.14}$$

then the concave static meniscus is stable.

Inequality (3.14) is a sufficient condition of stability for a static concave meniscus which exist for a certain value of the pressure difference p . Note that inequality (3.14) is not a necessary condition of stability i.e. inequality

$$r_0 - r_1 > \pi \times \sqrt{\frac{\gamma \times (cos\alpha_c)^3}{g \times \rho}} \tag{3.15}$$

not imply instability.

For find sufficient condition concerning instability we follow the way described above.

Jacobi requirement of instability condition is that the solution of equation (3.7) which verifies the initial condition $\eta (r_0) = 0$ and $\eta' (r_0) = 1$ vanishes at least twice on the interval $[r_1, r_0]$.
For this purpose, according to a Sturm-type lower bound for the Jacobi equation (3.7) has to be find [10].

Remark now that for the coefficients of (3.7) the following inequalities hold:

$$\frac{r \times \gamma}{(1 + (z')^2)^{\frac{3}{2}}} < r_0 \times \gamma \times (sin\alpha_g)^3 and - g \times \rho \times r_0 < -g \times \rho \times r \tag{3.16}$$

Hence

$$(\theta' \times r_0 \times \gamma \times (sin\alpha_g)^3)' - g \times \rho \times r_0 = 0 \tag{3.17}$$

is a Sturm-type lower bound for the Jacobi equation.

Using the Sturm –type lower bound $\theta'' = \frac{g \times \rho}{\gamma \times (sin\alpha_g)^3}$, according to condition has to be found assuring that the solution of

(3.17) vanish at least twice on the interval $[r_1, r_0]$ [10].

Remark that an arbitrary nonzero solution of (3.17) is given by

$$\theta(r) = A \times \sin(\omega \times r + \varphi) \tag{3.18}$$

where A and φ are real constants and

$$\omega^2 = \frac{g \times \rho}{\gamma \times (sin\alpha_g)^3} \tag{3.19}$$

The period of a non-zero solution $\theta(r)$ defined by (3.10) is given by

$$\frac{2\pi}{\omega} = 2\pi \times \sqrt{\frac{\gamma \times (sin\alpha_g)^3}{g \times \rho}} \tag{3.20}$$

If the period of the function $\theta(r)$, verifies inequality:

$$2\pi \times \sqrt{\frac{\gamma \times (sin\alpha_g)^3}{g \times \rho}} < r_0 - r_1 \tag{3.21}$$

then the function $\theta(r)$ vanish at least twice on the interval $[r_1, r_0]$.

Therefore, according to the function $\eta(r)$ which verifies (3.7) and the initial conditions $\eta(r_0) = 0$, $\eta'(r_0) = 1$, vanishes at least twice on the interval $[r_1, r_0]$. Hence, the Jacoby condition of instability is verified [10].

The conclusion is:
Statement 3.2. If for a certain value of the pressure difference $p = -p_g - \rho \times g \times H$ a concave static meniscus exist and the following inequality holds

$$r_0 - r_1 > 2\pi \times \sqrt{\frac{\gamma \times (sin\alpha_g)^3}{g \times \rho}} \tag{3.22}$$

then the concave static meniscus is unstable.
For an InSb laser thin rod the sufficient condition of stability is $r_0 - r_1 < 0.002350673620$[m]
and the sufficient condition of instability is $r_0 - r_1 > 0.005383842371$[m]
According to Statement 3.1 the InSb concave static meniscus represented in Fig.3.2 is stable.

4. Results
Necessary conditions for the existence and sufficient conditions for the stability or instability of the static meniscus (liquid bridge) appearing in the thin rod single crystal growth from the melt, of predetermined sizes, by using the edge-defined-film- fed (EFG) growth method, are presented. Theoretical results are illustrated numerically in case of NDYAG laser thin rod single crystal and InSb laser single crystal growth by EFG method.

5. Comments and Conclusions
The main novelty in this article consists in the obtained inequalities. These represent limits for what can and cannot be achieved. Experimentally, only stable static liquid bridges can be created if they exist theoretically. Unstable static liquid bridges could exist just in theory; in reality, they collapse; therefore, they are not appropriate for crystal growth.

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