PAST KNOWLEDGE PEOPLE

Engineering : Open Access ISSN: 2993-8643

Research Article,

Engineering: Open Access

Estimation of the Generalized Weibull Distribution Parameters based on the Kernel and Bayes Methods with Real Data Applications

M. Maswadah*

Department of Mathematics, Faculty of Science Aswan University Aswan, Egypt

***Corresponding Author**
M. Maswadah, Department of Mathematics, Faculty of Science Aswan University Aswan, Egypt.

Submitted: 2025, Dec 31; **Accepted**: 2026, Mar 23; **Published**: 2026, Apr 06

Citation: Maswadah, M. (2026). Estimation of the Generalized Weibull Distribution Parameters based on the Kernel and Bayes Methods with Real Data Applications. *Eng OA, 4*(4), 01-20.

Abstract

In this work, a new estimation method using the kernel iteration technique based on the kernel estimation function has been studied. This estimation method is presented as a new tool for estimation in statistical inference that has been applied for estimating the generalized Weibull distribution parameters. The generalized Weibull model parameter estimations were derived using the kernel and Bayes methods based on the generalized progressive hybrid censoring scheme via a Monte Carlo simulation. The simulation results indicated that the kernel estimation method is highly efficient and outperforms the Bayes estimation method based on the informative gamma and kernel priors using two different loss function Finally, two real data sets were studied to ensure the kernel estimation method can be used more effectively than the most popular estimation methods in fitting and analyzing real lifetime data.

Keywords: Bayesian Inference, Informative Prior, Kernel Inference, Kernel Prior

1. Introduction

In statistical inference, the Bayesian estimation method is the most popular method widely used in the social sciences and psychology, although it is subject to prior information other than data. This information can lead to undesirable inferences. Therefore, the main objective of this study is to introduce an improvement estimation method by using the kernel density function. The simulation results indicated that the improved estimation method is highly efficient and outperforms the Bayes method based on the informative gamma and kernel priors using, the squared error loss function and the Linex loss function. Thus, the statistical significance of this method is that it is more efficient than the most popular estimation methods in statistical inference and it is reliable and easy to apply, especially for social sciences and psychology researchers.

In recent years, an extensive effort has been made to present new models in distribution theory and related statistical applications. Some of the new distributions were developed as generalizations or modifications of the Weibull distribution that have been used extensively for data modeling in many fields, such as engineering and medical science [1,2]. For a review of some generalized Weibull distributions, one may refer to developed a new class of distributions called the T-X family, given a random variable **X** with a cumulative distribution function **G(x)**, and a generator random variable T defined on $[0,\infty)$ with **h(t)** and **H(t)** be the probability density function (PDF) and the cumulative distribution function (CDF), respectively. Thus, the CDF of the generalized T-X family is given by: $F(x)=H(-\log(1-G(x)))$.

This family extends some lifetime distributions, such as the generalized Weibull, Weibull, Weibull extension, Lomax, logistic, and log-logistic distributions. Several new distributions within this family, including the Weibull-Pareto distribution, have been introduced and studied in, the Gamma-Pareto distribution in and the Gamma-Normal distribution in. For a review of this family and other distributions, see [3-7].

For deriving the generalized Weibull distribution, let $\mathbf{h(t)}$ be the PDF of the Weibull distribution, which is defined as follows:

$$\mathbf{h(t) = \alpha\beta t^{\alpha-1} \exp[-\beta t^{\alpha}]}, \qquad \mathbf{t \geq 0}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} > 0.$$

Then, we have the Weibull (T-X family) with CDF defined as follows:

$$\mathbf{F(x) = 1 - \exp[-\beta(-\log(1 - G(x)))^{\alpha}]}.$$

Letting $\mathbf{G(x) = 1 - \exp(-(e^{\gamma x} - 1))}$, be the CDF of the exponential extension model.
Thus, the CDF of the generalized Weibull distribution (GWD) can be derived as follows:

$$\mathbf{F(x) = 1 - \exp[-\beta(e^{x\gamma} - 1)^{\alpha}]}, \qquad \boldsymbol{x \geq 0}, \ \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} > 0. \tag{1}$$

The corresponding PDF is given as follows:

$$\mathbf{f(x) = \alpha\beta\gamma(e^{x\gamma} - 1)^{\alpha-1} \exp[x\gamma - \beta(e^{x\gamma} - 1)^{\alpha}]}, \quad \mathbf{x \geq 0, \alpha, \beta, \gamma} > 0 \tag{2}$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are scale parameters, $\boldsymbol{\alpha}$ is a shape parameter.

This distribution has higher skewness compared with the Weibull, inverse Weibull, and the log-logistic distributions, and therefore it is more suitable to model heavily skewed data that often arise in reliability and survival analysis, see [3,8]. The GWD has some special cases, when the random variable $\mathbf{Y=e^{x\gamma}-1}$, we get $\mathbf{Y}$ has the two-parameter Weibull distribution, with $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are the scale and shape parameters, respectively. When $\boldsymbol{\alpha}$**=1,** we get the exponential extension model, which is a special case from the Weibull extension model, see [9]. For comparison between the two proposed estimation methods, the generalized Weibull distribution parameters have been estimated using the kernel and Bayes estimation methods based on the informative gamma and the informative kernel priors with different loss functions based on the generalized progressive hybrid censoring scheme.

In reliability analysis, the progressive Type-II censoring scheme is the most applicable in life test experiments. It is useful for both industrial life test applications and clinical trials and allows removing some of the surviving experimental units at various stages before testing is terminated. However, the trial time can be quite long due to some highly reliable units. Thus, recently proposed a censoring scheme, which is the Type-II progressive hybrid censoring scheme. However, the disadvantage of this scheme is very few failures may occur before the time point $\mathbf{T}$. In order to provide a guarantee to the number of failures observed as well as the time to complete the test, proposed the generalized progressive hybrid censoring scheme (GPHCS) that modifies the progressive hybrid censoring scheme [10-12]. It allows the experiment to continue beyond time $\mathbf{T}$ to observe at least $\mathbf{k}$ failures, if the number of failures is less than $\mathbf{m}$. The algorithm for generating the GPHCS has been described in [13,14].

Thus, given a generalized progressive hybrid censored sample, the likelihood function can be written in a unified form as follows:

$$\mathbf{L(\bar{X}\,;\theta) = C\prod_{i=1}^{n} f(x_i)\,[1 - F(x_i)]^{R_i}[1 - F(T)]^{\delta R_T^*}}, \tag{3}$$

$$\mathbf{n} = \begin{cases} \mathbf{m}, \quad \boldsymbol{\delta} = \mathbf{0} \quad , \quad \mathbf{if} \quad \mathbf{X_{k:m:n} < X_{m:m:n} < T} \\ \mathbf{k}, \quad \boldsymbol{\delta} = \mathbf{0} \quad , \quad \mathbf{if} \quad \mathbf{T < X_{K:m:n} < X_{m:m:n}}, \\ \mathbf{J}, \quad \boldsymbol{\delta} = \mathbf{1} \quad , \quad \mathbf{if} \quad \mathbf{X_{k:m:n} < T < X_{m:m:n}} \end{cases}$$

where $\underline{\boldsymbol{X}} = (\mathbf{X_1, X_2, \ldots, X_n})$ and $\mathbf{R_T}^*$ the number of surviving units that are removed at the stopping time

$\mathbf{T}^* = \mathbf{\max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}}$.

The GPHCS has been applied for some distributions such as the Weibull distribution, see, the inverse Weibull distribution, see the exponential distribution, see and the Rayleigh distribution, see the shape-scale family, see the generalized Weibull distribution, see and the Burr-XII distribution, see [15-19].

2. Estimation Method

2.1 Kernel Method

We propose a simple and tractable algorithm for estimating the distribution parameters based on the kernel density estimate as the following:

The kernel estimate for the function **g(α,β,γ)** can be derived by using the trivariate kernel density estimator for the unknown probability density function with support on [**0**,∞), which is defined as follows:

$$\hat{g}(\alpha,\beta,\gamma) = \frac{1}{nh_1h_2h_3}\sum_{i=1}^{n} K\left(\frac{\alpha-\hat{\alpha}_i}{h_1},\frac{\beta-\hat{\beta}_i}{h_2},\frac{\gamma-\hat{\gamma}_i}{h_3}\right), \quad (4)$$

h_i, $i=1,2,3$ are called the bandwidths or smoothing parameters, which chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n\to\infty$, where n is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i which minimizes the mean squared errors is $h_i=1.06S_i\, n^{-0.2}$, S_i the sample standard deviations. The optimal choice for the kernel function $K(.,.)$can be used as the trivariate standard normal distribution for the parameters α , β, and γ.

1. Generate a random sample $X=(x_1,x_2,\dots,xn)$ from the generalized Weibull distribution.
2. Bootstrapping with replacement n samples from the random sample in (1) as follows: $X_1=(X_{11},\dots,X_{1n})$, $X_2=(X_{21},\dots,X_{2n})$, …, $X_n=(X_{n1},\dots,X_{nn})$.
3. For each sample in step 2, find the MLEs for the parameters α , β, and γ. Thus, we get the following random variables: $A=(\hat{\alpha}_1,\hat{\alpha}_2,\dots,\hat{\alpha}_n)$, $B=(\hat{\beta}_1,\hat{\beta}_2,\dots,\hat{\beta}_n)$, and $C=(\hat{\gamma}_1,\hat{\gamma}_2,\dots,\hat{\gamma}_n)$.

Integrate the density function (4) with respect to θ from θ_0 to θ_1 for $\theta=(\alpha,\beta,\gamma)$ as follows:

$$\int_{\theta_0}^{\theta_1} g(\theta)d\theta = \frac{1}{nh}\sum_{i=1}^{n}\int_{\theta_0}^{\theta_1} K\left(\frac{\theta-\theta_i}{h}\right)d\theta$$

$$\text{Thus, } G(\theta_1)-G(\theta_0) = \frac{1}{nh}\sum_{i=1}^{n}\int_{\theta_0}^{\theta_1} K\left(\frac{\theta-\theta_i}{h}\right)d\theta$$

$$G(\theta_1)-G(\theta_0) = [\gamma - \frac{1}{nh}\sum_{i=1}^{n}\int_{0}^{\theta_0} K\left(\frac{\theta-\theta_i}{h}\right)d\theta]$$

$$= [n\gamma - \sum_{i=1}^{n}\int_{\frac{-\theta_i}{h}}^{\frac{\theta_0-\theta_i}{h}} K(y)\,dy]/n$$

$$\text{where } \gamma \in U(0,1).\ \text{Let } W(\frac{-\theta_i}{h},\frac{\theta_0-\theta_i}{h}) = \int_{\frac{-\theta_i}{h}}^{\frac{\theta_0-\theta_i}{h}} K(y)\,dy.$$

Thus,

$$G(\theta_1)-G(\theta_0) = \left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h},\frac{\theta_0-\theta_i}{h}\right)\right]/n. \quad (5)$$

It is known that the second order approximation of the first derivative, which is defined by the centered differencing, can be written as follows:

$$\frac{dG(\theta_m)}{d\theta} = \frac{G(\theta_1)-G(\theta_0)}{\theta_1-\theta_0} = g(\theta_m), \quad \text{for} \quad \theta_0 < \theta_m < \theta_1. \quad (6)$$

From (5) and (6) we get the integral equation

$$\tilde{\theta}_1 = \tilde{\theta}_0 + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h}, \frac{\theta_0 - \theta_i}{h}\right)\right],$$

Thus, the iterative process for the kernel method can be derived as follows:

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h}, \frac{\theta_n - \theta_i}{h}\right)\right], \quad \text{for} \quad n = 1,2,3,\ldots \quad (7)$$

where $0 < C \leq \frac{2h}{nL_1}$, **and** $L_1 = K(0)$.

The convergence of (7) is continued until two consecutive numerical solutions are almost the same, that is if $\left|\hat{\theta}_{n+1} - \hat{\theta}_n\right| < 10^{-5}$. .

2.2 Bayesian Method

In this section, the Bayes estimations will be derived using the informative gamma and kernel prior distributions based on two loss functions:

Firstly, the squared error loss function (SLF), **$L(\theta,\theta^*)=(\theta-\theta^*)^2$.** For this loss function, the Bayes estimator that minimizes the risk function is given by **$\theta^*=E(\theta|x)$**.

Secondly, the compound LINEX loss function, which is defined as follows:

$$L(\Delta) = L_\delta(\Delta) + L_{-\delta}(\Delta) = e^{\delta\Delta} + e^{\delta\Delta} - 2, \qquad \delta > 0$$

It is named LINEX-based loss function, see [31], where

$$L_\delta(\Delta) = \exp[\delta\Delta] - \delta\Delta - 1, \qquad \Delta = \theta^* - \theta, \quad \delta \neq 0.$$

is the LINEX loss function (LLF) that has been introduced in [22] and [24]. The Bayes estimator of the parameter θ that minimizes the risk function can be derived as follows:

$$\theta_L^* = \frac{1}{2\delta}\ln\left[\frac{E(e^{\delta\theta}|x)}{E(e^{-\delta\theta}|x)}\right].$$

• The Informative gamma prior

We consider the unknown parameters *α, β* and *γ* have independent gamma prior distributions with the joint probability density function, which is given by:

$$\mathbf{h}(\alpha, \beta, \gamma) \propto \alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha-d\beta-f\gamma}, \qquad (8)$$

where the hyper-parameter **a,b,c,d,e** and **f** are assumed to be known and positive and chosen to reflect the prior belief about the unknown parameters.

• The Informative Kernel Prior

For deriving the kernel prior, we introduce the trivariate kernel density estimator for the unknown probability density function ***g(α,β,γ)*** with support on **(0,∞)**, which is defined as

$$\hat{\mathbf{g}}(\alpha, \beta, \gamma) = \frac{1}{nh_1h_2h_3}\sum_{i=1}^{n} K\left(\frac{\alpha - \hat{\alpha}_i}{h_1}, \frac{\beta - \hat{\beta}_i}{h_2}, \frac{\gamma - \hat{\gamma}_i}{h_3}\right), \qquad (9)$$

h_i,i=1,2,3 are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$, where n is the sample size. The influence of the smoothing parameter **h** is critical because it determines the amount of smoothing. However, the optimal choice for h_i, which minimizes the mean squared errors, is given by $h_i = 1.06 S_i\, n^{-0.2}$, and S_i is the sample standard deviation. The optimal choice for the kernel function **K(.,.)** can be used as the trivariate standard normal distribution for the parameters ***α, β,*** and *γ*. Based on the properties of the MLEs of the parameters, which are converging in probability to the original parameters, the algorithm for deriving the kernel prior can be found in [13,14].

Thus, using the joint prior of (9) and (10) with the likelihood function of the GPHCS (3), the posterior density for the parameters α, β, and γ can be written in a unified form as follows:

$$f(\alpha,\beta,\gamma|\underline{x}) = K l(\alpha,\beta,\gamma) L(\bar{X};\theta), \text{ where}$$

$$\boldsymbol{l}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \mathbf{h}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\hat{\mathbf{g}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$$

$$= \hat{\mathbf{g}}_1^{p_1}(\boldsymbol{\alpha})\,\hat{\mathbf{g}}_2^{p_2}(\boldsymbol{\beta})\,\hat{\mathbf{g}}_3^{p_3}(\boldsymbol{\gamma})\boldsymbol{\alpha}^{a-1}\boldsymbol{\beta}^{c-1}\boldsymbol{\gamma}^{e-1}e^{-b\alpha-d\beta-f\gamma}$$

is the general prior distribution function with $p_1=p_2=p_3=0$ for the informative prior (8), and $p_1=p_2=p_3=1$, **a=c=e=1,** and **b=d=f=0** for the kernel prior (9).

Thus, the posterior density function can be written as follows:

$$\mathbf{f}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}|\underline{\mathbf{x}}) = \mathbf{K}\,\hat{\mathbf{g}}_1^{p_1}(\boldsymbol{\alpha})\,\hat{\mathbf{g}}_2^{p_2}(\boldsymbol{\beta})\,\hat{\mathbf{g}}_3^{p_3}(\boldsymbol{\gamma})\,\boldsymbol{\alpha}^{D+a-1}\,\boldsymbol{\beta}^{D+c-1}\,\boldsymbol{\gamma}^{D+e-1}$$

$$\times \exp\Big(-\boldsymbol{\beta}\big[\mathbf{d} + \textstyle\sum_{i=1}^{n}(\mathbf{R}_i + 1)\,(e^{x_i\gamma} - 1)^{\alpha} + \boldsymbol{\delta}\mathbf{R}_T^*(e^{T\gamma} - 1)^{\alpha}\big]$$

$$-\boldsymbol{\gamma}(\mathbf{f} - \textstyle\sum_{i=1}^{n} x_i) - \boldsymbol{\alpha}\mathbf{b} + (\boldsymbol{\alpha} - 1)\textstyle\sum_{i=1}^{n}\ln(e^{x_i\gamma} - 1)\Big]. \tag{10}$$

Thus, based on (10) we can use the Tierney and Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. Iintroduced an easily computable approximation for the posterior mean and variance of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space [15]. For detail, let **q(α,β,γ)** be a smooth, positive function on the parameter space. The posterior expectation of **q(α,β,γ)** can be obtained as

$$\mathbf{q}^* = \mathbf{E}(\mathbf{q}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})|\mathbf{x}) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty e^{nH^*(\alpha,\beta,\gamma)}d\alpha d\beta d\gamma}{\int_0^\infty\int_0^\infty\int_0^\infty e^{nH(\alpha,\beta,\gamma)}d\alpha d\beta d\gamma}, \tag{11}$$

where $\mathbf{H} = \ln \mathbf{f}\,(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}|\mathbf{x})/\mathbf{n}$, and $\mathbf{H}^* = \mathbf{H} + \ln \mathbf{q}\,(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})/\mathbf{n}$.

For **(α,β,γ)** the Bayes estimator using Tierney and Kadane approximation for **q(α,β,γ)** can be obtained as

$$\mathbf{q}^* = \sqrt{|\textstyle\sum^*|/|\sum|}\,\exp[\,\mathbf{n}[\mathbf{H}^*(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) - \mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})],$$

where $(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$ and $(\hat{\boldsymbol{\alpha}}^*,\hat{\boldsymbol{\beta}}^*,\hat{\boldsymbol{\gamma}}^*)$ maximize the $\mathbf{H}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\gamma}})$ and $\mathbf{H}^*(\hat{\boldsymbol{\alpha}}^*,\hat{\boldsymbol{\beta}}^*,\hat{\boldsymbol{\gamma}}^*)$, respectively

$$|\Sigma| = \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}^{-1}, \text{ and } |\Sigma*| = \begin{vmatrix} H_{11}^* & H_{12}^* & H_{13}^* \\ H_{21}^* & H_{22}^* & H_{23}^* \\ H_{31}^* & H_{32}^* & H_{33}^* \end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha,\beta,\gamma)$ and $H^*(\alpha,\beta,\gamma)$ at $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ respectively.

Using (11) we can define H=H(α,β,γ) as follows:

$$\mathbf{H}(\alpha,\beta,\gamma|\underline{x}) = [p_1\ln\hat{g}_1(\alpha) + p_2\ln\hat{g}_2(\beta) + p_3\ln\hat{g}_3(\gamma) + (n+a-1)\ln\alpha + (n+c-1)\ln\beta$$

$$+(n+e-1)\ln\gamma + (n+a-1)\ln\alpha + (n+c-1)\ln\beta + (n+e-1)\ln\gamma$$

$$-\beta[d + \sum_{i=1}^{n}(R_i+1)(e^{x_i\gamma}-1)^{\alpha} + \delta R_T^*(e^{T\gamma}-1)^{\alpha}] + \gamma\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\ln(e^{x_i\gamma}-1). \quad (12)$$

The derivatives of $H(\alpha,\beta,\gamma)$ and $\mathbf{H}^*(\alpha,\beta,\gamma)$ have been derived in the Appendix A.

3. Simulation Study

The purpose of the simulation study is to compare the performance of the estimates using the kernel and Bayes methods based on the informative gamma and the informative kernel priors with two different loss functions, through two criteria the average bias (AVB) and the mean squared error (MSE) as given by:

$$\mathbf{AVB} = \frac{1}{L}\sum_{i=1}^{L}|\hat{\theta}_i - \theta| \quad \text{and} \quad \mathbf{MSE} = \sum_{i=1}^{L}\left(\hat{\theta}_i - \theta\right)^2/L,$$

$\hat{\theta}$ is the estimate of θ and **L** is the number of replications.

In the simulation study we choose different combinations for the hyperparameters of **α** and **β** say: **a=(2,4), b=(8,7), c=(3,5), d=(8,10), e=(4,7)** and **f=(8,9).** Thus, we can generate from the gamma distribution two values for the parameter **α=(0.59,1.11),** two values for the parameter **β=(0.79,0.92),** and two values for the parameter **γ=(0.97,1.32)** respectively. Using the above parameter values for generating different samples from the generalized Weibull distribution with sizes n= 20, 40, and 60 to represent small, moderate, and large sizes. To assess the performance of these estimates, the average Bias (AVB) and the MSE for each were calculated using 1000 replicates.

The algorithm for generating the generalized progressive hybrid censoring scheme has been derived, see [16,17].

From the simulation results in Tables, some of the points are quite clear based on these estimates, and the others have been summarized in the following main points:

1. Generally, for both parameters β and γ , the AVB and MSE values based on the kernel method are less than the comparable values based on the Bayes' method for the different loss functions. However, for the parameter α, the kernel and Bayes methods have almost the same AVB and MSE values, especially based on the LINEX loss function.
2. It is evident that the estimated AVB and MSE values decrease with increasing the hyperparameters, the termination time of the experiment T, and the sample sizes as expected for all methods.
3. For the parameter α, the estimated MSE values increase with increasing the value of α, while decreasing as the value of β and γ increase.
4. For the parameters β and γ , the estimated MSE values increase with increasing the values of β and γ, while decreasing as the value of α increases .
5. In general, the estimated MSE values for the Bayes method based on the LINEX loss function are more efficient than those based on the squared error loss function.
6. In conclusion, the kernel estimates outperform the Bayesian estimation method based on the informative gamma and kernel priors.

4. Real Data Application

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the generalized Weibull model, which is suitable for fitting several types of data and can be adapted to fit the data set with the monotone hazard rate function. It also demonstrates that the GW distribution can be used in many applications in reliability engineering and in new fields such as biomedical sciences and survival analysis to describe the age of the specific mortality and failure rates. Hence, we have fitted these datasets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Chi-square (CH2) tests for a significance level equals to 0.05.

4.1 Vinyl Chloride Dataset

Since vinyl chloride is a known human carcinogen, exposure to this compound should be avoided to the maximum practicable extent, and levels should be kept as low as technically possible. Whereas, it is known that the concentration of vinyl chloride in drinking water of 0.5 mg/liter was calculated to be associated with an increased risk of liver and brain tumors for exposure starting from adulthood and would double the risk of developing cancer from continuous exposure from birth. Therefore, we consider the dataset used by, which represents 34 data points in mg/L of vinyl chloride obtained from clean upgrade monitoring wells, as follows [18]:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3,
3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

We found that the generalized Weibull model is a good fit for this dataset, as shown in Table 2 and Figure (1 a). To study the concentration of vinyl chloride in the water for these wells based on this dataset, we estimated the scale and shape parameters of the concentration to determine the average concentration in the water. We observed that the kernel and Bayes estimates for α are very close to one, indicating that this dataset is right-skewed, and the concentration decreases with increasing time, see Figure (1 b). Also, the kernel and Bayes estimates for **β** and **γ** are close to 2, ensuring that the dataset is right-skewed, and that the vinyl chloride concentration will decrease with increasing time, so monitoring these wells is very significant.

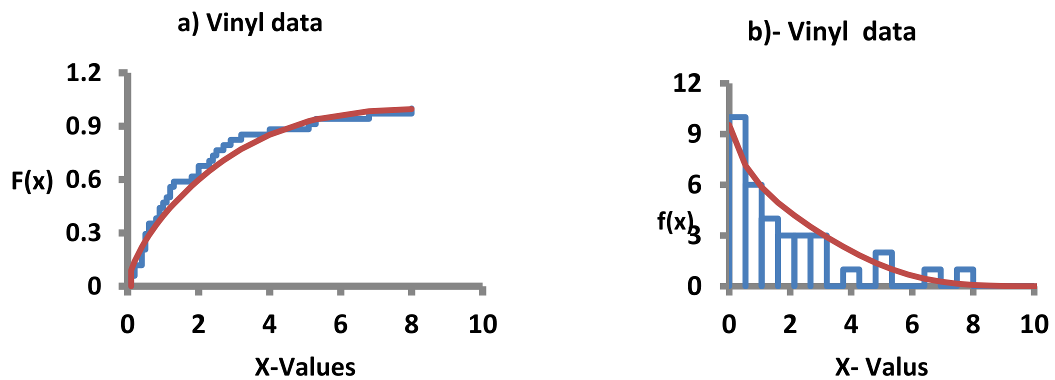

Figure 1: a) The Empirical CDF and the Fitted CDF. b) The Histogram and the Fitted PDF.

4.2 Leukemia Dataset

In healthcare, leukemia affects blood status and can be detected with a blood cell counter (CBC). Mostly, leukemia patients undergo chemotherapy. Therefore, we study the effect of this treatment on leukemia patients based on a dataset collected by the Ministry of Health Hospital in Saudi Arabia and used in which indicates the lifetimes in days for forty-three blood patients with leukemia after chemotherapy as follows [19,20]:

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1025, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852, 1899, 1925, 1965.

We found that the generalized Weibull model is the best fitting for this dataset as shown in Figure (2 a). To study the effect of chemotherapy on patients based on this dataset, we found the estimates of the distribution parameters, which represent the scale and shape of the lifetime. We observed that the estimates of the kernel and Bayes methods for **β** and **γ** are close to 2, while for **α** is less than one in most cases, which indicates a decreasing hazard rate, and the graph is approximately symmetric, see Figure (2 b). Thus, the parameter estimates indicate the decreasing hazard rate for cancer, and that means the longer the patient survives, the more likely they are to reach the upper limit of their natural lifespan. So overall, this dataset indicates that the patient's lifespan is more stable and lives longer due to the chemotherapy dose, and it is highly effective in giving patients more antibodies against cancer.

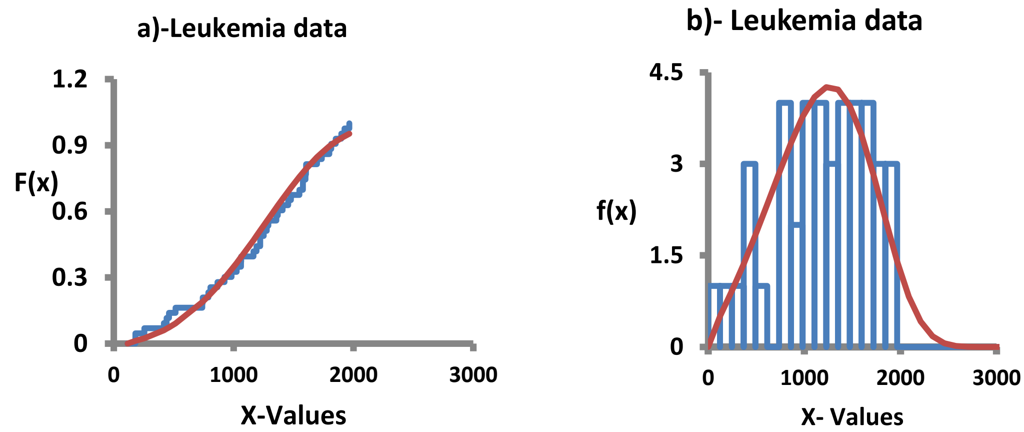

Figure 2: a) The Empirical CDF and the Fitted CDF. b) The Histogram and the Fitted PDF.

Data	The Tests	Critical values	Calculated values	p-values	MLES		
					α	β	γ
The vinyl Chloride data N=34	K-S	0.8579	0.7349	0.2389	0.6849	0.9507	0.3322
	A-D	0.7464	0.7170	0.1583			
	CH2	8.6745	4.2347	0.3516			
The Leukemia data N=43	K-S	0.8666	0.5402	0.6512	1.6882	0.27003	8.4E-04
	A-D	0.7462	0.4421	0.3044			
	CH2	8.6691	2.1013	0.6983			

Table 1: The Critical and the Calculated Values for the K-S, A-D, and CH2 Tests and their Powers (p-values).

The MLE for the parameters for these data sets have been calculated.

Using gamma and kernel priors under the squared error loss function based on the GPHCS

for *m*=*n*/2,k=m/2 . The hyperparameters are: ***a*=*e*=2,*b*=*d*=*f*=4,*c*=8..**

Samples	T	Par.	MLE	Kernel estimate		Gamma Prior		Kernel Prior	
				Estimate	MSE	Estimate	MSE	Estimate	MSE
The vinyl Chloride data N=34	0.75	α	0.8101	0.6748	0.018316	0.6513	0.02523	0.6504	0.02549
		β	0.2497	0.2609	0.00124	0.1936	0.00315	0,1889	0.00369
		γ	0.1908	0.1557	0.001228	0.1526	0.00146	0.1526	0.00368
	3.5	α	0.6053	0.5607	0.00198	0.4855	0.01434	0.4854	0.01437
		β	0.3314	0.3358	0.000194	0.2596	0.00515	0.2575	0.00546
		γ	0.0689	0.0589	0.00219	0.0551	0.00189	0.0551	0.00189
The Leukemia Data N=43	50	α	02834	0.3467	0.00401	0.2400	0.01881	0.2412	0.01773
		β	5.5434	5.2660	0.0769	4.9662	0.3331	5.0259	0.26782
		γ	0.2531	0.3731	0.01440	0.2132	0.00159	0.2135	0.00156
	85	α	0.3675	0.3568	0.00052	0.3112	0.03174	0.3106	0.03248
		β	8.7155	8.4562	0.06723	7.6969	0.10374	7.7927	0.8516
		γ	0.3473	0.3862	0.00833	0.3025	0.03005	0.30187	0.03077

Table 2:The Estimate and the MSEs for the Parameters α, β, and γ based on Kernel and Bayes Methods

From the results in Table 1, the GWD is a good fit for these datasets, as the power of the tests (p-values) is greater than the significance level of the tests, and the calculated values are less than the critical values. Figure (1 a) and Figure (2 a) display the empirical CDF and the CDF of the GWD distribution for these data sets, which confirm the goodness-of-fit tests. The results in Table 2 indicate that the kernel estimates of the parameters have AVB and MSE values lower than the Bayesian estimates based on the informative gamma prior and almost close to the ones based on the informative kernel prior. For both datasets, the MSE values for the parameters α, β and γ decrease as the T values increase.

5. Conclusions

In this study, we applied the kernel and Bayesian estimation methods for estimating the generalized Weibull distribution parameters as a new lifetime distribution. The simulation results indicated that the average bias and MSEs of the parameters based on the kernel method are more efficient than the Bayesian method based on the informative gamma and kernel priors using two different loss functions. However, Bayes estimates based on the informative kernel prior are more efficient than their counterparts based on the informative gamma prior and are close to those based on the kernel estimates. Thus, the statistical significance of the kernel method is its efficiency compared to the most popular estimation methods. It is a viable estimation method for any lifetime model and is reliable and easy to apply especially for medical, biological, social sciences, psychology, and engineering researchers [20-24].

Disclosure statement

No potential conflict of interest was reported by the author.

References

1. Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron, 71*(1), 63-79.
2. Aljarrah, M. A., Lee, C., & Famoye, F. (2014). On generating TX family of distributions using quantile functions. Journal of Statistical Distributions and Applications, 1(1), 2.
3. Almheidat, M., Lee, C., & Famoye, F. (2016). A generalization of the Weibull distribution with applications. Journal of Modern Applied Statistical Methods, 15(2), 47.
4. Alzaatreh A, Famoye F, Lee C. Gamma-Pareto distribution and its applications. Journal of Modern Applied Statistical Methods. 2012; 11, issue 1:78-94.
5. Alzaatreh, A., Famoye, F., & Lee, C. (2013). Weibull-Pareto distribution and its applications. Communications in Statistics-Theory and Methods, 42(9), 1673-1691.
6. Alzaatreh, A., Famoye, F., & Lee, C. (2014). The gamma-normal distribution: Properties and applications. Computational Statistics & Data Analysis, 69, 67-80.
7. Cho, Y., Sun, H., & Lee, K. (2014). An estimation of the entropy for a Rayleigh distribution based on doubly-generalized Type-II hybrid censored samples. *Entropy, 16*(7), 3655-3669.
8. Cho, Y., Sun, H., & Lee, K. (2015). Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme. Statistical Methodology, 23, 18-34.
9. Maswadah, M. (2021). An Optimal Point Estimation Method for the Inverse Weibull Model Parameters Using the Runge-Kutta Method. Aligarh Journal of Statistics, 41.
10. Ahsanullah, M., Maswadah, M., & Ali, S. M. (2013). Kernel inference on the generalized gamma distribution based on generalized order statistics. Journal of Statistical Theory and Applications, 12(2), 152-172.
11. Cho, Y., Sun, H., & Lee, K. (2015). Estimating the entropy of a Weibull distribution under generalized progressive hybrid censoring. Entropy, 17(1), 102-122.
12. Maswadah, M. (2022). Improved maximum likelihood estimation of the shape-scale family based on the generalized progressive hybrid censoring scheme. Journal of Applied Statistics, 49(11), 2825-2844.
13. Tierney, L., & Kadane, J. B. (1986). Accurate approximations for posterior moments and marginal densities. Journal of the american statistical association, 81(393), 82-86.
14. Xie, M., Tang, Y., & Goh, T. N. (2002). A modified Weibull extension with bathtub-shaped failure rate function. Reliability Engineering & System Safety, 76(3), 279-285.
15. Maswadah, M. Numerical estimation method for the generalized Weibull distribution parameters. International Journal of Applied Mathematics, Computational science and system engineering (WSEAS). Vol.5, No. 1, 2023 a, pp.1-17.
16. Maswadah, M. Numerical estimation for the three-parameter Burr-XII model parameters based on Picard's method. Pak. J. Statist. Vol. 40, No. 4, 2023 b, pp.335-357.
17. Maswadah, M. (2006). Kernel inference on the inverse Weibull distribution. Communications for Statistical Applications and Methods, 13(3), 503-512.
18. Li, X., Shi, Y., Wei, J., & Chai, J. (2007). Empirical Bayes estimators of reliability performances using LINEX loss under progressively type-II censored samples. *Mathematics and Computers in Simulation, 73*(5), 320-326.
19. Bhaumik, D. K., Kapur, K., & Gibbons, R. D. (2009). Testing parameters of a gamma distribution for small samples. Technometrics,

51(3), 326-334.
20. Abd-Elrahman, A. M., & Sultan, K. S. (2007). Reliability estimation based on general progressive censored data from theWeibull model: comparison between Bayesian and classical approaches. METRON-International Journal of Statistics, 65(2), 239-257.
21. Varian HRA. Bayesian approach to real estimate assessment. In studies in Bayesian Econometrics Studies in Honor of Leonard J. Savage, North Holland; Amesterdam:1975; 195-208.
22. Górny, J., & Cramer, E. (2016). Exact likelihood inference for exponential distributions under generalized progressive hybrid censoring schemes. Statistical Methodology, 29, 70-94.
23. Kundu, D., & Joarder, A. (2006). Analysis of Type-II progressively hybrid censored data. Computational Statistics & Data Analysis, 50(10), 2509-2528.
24. Wei, C. D., Wei, S., & Su, H. (2010). Bayes estimation and application of Poisson distribution parameter under compound LINEX symmetric loss [J]. Statistics and Decision, 7, 156-157.

Appendix A:

From (12) we can derive the derivatives in (11) as follows:

$$H(\alpha,\beta,\gamma|\underline{x}) = [p_1 ln\hat{g}_1(\alpha) + p_2 ln\hat{g}_2(\beta) + p_3 ln\hat{g}_3(\gamma)$$

$$+(n+a-1)\ln\alpha + (n+c-1)\ln\beta + (n+e-1)\ln\gamma$$
$$-\beta[d + \sum_{i=1}^{n}(R_i+1)(e^{x_i\gamma}-1)^\alpha + \delta R_T^*(e^{T\gamma}-1)^\alpha]$$

$$+\gamma\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\ln(e^{x_i\gamma}-1)$$

$$H_1 = \frac{\partial H}{\partial \alpha} = [p_1\frac{\hat{g}_1^{/}(\alpha)}{\hat{g}_1(\alpha)} + (n+a-1)/\alpha + \sum_{i=1}^{n}\ln(e^{x_i\gamma}-1)$$

$$-\beta[\sum_{i=1}^{n}(1+R_i))^\alpha ln(e^{x_i\gamma}-1) + \delta R_T^*(e^{T\gamma}-1)^\alpha ln(e^{T\gamma}-1)]]/n$$

$$H_{12} = \frac{\partial^2 H}{\partial\alpha\partial\beta} = -[\sum_{i=1}^{n}(1+R_i)(e^{x_i\gamma}-1)^\alpha ln(e^{x_i\gamma}-1)$$

$$+\delta R_T^*(e^{T\gamma}-1)^\alpha ln(e^{T\gamma}-1)]]/n$$

$$H_{13} = \frac{\partial^2 H}{\partial\alpha\partial\gamma} = [\sum_{i=1}^{n}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$

$$-\beta[\sum_{i=1}^{n}(1+R_i)(e^{x_i\gamma}-1)^\alpha\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)} + \delta R_T^*(e^{T\gamma}-1)^\alpha\frac{Te^{T\gamma}}{(e^{T\gamma}-1)}]$$
$$-\alpha\beta[\sum_{i=1}^{n}(1+R_i)x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1} ln(e^{x_i\gamma}-1)$$
$$+\delta R_T^* T e^{T\gamma}(e^{T\gamma}-1)^{\alpha-1} ln(e^{T\gamma}-1)]]/n]$$

$$H_{11}=\frac{\partial^2 H}{\partial\alpha^2}=[p_1\frac{\hat{g}_1(\alpha)\hat{g}_1^{//}(\alpha)-\hat{g}_1^{/2}(\alpha)}{\hat{g}_1^2(\alpha)}-(n+a-1)/\alpha^2$$

$$-\beta[\sum_{i=1}^{n}(1+R_i)(e^{x_i\gamma}-1)^{\alpha}(ln(e^{x_i\gamma}-1))^2+\delta R_T^*(e^{T\gamma}-1)^{\alpha}(ln(e^{T\gamma}-1))^2]]/n$$

$$H_2=\frac{\partial H}{\partial\beta}=[p_2\frac{\hat{g}_2'(\beta)}{\hat{g}_2(\beta)}+(n+c-1)/\beta-[d+\sum_{i=1}^{n}(R_i+1)(e^{x_i\gamma}-1)^{\alpha}$$

$$+\delta R_T^*(e^{T\gamma}-1)^{\alpha}]]/n$$

$$H_{22}=\frac{\partial^2 H}{\partial\beta^2}=[p_2\frac{\hat{g}_2(\beta)g_2^{//}(\beta)-\hat{g}_2^{/2}(\beta)}{\hat{g}_2{}^2(\beta)}-(n+c-1)/\beta^2]/n.$$

$$H_{12}=\frac{\partial^2 H}{\partial\alpha\partial\beta}=-[\sum_{i=1}^{n}(R_i+1)(e^{x_i\gamma}-1)^{\alpha}\ln(e^{x_i\gamma}-1)$$

$$+\delta R_T^*(e^{T\gamma}-1)^{\alpha}\ln e^{T\gamma}-1)]/n$$

$$H_{23}=\frac{\partial^2 H}{\partial\beta\partial\gamma}=[-\alpha[\sum_{i=1}^{n}(R_i+1)x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}$$

$$+\delta R_T^* T e^{T\gamma}(e^{T\gamma}-1)^{\alpha-1}]]/n$$

$$H_3=\frac{\partial H}{\partial\gamma}=[p_3\frac{\hat{g}_3'(\gamma)}{\hat{g}_3(\gamma)}+(n+e-1)/\gamma+\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$

$$-\alpha\beta[\sum_{i=1}^{n}(R_i+1)x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}]]/n$$

$$H_{31}=\frac{\partial H}{\partial\gamma\partial\alpha}=[\sum_{i=1}^{n}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$

$$-\beta[\sum_{i=1}^{n}(R_i+1)x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}]$$

$$-\alpha\beta[\sum_{i=1}^{n}(R_i+1)x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}ln(e^{x_i\gamma}-1)$$

$$+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}\ln(e^{T\gamma}-1)]]/n,$$

$$H_{32}=\frac{\partial H}{\partial\gamma\partial\beta}=[-\alpha[\sum_{i=1}^{n}(R_i+1)\,x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}$$

$$+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}]]/n. \quad (13)$$

$$H_{33}=\frac{\partial^2 H}{\partial\gamma^2}=[p_3\frac{\hat{g}_3(\gamma)g_3^{//}(\gamma)-\hat{g}_3^{/\,2}(\gamma)}{\hat{g}_3^{\,2}(\gamma)}-(n+e-1)/\gamma^2$$

$$+\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\frac{(e^{x_i\gamma}-1)x_i^2e^{x_i\gamma}-(x_ie^{x_i\gamma})^2}{(e^{x_i\gamma}-1)^2}$$

$$-\alpha\beta[\sum_{i=1}^{n}(R_i+1)\,[x_i^2e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}+(\alpha-1)(x_ie^{x_i\gamma})^2(e^{x_i\gamma}-1)^{\alpha-2}$$

$$+\delta R_T^*[T^2e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}+(\alpha-1)(Te^{T\gamma})^2(e^{T\gamma}-1)^{\alpha-2}]]/n,$$

where the r^{th} derivative of the kernel density estimation can be defined as follows:

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r}=\hat{g}_1{}^r(\alpha)=\frac{1}{nh_1^{r+1}}\sum_{i=1}^{n}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right), \quad (13)$$

where r=0,1,2,3,-----.
Using the Gaussian kernel and (13), we have

$$\hat{g}_1(\alpha)=\frac{1}{n\sqrt{2\pi}}\sum_{i=1}^{n}e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},\ \hat{g}_1'(\alpha)=-\frac{1}{nh_1^2\sqrt{2\pi}}\sum_{i=1}^{n}\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1''(\alpha)=\frac{1}{nh_1^3\sqrt{2\pi}}\sum_{i=1}^{n}[(\frac{\alpha-\alpha_i}{h_1})^2-1]\,e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$ and $\hat{g}_3(\gamma)$.

n	m	k	α	β	γ	Kernel estimations	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.0818(0.0085)	0.1102(0.0133)	0.086(0.0080)	0.1085(0.0128)	0.087(0.0082)
				0.92	1.32	0.0794(0.0081)	0.1011(0.0117)	0.077(0.0068)	0.0989(0.0111)	0.078(0.0069)
			1.11	0.79	0.97	0.2453(0.0633)	0.2423(0.0599)	0.198(0.0397)	0.2357(0.0564)	0.201(0.0408)
				0.92	1.32	0.2415(0.0617)	0.2339(0.0562)	0.191(0.0371)	0.2268(0.0524)	0.194(0.0382)
		8	0.59	0.79	0.97	0.0822(0.0084)	0.1073(0.0123)	0.089(0.0086)	0.1070(0.0122)	0.090(0.0087)
				0.92	1.32	0.0830(0.0086)	0.0984(0.0106)	0.081(0.0072)	0.0981(0.0105)	0.082(0.0074)
			1.11	0.79	0.97	0.2542(0.0673)	0.2317(0.0543)	0.201(0.0407)	0.2295(0.0531)	0.203(0.0417)
				0.92	1.32	0.2526(0.0666)	0.2225(0.0502)	0.193(0.0378)	0.2207(0.0493)	0.196(0.0389)
	15	8	0.59	0.79	0.97	0.0789(0.0079)	0.1074(0.0123)	0.089(0.0086)	0.1071(0.0122)	0.090(0.0088)
				0.92	1.32	0.0817(0.0083)	0.1006(0.0109)	0.083(0.0075)	0.1002(0.0108)	0.084(0.0076)
			1.11	0.79	0.97	0.2511(0.0658)	0.2319(0.0543)	0.202(0.0410)	0.2297(0.0531)	0.204(0.0420)
				0.92	1.32	0.2537(0.0670)	0.2235(0.0508)	0.194(0.0381)	0.2215(0.0497)	0.197(0.0392)
		11	0.59	0.79	0.97	0.0897(0.0095)	0.1096(0.0124)	0.095(0.0094)	0.1097(0.0124)	0.096(0.0096)
				0.92	1.32	0.0892(0.0094)	0.1012(0.0108)	0.087(0.0081)	0.1015(0.0108)	0.088(0.0083)
			1.11	0.79	0.97	0.2526(0.0661)	0.2271(0.0519)	0.204(0.0419)	0.2263(0.0514)	0.206(0.0426)
				0.92	1.32	0.2525(0.0661)	0.2194(0.0485)	0.197(0.0392)	0.2191(0.0483)	0.200(0.0402)
			0.59	0.79	0.97	0.0863(0.0089)	0.0975(0.0101)	0.084(0.0076)	0.0974(0.0101)	0.085(0.0077)
40	20	10		0.92	1.32	0.0795(0.0078)	0.0899(0.0086)	0.077(0.0064)	0.0897(0.0086)	0.078(0.0065)
			1.11	0.79	0.97	0.2496(0.0647)	0.2213(0.0493)	0.199(0.0397)	0.2199(0.0487)	0.201(0.0404)
				0.92	1.32	0.2442(0.0620)	0.2142(0.0462)	0.193(0.0373)	0.2127(0.0455)	0.195(0.0381)
		15	0.59	0.79	0.97	0.0895(0.0092)	0.0965(0.0097)	0.086(0.0078)	0.0968(0.0098)	0.087(0.0080)
				0.92	1.32	0.0865(0.0088)	0.0894(0.0085)	0.080(0.0068)	0.0898(0.0085)	0.080(0.0069)
			1.11	0.79	0.97	0.2529(0.0662)	0.2166(0.0472)	0.200(0.0401)	0.2165(0.0471)	0.202(0.0408)
				0.92	1.32	0.2490(0.0640)	0.2094(0.0442)	0.193(0.0374)	0.2095(0.0442)	0.195(0.0383)
	30	15	0.59	0.79	0.97	0.0873(0.0089)	0.0975(0.0099)	0.087(0.0080)	0.0978(0.0100)	0.088(0.0081)
				0.92	1.32	0.0872(0.0088)	0.0897(0.0085)	0.080(0.0068)	0.0900(0.0086)	0.081(0.0069)
			1.11	0.79	0.97	0.2528(0.0658)	0.2188(0.0481)	0.202(0.0408)	0.2185(0.0479)	0.203(0.0414)
				0.92	1.32	0.2508(0.0649)	0.2096(0.0442)	0.193(0.0376)	0.2097(0.0442)	0.195(0.0384)
		23	0.59	0.79	0.97	0.0919(0.0095)	0.1029(0.0108)	0.096(0.0094)	0.1033(0.0109)	0.096(0.0095)
				0.92	1.32	0.0921(0.0095)	0.0946(0.0092)	0.088(0.0079)	0.0951(0.0093)	0.088(0.0081)
			1.11	0.79	0.97	0.2473(0.0629)	0.2160(0.0468)	0.205(0.0420)	0.2164(0.0469)	0.206(0.0426)
				0.92	1.32	0.2479(0.0632)	0.2089(0.0438)	0.198(0.0393)	0.2097(0.0441)	0.200(0.0399)
60	30	15	0.59	0.79	0.97	0.0893(0.0091)	0.0939(0.0091)	0.085(0.0075)	0.0941(0.0092)	0.086(0.0076)
				0.92	1.32	0.0850(0.0084)	0.0850(0.0076)	0.076(0.0062)	0.0852(0.0076)	0.077(0.0063)
			1.11	0.79	0.97	0.2514(0.0652)	0.2140(0.0460)	0.199(0.0398)	0.2138(0.0459)	0.201(0.0403)
				0.92	1.32	0.2490(0.0638)	0.2062(0.0428)	0.192(0.0369)	0.2061(0.0427)	0.193(0.0375)
		23	0.59	0.79	0.97	0.0931(0.0097)	0.0950(0.0093)	0.088(0.0080)	0.0954(0.0093)	0.089(0.0081)
				0.92	1.32	0.0908(0.0093)	0.0871(0.0079)	0.080(0.0067)	0.0875(0.0079)	0.081(0.0068)
			1.11	0.79	0.97	0.2472(0.0629)	0.2117(0.0450)	0.200(0.0403)	0.2121(0.0451)	0.202(0.0408)
				0.92	1.32	0.2433(0.0610)	0.2047(0.0421)	0.194(0.0376)	0.2053(0.0423)	0.195(0.0383)
	45	23	0.59	0.79	0.97	0.0905(0.0093)	0.0948(0.0092)	0.088(0.0079)	0.0951(0.0093)	0.088(0.0080)
				0.92	1.32	0.0895(0.0090)	0.0863(0.0078)	0.079(0.0066)	0.0867(0.0078)	0.080(0.0067)
			1.11	0.79	0.97	0.2469(0.0627)	0.2116(0.0449)	0.200(0.0403)	0.2120(0.0451)	0.202(0.0408)
				0.92	1.32	0.2455(0.0619)	0.2039(0.0417)	0.193(0.0373)	0.2046(0.0420)	0.195(0.0380)
		34	0.59	0.79	0.97	0.0973(0.0104)	0.0995(0.0100)	0.094(0.0091)	0.0998(0.0101)	0.095(0.0091)
				0.92	1.32	0.0966(0.0102)	0.0924(0.0087)	0.087(0.0078)	0.0929(0.0088)	0.088(0.0079)
			1.11	0.79	0.97	0.2467(0.0623)	0.2122(0.0451)	0.204(0.0418)	0.2127(0.0453)	0.205(0.0423)
				0.92	1.32	0.2473(0.0626)	0.2052(0.0422)	0.198(0.0391)	0.2060(0.0425)	0.199(0.0397)

Table 3: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for parameter α using Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and δ=2 for LINEX loss function.

n	m	k	α	β	γ	Kernel estimations	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.0823(0.0086)	0.1080(0.0124)	0.090(0.0087)	0.1078(0.0123)	0.091(0.0088)
				0.92	1.32	0.0814(0.0084)	0.0978(0.0105)	0.079(0.0070)	0.0971(0.0103)	0.080(0.0072)
			1.11	0.79	0.97	0.2492(0.0650)	0.2337(0.0554)	0.200(0.0404)	0.2306(0.0537)	0.203(0.0414)
				0.92	1.32	0.2450(0.0630)	0.2263(0.0521)	0.194(0.0379)	0.2232(0.0505)	0.197(0.0390)
		8	0.59	0.79	0.97	0.0820(0.0082)	0.1071(0.0121)	0.091(0.0088)	0.1071(0.0121)	0.092(0.0090)
				0.92	1.32	0.0805(0.0080)	0.0990(0.0106)	0.083(0.0075)	0.0989(0.0105)	0.084(0.0077)
			1.11	0.79	0.97	0.2530(0.0666)	0.2290(0.0529)	0.202(0.0411)	0.2275(0.0522)	0.204(0.0420)
				0.92	1.32	0.2517(0.0661)	0.2217(0.0497)	0.195(0.0384)	0.2205(0.0490)	0.198(0.0395)
	15	8	0.59	0.79	0.97	0.0811(0.0081)	0.1066(0.0121)	0.090(0.0087)	0.1066(0.0121)	0.091(0.0088)
				0.92	1.32	0.0823(0.0084)	0.1001(0.0109)	0.083(0.0075)	0.0998(0.0108)	0.084(0.0077)
			1.11	0.79	0.97	0.2543(0.0671)	0.2307(0.0538)	0.201(0.0408)	0.2286(0.0527)	0.204(0.0417)
				0.92	1.32	0.2520(0.0659)	0.2231(0.0505)	0.194(0.0380)	0.2212(0.0495)	0.197(0.0391)
		11	0.59	0.79	0.97	0.0879(0.0091)	0.1075(0.0121)	0.093(0.0092)	0.1077(0.0121)	0.094(0.0093)
				0.92	1.32	0.0880(0.0092)	0.1007(0.0108)	0.087(0.0081)	0.1009(0.0108)	0.088(0.0083)
			1.11	0.79	0.97	0.2541(0.0670)	0.2259(0.0514)	0.204(0.0417)	0.2252(0.0510)	0.206(0.0425)
				0.92	1.32	0.2536(0.0667)	0.2191(0.0484)	0.197(0.0390)	0.2188(0.0482)	0.200(0.0401)
			0.59	0.79	0.97	0.0879(0.0090)	0.0987(0.0101)	0.089(0.0084)	0.0990(0.0102)	0.090(0.0085)
40	20	10		0.92	1.32	0.0870(0.0088)	0.0908(0.0087)	0.081(0.0070)	0.0912(0.0087)	0.082(0.0071)
			1.11	0.79	0.97	0.2458(0.0625)	0.2174(0.0475)	0.202(0.0408)	0.2174(0.0474)	0.203(0.0414)
				0.92	1.32	0.2442(0.0619)	0.2087(0.0439)	0.193(0.0376)	0.2091(0.0440)	0.196(0.0384)
		15	0.59	0.79	0.97	0.0884(0.0090)	0.0994(0.0102)	0.090(0.0084)	0.0997(0.0103)	0.091(0.0085)
				0.92	1.32	0.0869(0.0088)	0.0914(0.0088)	0.082(0.0071)	0.0918(0.0088)	0.083(0.0073)
			1.11	0.79	0.97	0.2521(0.0654)	0.2168(0.0472)	0.202(0.0408)	0.2169(0.0472)	0.203(0.0414)
				0.92	1.32	0.2475(0.0633)	0.2092(0.0440)	0.195(0.0381)	0.2096(0.0442)	0.197(0.0388)
	30	15	0.59	0.79	0.97	0.0839(0.0082)	0.0993(0.0102)	0.090(0.0084)	0.0996(0.0102)	0.091(0.0086)
				0.92	1.32	0.0885(0.0091)	0.0899(0.0085)	0.081(0.0069)	0.0903(0.0086)	0.081(0.0070)
			1.11	0.79	0.97	0.2478(0.0634)	0.2163(0.0470)	0.200(0.0403)	0.2163(0.0470)	0.202(0.0410)
				0.92	1.32	0.2481(0.0635)	0.2091(0.0440)	0.194(0.0377)	0.2094(0.0441)	0.196(0.0385)
		23	0.59	0.79	0.97	0.0914(0.0094)	0.1026(0.0108)	0.095(0.0093)	0.1029(0.0108)	0.096(0.0094)
				0.92	1.32	0.0900(0.0092)	0.0945(0.0092)	0.088(0.0080)	0.0950(0.0093)	0.088(0.0081)
			1.11	0.79	0.97	0.2494(0.0640)	0.2162(0.0469)	0.205(0.0421)	0.2166(0.0470)	0.206(0.0426)
				0.92	1.32	0.2481(0.0633)	0.2089(0.0438)	0.198(0.0393)	0.2096(0.0441)	0.200(0.0400)
60	30	15	0.59	0.79	0.97	0.0925(0.0095)	0.0968(0.0096)	0.091(0.0084)	0.0972(0.0096)	0.091(0.0085)
				0.92	1.32	0.0927(0.0096)	0.0874(0.0079)	0.081(0.0069)	0.0878(0.0080)	0.082(0.0070)
			1.11	0.79	0.97	0.2483(0.0632)	0.2116(0.0449)	0.201(0.0406)	0.2121(0.0451)	0.202(0.0411)
				0.92	1.32	0.2469(0.0626)	0.2041(0.0418)	0.194(0.0377)	0.2048(0.0421)	0.196(0.0384)
		23	0.59	0.79	0.97	0.0911(0.0093)	0.0963(0.0095)	0.090(0.0083)	0.0967(0.0096)	0.091(0.0084)
				0.92	1.32	0.0913(0.0093)	0.0878(0.0080)	0.082(0.0069)	0.0883(0.0081)	0.082(0.0070)
			1.11	0.79	0.97	0.2497(0.0640)	0.2118(0.0450)	0.202(0.0407)	0.2123(0.0452)	0.203(0.0413)
				0.92	1.32	0.2455(0.0620)	0.2039(0.0417)	0.194(0.0378)	0.2047(0.0420)	0.196(0.0384)
	45	23	0.59	0.79	0.97	0.0924(0.0095)	0.0984(0.0099)	0.092(0.0087)	0.0987(0.0099)	0.093(0.0088)
				0.92	1.32	0.0920(0.0094)	0.0881(0.0080)	0.082(0.0070)	0.0886(0.0081)	0.083(0.0071)
			1.11	0.79	0.97	0.2475(0.0629)	0.2124(0.0452)	0.202(0.0411)	0.2129(0.0454)	0.204(0.0416)
				0.92	1.32	0.2455(0.0620)	0.2043(0.0419)	0.195(0.0380)	0.2051(0.0422)	0.196(0.0387)
		34	0.59	0.79	0.97	0.0984(0.0106)	0.1003(0.0102)	0.095(0.0092)	0.1006(0.0103)	0.096(0.0093)
				0.92	1.32	0.0948(0.0099)	0.0927(0.0088)	0.088(0.0079)	0.0931(0.0088)	0.088(0.0080)
			1.11	0.79	0.97	0.2476(0.0628)	0.2121(0.0451)	0.204(0.0418)	0.2127(0.0453)	0.205(0.0423)
				0.92	1.32	0.2428(0.0605)	0.2057(0.0424)	0.198(0.0393)	0.2066(0.0428)	0.199(0.0399)

Table 4: The Average Bias (AVB) and Mean Squared Errors (MSEs) in prarenthses for GWD parameter α using Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and δ=2 for LINEX loss function.

n	m	k	α	β	γ	Kernel estimations	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.0918(0.0102)	0.2177(0.0480)	0.171(0.0291)	0.2095(0.0444)	0.169(0.0286)
				0.92	1.32	0.1230(0.0170)	0.2732(0.0756)	0.210(0.0442)	0.2576(0.0670)	0.206(0.0425)
			1.11	0.79	0.97	0.1020(0.0123)	0.2208(0.0491)	0.166(0.0274)	0.2091(0.0440)	0.165(0.0271)
				0.92	1.32	0.1194(0.0162)	0.2682(0.0724)	0.198(0.0391)	0.2497(0.0626)	0.196(0.0382)
		8	0.59	0.79	0.97	0.0972(0.0111)	0.1858(0.0348)	0.165(0.0272)	0.1830(0.0337)	0.164(0.0270)
				0.92	1.32	0.1103(0.0137)	0.2257(0.0513)	0.198(0.0391)	0.2202(0.0488)	0.196(0.0384)
			1.11	0.79	0.97	0.0947(0.0106)	0.1879(0.0355)	0.163(0.0265)	0.1836(0.0338)	0.162(0.0263)
				0.92	1.32	0.1095(0.0137)	0.2262(0.0514)	0.193(0.0372)	0.2192(0.0482)	0.192(0.0367)
	15	8	0.59	0.79	0.97	0.0955(0.0108)	0.1857(0.0347)	0.165(0.0272)	0.1829(0.0336)	0.164(0.0270)
				0.92	1.32	0.1115(0.0141)	0.2254(0.0511)	0.197(0.0390)	0.2199(0.0486)	0.196(0.0383)
			1.11	0.79	0.97	0.0954(0.0107)	0.1890(0.0359)	0.163(0.0265)	0.1845(0.0342)	0.162(0.0263)
				0.92	1.32	0.1103(0.0137)	0.2261(0.0513)	0.193(0.0372)	0.2191(0.0481)	0.192(0.0367)
		11	0.59	0.79	0.97	0.0905(0.0096)	0.1719(0.0296)	0.163(0.0265)	0.1708(0.0292)	0.162(0.0264)
				0.92	1.32	0.1099(0.0135)	0.2062(0.0427)	0.193(0.0373)	0.2039(0.0417)	0.192(0.0369)
			1.11	0.79	0.97	0.0898(0.0094)	0.1754(0.0308)	0.162(0.0261)	0.1734(0.0301)	0.161(0.0260)
				0.92	1.32	0.1108(0.0137)	0.2094(0.0439)	0.190(0.0363)	0.2060(0.0425)	0.190(0.0360)
40	20	10	0.59	0.79	0.97	0.0896(0.0095)	0.1927(0.0373)	0.170(0.0290)	0.1895(0.0360)	0.169(0.0287)
				0.92	1.32	0.1106(0.0138)	0.2374(0.0565)	0.207(0.0430)	0.2310(0.0535)	0.205(0.0420)
			1.11	0.79	0.97	0.0938(0.0102)	0.1900(0.0362)	0.165(0.0272)	0.1858(0.0346)	0.164(0.0270)
				0.92	1.32	0.1117(0.0140)	0.2303(0.0531)	0.197(0.0386)	0.2232(0.0499)	0.195(0.0381)
		15	0.59	0.79	0.97	0.0928(0.0098)	0.1778(0.0317)	0.166(0.0277)	0.1764(0.0312)	0.166(0.0275)
				0.92	1.32	0.1091(0.0131)	0.2157(0.0466)	0.200(0.0400)	0.2127(0.0453)	0.198(0.0394)
			1.11	0.79	0.97	0.0929(0.0099)	0.1775(0.0315)	0.163(0.0266)	0.1755(0.0308)	0.163(0.0265)
				0.92	1.32	0.1097(0.0133)	0.2124(0.0451)	0.193(0.0374)	0.2089(0.0437)	0.192(0.0371)
	30	15	0.59	0.79	0.97	0.0906(0.0095)	0.1773(0.0315)	0.166(0.0277)	0.1759(0.0310)	0.166(0.0275)
				0.92	1.32	0.1086(0.0130)	0.2156(0.0466)	0.200(0.0400)	0.2126(0.0453)	0.198(0.0394)
			1.11	0.79	0.97	0.0901(0.0093)	0.1770(0.0314)	0.163(0.0266)	0.1750(0.0307)	0.163(0.0265)
				0.92	1.32	0.1084(0.0130)	0.2126(0.0452)	0.193(0.0374)	0.2091(0.0438)	0.192(0.0370)
		23	0.59	0.79	0.97	0.0904(0.0092)	0.1660(0.0276)	0.163(0.0266)	0.1656(0.0274)	0.163(0.0265)
				0.92	1.32	0.1036(0.0118)	0.1983(0.0394)	0.194(0.0376)	0.1973(0.0390)	0.193(0.0373)

			1.11	0.79	0.97	0.0899(0.0092)	0.1671(0.0279)	0.161(0.0261)	0.1664(0.0277)	0.161(0.0260)
				0.92	1.32	0.1036(0.0118)	0.1978(0.0391)	0.190(0.0362)	0.1966(0.0387)	0.190(0.0360)
60	30	15	0.59	0.79	0.97	0.0891(0.0091)	0.1826(0.0334)	0.169(0.0286)	0.1809(0.0328)	0.168(0.0284)
				0.92	1.32	0.1048(0.0122)	0.2263(0.0513)	0.206(0.0426)	0.2223(0.0495)	0.205(0.0418)
			1.11	0.79	0.97	0.0886(0.0091)	0.1816(0.0330)	0.165(0.0271)	0.1792(0.0321)	0.164(0.0270)
				0.92	1.32	0.1036(0.0120)	0.2182(0.0476)	0.196(0.0385)	0.2141(0.0458)	0.195(0.0381)
		23	0.59	0.79	0.97	0.0899(0.0092)	0.1731(0.0300)	0.166(0.0276)	0.1723(0.0297)	0.166(0.0274)
				0.92	1.32	0.1037(0.0118)	0.2094(0.0439)	0.199(0.0397)	0.2075(0.0431)	0.198(0.0393)
			1.11	0.79	0.97	0.0882(0.0089)	0.1723(0.0297)	0.163(0.0266)	0.1712(0.0293)	0.163(0.0265)
				0.92	1.32	0.1047(0.0120)	0.2057(0.0423)	0.193(0.0373)	0.2036(0.0415)	0.192(0.0371)
		23	0.59	0.79	0.97	0.0893(0.0091)	0.1731(0.0300)	0.166(0.0276)	0.1722(0.0297)	0.166(0.0274)
				0.92	1.32	0.1053(0.0122)	0.2099(0.0441)	0.199(0.0398)	0.2079(0.0433)	0.198(0.0394)
	45		1.11	0.79	0.97	0.0890(0.0090)	0.1724(0.0297)	0.163(0.0266)	0.1712(0.0293)	0.163(0.0265)
				0.92	1.32	0.1023(0.0115)	0.2055(0.0423)	0.193(0.0373)	0.2035(0.0414)	0.192(0.0371)
		34	0.59	0.79	0.97	0.0870(0.0085)	0.1651(0.0273)	0.163(0.0267)	0.1648(0.0272)	0.163(0.0266)
				0.92	1.32	0.1059(0.0121)	0.1968(0.0387)	0.194(0.0376)	0.1960(0.0384)	0.193(0.0374)
			1.11	0.79	0.97	0.0894(0.0089)	0.1653(0.0273)	0.161(0.0261)	0.1649(0.0272)	0.161(0.0260)
				0.92	1.32	0.1052(0.0120)	0.1956(0.0383)	0.190(0.0362)	0.1948(0.0379)	0.190(0.0361)

Table 5: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the GWD Parameter β using Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and δ=2 for LINEX loss function.

n	m	k	α	β	γ	Kernel estimations	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.0937(0.0106)	0.1851(0.0345)	0.165(0.0272)	0.1823(0.0334)	0.164(0.0270)
				0.92	1.32	0.1227(0.0168)	0.2381(0.0571)	0.200(0.0402)	0.2303(0.0534)	0.198(0.0393)
			1.11	0.79	0.97	0.0937(0.0106)	0.1960(0.0386)	0.163(0.0267)	0.1901(0.0363)	0.163(0.0265)
				0.92	1.32	0.1184(0.0158)	0.2359(0.0559)	0.194(0.0376)	0.2265(0.0515)	0.193(0.0371)
		8	0.59	0.79	0.97	0.0921(0.0100)	0.1795(0.0324)	0.164(0.0269)	0.1775(0.0317)	0.163(0.0267)
				0.92	1.32	0.1144(0.0147)	0.2174(0.0475)	0.196(0.0383)	0.2133(0.0457)	0.194(0.0377)
			1.11	0.79	0.97	0.0931(0.0102)	0.1831(0.0336)	0.162(0.0263)	0.1798(0.0324)	0.162(0.0262)
				0.92	1.32	0.1150(0.0148)	0.2192(0.0482)	0.192(0.0368)	0.2137(0.0458)	0.191(0.0364)
	15	8	0.59	0.79	0.97	0.0937(0.0103)	0.1812(0.0330)	0.164(0.0270)	0.1789(0.0322)	0.164(0.0268)
				0.92	1.32	0.1149(0.0148)	0.2254(0.0512)	0.197(0.0390)	0.2199(0.0486)	0.196(0.0383)
			1.11	0.79	0.97	0.0932(0.0103)	0.1888(0.0358)	0.163(0.0265)	0.1844(0.0341)	0.162(0.0263)
				0.92	1.32	0.1131(0.0143)	0.2262(0.0514)	0.193(0.0372)	0.2191(0.0482)	0.192(0.0367)
		11	0.59	0.79	0.97	0.0884(0.0093)	0.1722(0.0297)	0.163(0.0265)	0.1711(0.0293)	0.162(0.0264)
				0.92	1.32	0.1102(0.0136)	0.2065(0.0428)	0.193(0.0373)	0.2041(0.0418)	0.192(0.0369)
			1.11	0.79	0.97	0.0871(0.0091)	0.1762(0.0311)	0.162(0.0261)	0.1741(0.0304)	0.161(0.0260)
				0.92	1.32	0.1102(0.0135)	0.2091(0.0438)	0.190(0.0363)	0.2057(0.0424)	0.190(0.0360)
40	20	10	0.59	0.79	0.97	0.0904(0.0094)	0.1736(0.0302)	0.165(0.0273)	0.1726(0.0298)	0.165(0.0271)
				0.92	1.32	0.1103(0.0135)	0.2120(0.0450)	0.199(0.0394)	0.2094(0.0439)	0.197(0.0389)
			1.11	0.79	0.97	0.0878(0.0090)	0.1754(0.0308)	0.163(0.0265)	0.1737(0.0302)	0.162(0.0264)
				0.92	1.32	0.1092(0.0132)	0.2097(0.0440)	0.193(0.0372)	0.2067(0.0427)	0.192(0.0369)
		15	0.59	0.79	0.97	0.0871(0.0088)	0.1729(0.0299)	0.165(0.0273)	0.1720(0.0296)	0.165(0.0271)
				0.92	1.32	0.1067(0.0126)	0.2091(0.0438)	0.198(0.0391)	0.2069(0.0429)	0.196(0.0386)
			1.11	0.79	0.97	0.0877(0.0089)	0.1735(0.0301)	0.163(0.0264)	0.1720(0.0296)	0.162(0.0263)
				0.92	1.32	0.1059(0.0124)	0.2072(0.0430)	0.192(0.0370)	0.2047(0.0419)	0.192(0.0367)
	30	15	0.59	0.79	0.97	0.0871(0.0089)	0.1727(0.0299)	0.165(0.0272)	0.1717(0.0295)	0.165(0.0271)
				0.92	1.32	0.1069(0.0127)	0.2125(0.0452)	0.199(0.0395)	0.2099(0.0441)	0.197(0.0390)
			1.11	0.79	0.97	0.0909(0.0095)	0.1756(0.0309)	0.163(0.0265)	0.1738(0.0302)	0.162(0.0264)
				0.92	1.32	0.1062(0.0125)	0.2096(0.0440)	0.193(0.0372)	0.2066(0.0427)	0.192(0.0369)
		23	0.59	0.79	0.97	0.0878(0.0088)	0.1661(0.0276)	0.163(0.0266)	0.1657(0.0275)	0.163(0.0265)
				0.92	1.32	0.1027(0.0116)	0.1985(0.0394)	0.194(0.0376)	0.1975(0.0390)	0.193(0.0373)
			1.11	0.79	0.97	0.0885(0.0088)	0.1670(0.0279)	0.161(0.0261)	0.1664(0.0277)	0.161(0.0260)
				0.92	1.32	0.1017(0.0114)	0.1980(0.0392)	0.190(0.0362)	0.1968(0.0387)	0.190(0.0360)
60	30	15	0.59	0.79	0.97	0.0873(0.0086)	0.1688(0.0285)	0.165(0.0271)	0.1683(0.0283)	0.164(0.0270)
				0.92	1.32	0.1055(0.0122)	0.2062(0.0426)	0.198(0.0392)	0.2047(0.0419)	0.197(0.0388)
			1.11	0.79	0.97	0.0841(0.0081)	0.1706(0.0291)	0.163(0.0264)	0.1696(0.0288)	0.162(0.0264)
				0.92	1.32	0.1058(0.0122)	0.2030(0.0412)	0.192(0.0370)	0.2013(0.0405)	0.192(0.0368)
		23	0.59	0.79	0.97	0.0895(0.0090)	0.1693(0.0287)	0.165(0.0272)	0.1687(0.0285)	0.165(0.0271)
				0.92	1.32	0.1075(0.0126)	0.2046(0.0419)	0.197(0.0390)	0.2032(0.0413)	0.196(0.0386)
			1.11	0.79	0.97	0.0854(0.0084)	0.1696(0.0288)	0.162(0.0264)	0.1688(0.0285)	0.162(0.0263)
				0.92	1.32	0.1060(0.0122)	0.2018(0.0407)	0.192(0.0369)	0.2003(0.0401)	0.192(0.0367)
	45	23	0.59	0.79	0.97	0.0885(0.0088)	0.1683(0.0283)	0.164(0.0270)	0.1678(0.0282)	0.164(0.0269)
				0.92	1.32	0.1040(0.0118)	0.2032(0.0413)	0.197(0.0387)	0.2020(0.0408)	0.196(0.0384)
			1.11	0.79	0.97	0.0912(0.0093)	0.1688(0.0285)	0.162(0.0263)	0.1680(0.0282)	0.162(0.0263)
				0.92	1.32	0.1039(0.0118)	0.2006(0.0403)	0.192(0.0368)	0.1993(0.0397)	0.191(0.0366)
		34	0.59	0.79	0.97	0.0871(0.0085)	0.1649(0.0272)	0.163(0.0266)	0.1646(0.0271)	0.163(0.0266)
				0.92	1.32	0.1055(0.0120)	0.1968(0.0387)	0.194(0.0376)	0.1961(0.0385)	0.193(0.0374)
			1.11	0.79	0.97	0.0872(0.0085)	0.1654(0.0273)	0.161(0.0261)	0.1649(0.0272)	0.161(0.0260)
				0.92	1.32	0.1044(0.0118)	0.1956(0.0382)	0.190(0.0362)	0.1948(0.0379)	0.190(0.0361)

Table 6: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parantheses for GWD Parameter β using Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and δ=2 for LINEX loss function.

n	m	k	α	β	γ	Kernel estimations	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.1346(0.0199)	0.2145(0.0461)	0.215(0.0464)	0.2060(0.0425)	0.207(0.0428)
				0.92	1.32	0.1997(0.0418)	0.3199(0.1024)	0.321(0.1030)	0.2870(0.0824)	0.288(0.0830)
			1.11	0.79	0.97	0.1388(0.0212)	0.2017(0.0407)	0.202(0.0408)	0.2004(0.0401)	0.201(0.0403)
				0.92	1.32	0.2026(0.0429)	0.2807(0.0788)	0.281(0.0790)	0.2758(0.0761)	0.276(0.0763)
		8	0.59	0.79	0.97	0.1345(0.0198)	0.2022(0.0409)	0.203(0.0411)	0.1996(0.0398)	0.200(0.0401)
				0.92	1.32	0.1939(0.0392)	0.2852(0.0813)	0.286(0.0817)	0.2749(0.0756)	0.276(0.0760)
			1.11	0.79	0.97	0.1343(0.0196)	0.1984(0.0394)	0.199(0.0394)	0.1978(0.0391)	0.198(0.0392)
				0.92	1.32	0.1948(0.0396)	0.2733(0.0747)	0.273(0.0748)	0.2711(0.0735)	0.271(0.0736)
	15	8	0.59	0.79	0.97	0.1316(0.0190)	0.2022(0.0409)	0.203(0.0411)	0.1995(0.0398)	0.200(0.0400)
				0.92	1.32	0.1941(0.0393)	0.2848(0.0811)	0.285(0.0815)	0.2747(0.0755)	0.275(0.0759)
			1.11	0.79	0.97	0.1346(0.0197)	0.1984(0.0393)	0.199(0.0394)	0.1978(0.0391)	0.198(0.0392)
				0.92	1.32	0.1919(0.0385)	0.2733(0.0747)	0.273(0.0748)	0.2711(0.0735)	0.271(0.0736)
		11	0.59	0.79	0.97	0.1276(0.0178)	0.1979(0.0392)	0.198(0.0393)	0.1969(0.0388)	0.197(0.0389)
				0.92	1.32	0.1898(0.0374)	0.2742(0.0752)	0.275(0.0755)	0.2700(0.0729)	0.271(0.0732)
			1.11	0.79	0.97	0.1268(0.0176)	0.1969(0.0388)	0.197(0.0388)	0.1966(0.0386)	0.197(0.0387)
				0.92	1.32	0.1897(0.0375)	0.2699(0.0729)	0.270(0.0730)	0.2688(0.0722)	0.269(0.0723)
	20	10	0.59	0.79	0.97	0.1282(0.0179)	0.2142(0.0459)	0.214(0.0459)	0.2083(0.0434)	0.208(0.0434)
				0.92	1.32	0.1901(0.0376)	0.3129(0.0979)	0.313(0.0980)	0.2908(0.0846)	0.291(0.0847)
			1.11	0.79	0.97	0.1318(0.0189)	0.2011(0.0405)	0.201(0.0405)	0.2003(0.0401)	0.200(0.0402)
				0.92	1.32	0.1917(0.0382)	0.2792(0.0779)	0.279(0.0780)	0.2762(0.0763)	0.276(0.0763)
40		15	0.59	0.79	0.97	0.1283(0.0177)	0.2052(0.0421)	0.205(0.0422)	0.2026(0.0411)	0.203(0.0411)
				0.92	1.32	0.1830(0.0348)	0.2909(0.0846)	0.291(0.0847)	0.2807(0.0788)	0.281(0.0789)
			1.11	0.79	0.97	0.1280(0.0177)	0.1989(0.0395)	0.199(0.0396)	0.1984(0.0394)	0.198(0.0394)
				0.92	1.32	0.1839(0.0351)	0.2743(0.0752)	0.274(0.0752)	0.2726(0.0743)	0.273(0.0743)
	30	15	0.59	0.79	0.97	0.1258(0.0170)	0.2050(0.0420)	0.205(0.0421)	0.2025(0.0410)	0.203(0.0410)
				0.92	1.32	0.1814(0.0342)	0.2909(0.0846)	0.291(0.0847)	0.2808(0.0788)	0.281(0.0789)
			1.11	0.79	0.97	0.1270(0.0174)	0.1989(0.0396)	0.199(0.0396)	0.1984(0.0394)	0.198(0.0394)
				0.92	1.32	0.1805(0.0339)	0.2742(0.0752)	0.274(0.0752)	0.2725(0.0743)	0.273(0.0743)
		23	0.59	0.79	0.97	0.1253(0.0167)	0.1984(0.0393)	0.198(0.0394)	0.1977(0.0391)	0.198(0.0391)
				0.92	1.32	0.1792(0.0332)	0.2748(0.0755)	0.275(0.0756)	0.2719(0.0739)	0.272(0.0740)

			1.11	0.79	0.97	0.1254(0.0168)	0.1967(0.0387)	0.197(0.0387)	0.1966(0.0386)	0.197(0.0386)
				0.92	1.32	0.1801(0.0335)	0.2696(0.0727)	0.270(0.0727)	0.2689(0.0723)	0.269(0.0723)
60	30	15	0.59	0.79	0.97	0.1266(0.0172)	0.2120(0.0449)	0.212(0.0450)	0.2078(0.0432)	0.208(0.0432)
				0.92	1.32	0.1846(0.0353)	0.3104(0.0963)	0.310(0.0964)	0.2925(0.0856)	0.293(0.0856)
			1.11	0.79	0.97	0.1250(0.0168)	0.2009(0.0404)	0.201(0.0404)	0.2003(0.0401)	0.200(0.0401)
				0.92	1.32	0.1831(0.0347)	0.2787(0.0777)	0.279(0.0777)	0.2764(0.0764)	0.276(0.0764)
		23	0.59	0.79	0.97	0.1237(0.0164)	0.2046(0.0419)	0.205(0.0419)	0.2026(0.0411)	0.203(0.0411)
				0.92	1.32	0.1793(0.0332)	0.2897(0.0840)	0.290(0.0840)	0.2816(0.0793)	0.282(0.0793)
			1.11	0.79	0.97	0.1241(0.0165)	0.1988(0.0395)	0.199(0.0395)	0.1984(0.0394)	0.198(0.0394)
				0.92	1.32	0.1788(0.0330)	0.2740(0.0751)	0.274(0.0751)	0.2727(0.0744)	0.273(0.0744)
	45	23	0.59	0.79	0.97	0.1256(0.0169)	0.2047(0.0419)	0.205(0.0419)	0.2027(0.0411)	0.203(0.0411)
				0.92	1.32	0.1812(0.0339)	0.2899(0.0840)	0.290(0.0841)	0.2817(0.0794)	0.282(0.0794)
			1.11	0.79	0.97	0.1257(0.0169)	0.1988(0.0395)	0.199(0.0395)	0.1984(0.0394)	0.198(0.0394)
				0.92	1.32	0.1786(0.0330)	0.2740(0.0751)	0.274(0.0751)	0.2727(0.0744)	0.273(0.0744)
		34	0.59	0.79	0.97	0.1268(0.0170)	0.1986(0.0395)	0.199(0.0395)	0.1981(0.0392)	0.198(0.0392)
				0.92	1.32	0.1759(0.0319)	0.2752(0.0757)	0.275(0.0757)	0.2727(0.0743)	0.273(0.0743)
			1.11	0.79	0.97	0.1274(0.0172)	0.1968(0.0387)	0.197(0.0387)	0.1967(0.0387)	0.197(0.0387)
				0.92	1.32	0.1758(0.0318)	0.2698(0.0728)	0.270(0.0728)	0.2692(0.0725)	0.269(0.0725)

Table 7: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the GWD Parameter γ using Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and δ=2 for LINEX loss function.

n	m	k	α	β	γ	Kernel estimations	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.1334(0.0196)	0.2020(0.0408)	0.203(0.0410)	0.1994(0.0398)	0.200(0.0400)
				0.92	1.32	0.2031(0.0431)	0.2920(0.0853)	0.293(0.0857)	0.2776(0.0771)	0.278(0.0776)
			1.11	0.79	0.97	0.1321(0.0194)	0.1992(0.0397)	0.199(0.0397)	0.1984(0.0394)	0.199(0.0394)
				0.92	1.32	0.1981(0.0411)	0.2750(0.0756)	0.275(0.0758)	0.2723(0.0741)	0.273(0.0743)
		8	0.59	0.79	0.97	0.1263(0.0174)	0.2001(0.0400)	0.201(0.0402)	0.1983(0.0393)	0.199(0.0395)
				0.92	1.32	0.1903(0.0377)	0.2803(0.0786)	0.281(0.0789)	0.2728(0.0744)	0.273(0.0748)
			1.11	0.79	0.97	0.1289(0.0181)	0.1977(0.0391)	0.198(0.0392)	0.1973(0.0389)	0.197(0.0390)
				0.92	1.32	0.1874(0.0366)	0.2719(0.0739)	0.272(0.0740)	0.2702(0.0730)	0.270(0.0731)
	15	8	0.59	0.79	0.97	0.1292(0.0182)	0.2008(0.0403)	0.201(0.0405)	0.1987(0.0395)	0.199(0.0397)
				0.92	1.32	0.1896(0.0375)	0.2848(0.0811)	0.285(0.0815)	0.2747(0.0755)	0.275(0.0759)
			1.11	0.79	0.97	0.1261(0.0175)	0.1983(0.0393)	0.199(0.0394)	0.1978(0.0391)	0.198(0.0392)
				0.92	1.32	0.1893(0.0374)	0.2733(0.0747)	0.273(0.0748)	0.2711(0.0735)	0.271(0.0736)
		11	0.59	0.79	0.97	0.1266(0.0174)	0.1979(0.0392)	0.198(0.0393)	0.1969(0.0388)	0.197(0.0389)
				0.92	1.32	0.1878(0.0367)	0.2743(0.0752)	0.275(0.0755)	0.2700(0.0729)	0.271(0.0732)
			1.11	0.79	0.97	0.1278(0.0178)	0.1968(0.0387)	0.197(0.0388)	0.1965(0.0386)	0.197(0.0387)
				0.92	1.32	0.1878(0.0366)	0.2700(0.0729)	0.270(0.0730)	0.2688(0.0722)	0.269(0.0723)

40	20	10	0.59	0.79	0.97	0.1268(0.0174)	0.2026(0.0410)	0.203(0.0411)	0.2008(0.0403)	0.201(0.0403)
				0.92	1.32	0.1876(0.0365)	0.2874(0.0826)	0.287(0.0827)	0.2789(0.0778)	0.279(0.0779)
			1.11	0.79	0.97	0.1315(0.0185)	0.1985(0.0394)	0.199(0.0394)	0.1981(0.0392)	0.198(0.0393)
				0.92	1.32	0.1850(0.0355)	0.2734(0.0748)	0.273(0.0748)	0.2719(0.0739)	0.272(0.0740)
		15	0.59	0.79	0.97	0.1276(0.0175)	0.2025(0.0410)	0.203(0.0410)	0.2007(0.0403)	0.201(0.0403)
				0.92	1.32	0.1858(0.0358)	0.2847(0.0811)	0.285(0.0811)	0.2776(0.0770)	0.278(0.0771)
			1.11	0.79	0.97	0.1314(0.0185)	0.1982(0.0393)	0.198(0.0393)	0.1978(0.0391)	0.198(0.0391)
				0.92	1.32	0.1852(0.0355)	0.2727(0.0744)	0.273(0.0744)	0.2713(0.0736)	0.271(0.0737)
	30	15	0.59	0.79	0.97	0.1281(0.0176)	0.2021(0.0409)	0.202(0.0409)	0.2005(0.0402)	0.201(0.0402)
				0.92	1.32	0.1869(0.0362)	0.2876(0.0827)	0.288(0.0828)	0.2791(0.0779)	0.279(0.0780)
			1.11	0.79	0.97	0.1285(0.0177)	0.1985(0.0394)	0.199(0.0394)	0.1981(0.0392)	0.198(0.0393)
				0.92	1.32	0.1850(0.0354)	0.2734(0.0748)	0.273(0.0748)	0.2719(0.0739)	0.272(0.0740)
		23	0.59	0.79	0.97	0.1234(0.0163)	0.1983(0.0393)	0.198(0.0394)	0.1977(0.0391)	0.198(0.0391)
				0.92	1.32	0.1809(0.0337)	0.2749(0.0756)	0.275(0.0756)	0.2719(0.0740)	0.272(0.0740)
			1.11	0.79	0.97	0.1235(0.0164)	0.1968(0.0387)	0.197(0.0387)	0.1966(0.0386)	0.197(0.0386)
				0.92	1.32	0.1809(0.0338)	0.2696(0.0727)	0.270(0.0727)	0.2689(0.0723)	0.269(0.0723)
60	30	15	0.59	0.79	0.97	0.1267(0.0170)	0.2016(0.0406)	0.202(0.0406)	0.2004(0.0402)	0.200(0.0402)
				0.92	1.32	0.1785(0.0329)	0.2859(0.0817)	0.286(0.0817)	0.2794(0.0781)	0.279(0.0781)
			1.11	0.79	0.97	0.1268(0.0171)	0.1983(0.0393)	0.198(0.0393)	0.1980(0.0392)	0.198(0.0392)
				0.92	1.32	0.1801(0.0334)	0.2729(0.0745)	0.273(0.0745)	0.2718(0.0739)	0.272(0.0739)
		23	0.59	0.79	0.97	0.1244(0.0165)	0.2020(0.0408)	0.202(0.0408)	0.2007(0.0403)	0.201(0.0403)
				0.92	1.32	0.1808(0.0337)	0.2841(0.0807)	0.284(0.0808)	0.2783(0.0775)	0.278(0.0775)
			1.11	0.79	0.97	0.1284(0.0175)	0.1981(0.0392)	0.198(0.0392)	0.1978(0.0391)	0.198(0.0391)
				0.92	1.32	0.1792(0.0331)	0.2725(0.0742)	0.273(0.0743)	0.2714(0.0737)	0.271(0.0737)
	45	23	0.59	0.79	0.97	0.1253(0.0167)	0.2010(0.0404)	0.201(0.0404)	0.1999(0.0400)	0.200(0.0400)
				0.92	1.32	0.1814(0.0339)	0.2826(0.0799)	0.283(0.0799)	0.2774(0.0770)	0.277(0.0770)
			1.11	0.79	0.97	0.1266(0.0170)	0.1979(0.0392)	0.198(0.0392)	0.1976(0.0391)	0.198(0.0391)
				0.92	1.32	0.1800(0.0334)	0.2721(0.0740)	0.272(0.0740)	0.2711(0.0735)	0.271(0.0735)
		34	0.59	0.79	0.97	0.1292(0.0176)	0.1985(0.0394)	0.199(0.0394)	0.1980(0.0392)	0.198(0.0392)
				0.92	1.32	0.1757(0.0317)	0.2752(0.0757)	0.275(0.0757)	0.2726(0.0743)	0.273(0.0743)
			1.11	0.79	0.97	0.1277(0.0172)	0.1968(0.0387)	0.197(0.0387)	0.1967(0.0387)	0.197(0.0387)
				0.92	1.32	0.1761(0.0319)	0.2698(0.0728)	0.270(0.0728)	0.2692(0.0725)	0.269(0.0725)

Table 8: The Average Bias (AVB) and Mean Squared Errors (MSEs) in Parentheses for the GWD Parameter γ using Kernel and Bayess methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and δ=2 for LINEX loss function.