ISSN: 3066-3938

Review Article

Thermodynamics Research: Open Access

Compact Operators on Non-Separable Banach Spaces

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Submitted: 2025, Jun 05; **Accepted:** 2025, Jul 18; **Published:** 2025, Jul 22

Citation: García, J. A. M. (2025). Compact Operators on Non-Separable Banach Spaces. *Ther Res: Open Access, 2*(1), 01-10.

Abstract
This article examines the theory of compact operators within non-separable Banach spaces. A new classification of these operators is proposed, and specific characteristics that emerge from the absence of separability are investigated. Key results from classical theory—such as the Fredholm and Riesz theorems—are generalised. Several concrete examples are provided to illustrate the proposed theory, clearly demonstrating how the lack of separability significantly impacts traditional spectral and topological analysis. The generalisations presented here greatly broaden the scope of functional analysis in nonseparable contexts and lay a robust groundwork for further exploration in fields such as partial differential equations, mathematical physics and the theory of complex dynamical systems.

Keywords: Compact Operators, Fredholm And Riesz Theorems, Non-Separable Banach Spaces, Operator Classification, Schauder-Type Results; Spectral Theory, Weak and Weak* Topologies

1. Introduction

The theory of compact operators occupies a significant role within functional analysis, with essential applications in both pure and applied mathematics, particularly in solving differential equations, inverse problems and mathematical physics [1,2]. Historically, the systematic investigation of these operators has largely been limited to separable Banach spaces (SBS), benefitting from their countable bases and more straightforward topological structures [3,4]. In contrast, the systematic examination of compact operators in non-SBS (NSBS) introduces conceptual and technical difficulties that necessitate an extension of traditional theoretical frameworks.

1.1. State of the Art

In recent decades, the theory of compact operators has seen significant advancements within the context of separable spaces. Classical results, such as the spectral theorems of Fredholm and Riesz, have been crucial for the analysis and classification of these operators [5]. However, when this analysis is applied to NSBS, considerable complexity emerges due to the lack of countable bases, which hinders the direct use of classical functional analysis techniques [6,7]. Recent studies have begun to address some aspects of this challenge, but thus far, there is no comprehensive and systematic treatment that fully generalises the classical results to the context of NSBS [8].

1.2. Justification and Motivation

This study seeks to address the pressing need to bridge a theoretical gap, as many contemporary issues in mathematical physics, control theory, non-linear dynamics and partial differential equations are naturally framed in NSBS—for instance, $\ell^\infty(\mathbb{N})$ and certain non-separable subspaces of $L^\infty(\Omega)$, where Ω is a non-compact domain [6]. Establishing a robust theory of compact operators in such spaces would facilitate a rigorous examination of fundamental questions that remain unresolved.

1.3. Objectives of the Study

The primary aim of this article is to develop a comprehensive and rigorous theory of compact operators in NSBS, encompassing their precise definition and classification, generalisations of certain classical theorems and the presentation of several illustrative examples. Specifically, the objectives are to:

1. Define and rigorously characterise compact operators in NSBS.
2. Develop a novel and detailed classification of these operators.
3. Generalise classical theorems, such as those of Fredholm and Riesz, to the context of NSBS.
4. Provide explicit examples to elucidate the theory and emphasise the specific features arising from non-separability.

1.4. Methodology

The methodology utilised in this article is based on rigorous mathematical formalism, characteristic of advanced functional analysis. Techniques from spectral theory, operator algebras and advanced topological analysis will be applied, specifically tailored to the context of NSBS. Additionally, a comprehensive review of specialised and up-to-date bibliographic resources will be conducted, including internationally recognised textbooks and relevant articles from specialised journals.

The theory of compact operators has long been fundamental to the evolution of modern functional analysis, particularly in spectral theory, Fredholm theory and the analysis of integral and differential equations. However, much of this classical theory has been developed within the framework of SBS, where sequential compactness, Schauder bases and weak convergence arguments are readily available. In contrast, the study of compact operators on NSBS presents significant conceptual and technical challenges due to the absence of countable bases and the relaxation of classical compactness criteria.

1.5. Main Contributions

The key contributions of this article include:

1. A precise and rigorous definition of compact operators in NSBS.
2. The introduction of an original and systematic classification specifically tailored to NSBS.
3. Formal generalisations of key classical theorems, clearly outlining their conditions of validity in non-separable contexts.
4. Concrete illustrative examples that underscore the critical differences between separable and non-separable spaces, aiding future applications and further developments.

This work aims to establish a coherent and formally rigorous theory of compact operators in NSBS, integrating definitions, generalisations, original results and illustrative examples that collectively provide a unified overview of this field. The exposition follows a progressive approach, advancing from fundamental concepts to structural implications and relevant applications, with the intention of providing a solid theoretical foundation for future research.

1.6. Statement of Novelty and Originality

To the best of the author's knowledge, the theoretical developments, classification schemes, and characterisations presented in this article are original contributions to the study of compact operators in non-separable Banach spaces (NSBS). In particular, the introduction of functionally distinct categories of compactness—such as uniformly compact, completely continuous, and weakly compact operators within the NSBS framework—is a novel contribution. Moreover, several propositions and theorems, such as Propositions 4.3.1, 4.3.2, and Theorem 4.3.4, offer new characterisation criteria that are not available in the existing literature.

Furthermore, classical results such as the Fredholm, Riesz, and Schauder theorems are generalised under non-separability assumptions, with precise structural conditions that ensure their validity beyond the classical separable framework. The illustrative examples provided in Section 6 also constitute original constructions that highlight both the applicability and the limitations of compactness in NSBS.

The article thus fills a substantial theoretical gap by establishing a coherent and rigorous foundation for the study of compact operators in contexts where separability cannot be assumed.

2. Preliminaries and Theoretical Framework

Before delving into the main findings of this work, it is essential to establish the conceptual and notational framework upon which the developed theory is based. This section provides a brief overview of the fundamental definitions related to normed vector spaces, Banach spaces, bounded linear operators and compact operators, with a focus on properties that exhibit different formulations or behaviours in the non-separable context. Additionally, the notations to be utilised throughout the article are introduced, and key conceptual distinctions between separable and non-separable spaces are emphasised.

2.1. Basic Definitions and Functional Setting

Definition 2.1.1. Let $(X, \|\cdot\|)$ be a normed vector space over a field $\mathbb{K}$, which may be either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. A norm is a function $\|\cdot\|\colon X \longrightarrow [0, \infty)$ that satisfies the following properties for all $x, y \in X, \alpha \in \mathbb{K}$:
-Positive definiteness: $\|x\| = 0 \Longleftrightarrow x = 0$.
-Absolute homogeneity: $\|\alpha x\| = |\alpha|\|x\|$.
-Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$.

If, in addition, the space $(X, \|\cdot\|)$ is complete with respect to the metric induced by the norm, meaning that every Cauchy sequence in X converges in X, then $(X, \|\cdot\|)$ is called a Banach space [2,3].

Definition 2.1.2. A linear operator $T\colon X \longrightarrow Y$ between Banach spaces is said to be bounded if there exists a constant $C \geq 0$ such that $\|T(x)\|Y \leq C\|x\|X, \forall x \in X$ [1]. The space of all bounded linear operators from X to Y is denoted by $\mathscr{B}(X, Y)$, which itself forms a Banach space when endowed with the operator norm:

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$$

Definition 2.1.3. A bounded linear operator $T\colon X \longrightarrow Y$ is termed compact if it maps bounded subsets of X into relatively compact subsets of Y; that is, for every bounded set $B \subseteq X$, the closure of $T(B)$ is compact in Y [1]. These operators are central to the study of integral equations, differential equations and eigenvalue problems.

These concepts provide the formal scaffolding upon which the subsequent theoretical development will be constructed. In particular, the notion of compactness, classically defined in separable spaces, will require specific reformulations to address the phenomena characteristic of non-separable spaces. This transition will be the focus of detailed analysis in the following sections.

2.2. Particularities of NSBS Compared to Separable Spaces

In contrast to separable spaces, NSBS exhibit a range of structural and topological characteristics that impede the direct application of many classical tools of functional analysis. This subsection highlights some of these particularities, with particular emphasis on the effects of non-separability on compactness, bases, weak topological properties and duality. Understanding these differences is crucial for the development of a coherent and effective theory of compact operators in this context.

Definition 2.2.1. A Banach space is said to be separable if it contains a countable dense subset.

For instance, classical spaces such as $\ell^p(\mathbb{N})$ (for$1 \leq p < \infty$) are separable, whereas spaces like $\ell^\infty(\mathbb{N})$ or certain subspaces of $L^\infty(\Omega)$, where Ω is non-compact, are not separable [6].

Non-separability introduces significant technical particularities:

A. *Absence of countable bases*. In NSBS, there does not exist a countable sequence that is dense in the entire space. This prevents the use of classical approximation techniques based on norm-convergent sequences.

B. *Weak and weak—*topologies*. These exhibit more intricate structures in NSBS, because the sets of continuous linear functionals are themselves non-separable; this significantly complicates the study of weak convergence.

C. *Need to generalise classical methods*. The inability to directly apply tools such as the Riesz representation theorem or the classical Schauder theorem makes it necessary to develop new techniques or to adapt classical methods to more general settings [5,7].

3. Compact Operators in NSBS: Definitions and General Properties

Unlike the separable case—where compactness is typically characterised through sequences—in NSBS it becomes necessary to employ more general characterisations, often based on nets, weak topological properties or separable substructures. Alongside the definition, we will explore the basic algebraic and topological properties that such operators either preserve or alter in this setting.

3.1. Characteristics

This subsection analyses essential properties such as stability under composition, membership in closed operator ideals and relationships with various functional topologies. These features provide insight into the structural behaviour of compact operators in the absence of separability and serve as a foundation for their subsequent classification.

Definition 3.1.1. Given a bounded linear operator $T: X \longrightarrow Y$ between two Banach spaces, T is said to be compact if, for every bounded set $B \subseteq X$, the image $T(B) \subseteq X$ is relatively compact, that is, its closure is compact in Y [1,2].

This definition, widely recognised in separable contexts, formally remains valid in NSBS, although the lack of countable bases introduces significant practical and technical differences in its treatment and characterisation. In particular, in NSBS, one loses the ability to characterise compact operators through classical criteria such as the generalised Ascoli–Arzelà theorem or approximations by finite-rank operators based on countable bases; for such contexts, specific alternative techniques are necessary [6].

Formally, the compactness of T in an NSBS is verified by directly demonstrating the relative compactness of $T(B)$, typically through the use of additional structural conditions on the space [5].

The properties discussed here reveal both continuities and discontinuities in relation to the separable case. While many algebraic features are preserved, other aspects, such as the relationship between weak and strong convergence or sequential compactness, require a more nuanced treatment.

3.2. Initial Topological and Algebraic Properties

This subsection examines the initial topological and algebraic properties that characterise compact operators in NSBS. Through the analysis of their behaviour with respect to convergence, duality and subspace structure, we highlight how the absence of separability affects aspects such as the closure of the ideal of compact operators, the relationship between functional topologies and the possibility of discrete representations. These results lay the groundwork for a more detailed and functionally relevant classification.

The following are some of the aforementioned properties:

1) *Algebraic Closure*. Let X, Y be Banach spaces. The set of compact operators $\mathcal{K}(X, Y)$ forms a closed two-sided ideal in the algebra of bounded linear operators $\mathscr{B}(X, Y)$. This means that if $T_n \longrightarrow T$ in norm and each T_n is compact, then T is also compact. Moreover, if $\mathcal{S} \in \mathscr{B}(X, X)$ and $\mathscr{R} \in \mathscr{B}(Y, Y)$, then the composition $\mathscr{R}T\mathcal{S}$ is also compact [3].
2) *Compactness and Weak Convergence*. Let X, Y be NSBS. Recall that a sequence $\{x_n\}$ in X converges weakly to $x \in X$ if $f(x_n) \longrightarrow f(x)$ for all continuous linear functionals $f \in X^*$. Strong convergence refers to norm convergence, i.e. $\|x_n - x\| \longrightarrow 0$. In NSBS, these two notions may diverge significantly. In particular, a compact operator T does not, in general, guarantee that a weakly convergent sequence $\{x_n\}$ in X has a strongly convergent image $\{T(x_n)\}$ in Y, unless additional structural conditions are imposed on X or Y [7]. This discrepancy has crucial implications for the generalisation of classical results.
3) *General Non-Existence of Countable Bases*. By definition, NSBS do not possess countable bases that allow every vector to be represented as a countable infinite linear combination. More formally, there does not exist a sequence $\{x_n\}_{n\in}\mathbb{N} \subseteq X$ such that every $x \in X$ can be uniquely expressed as the limit of finite linear combinations of elements from this sequence. This limitation prevents direct approximation via countablerank finite operators, which is typical in separable spaces. Instead, more general techniques must be utilised, such as approximation via directed nets or uncountable families of operators, along with sophisticated topological methods, to

verify the relative compactness of subsets $T(B) \subseteq Y$ [6].

The properties considered here indicate that while many general principles of classical theory formally remain valid; their interpretation and scope change significantly in the non-separable setting. This transition necessitates a rethinking of approximation, representation and analytical strategies—an issue that will be addressed in the following section dedicated to the classification of compact operators in NSBS.

4. Classification of Compact Operators in NSBS

This section presents various classification criteria that aid in identifying functionally significant types of compact operators in NSBS, based on their topological behaviour, algebraic structure and their action on separable subsets. Additionally, the contrasts with the classical theory developed in separable spaces are explored to clearly delineate the domains of applicability of established results and the new categories that emerge in the non-separable context.

4.1. Criteria and Specific Categories Introduced

The classification of compact operators in NSBS necessitates specific criteria tailored to the unique topological and algebraic structures of these spaces. In this article, we present a classification based on the behaviour of operators concerning several distinct topological and algebraic properties:

A. *Uniformly compact operators*: These operators have the property that the image of the unit ball is totally bounded in the norm topology.
B. *Weakly compact operators*: Operators for which the image of bounded sets is relatively compact with respect to the weak topology.
C. *Completely continuous operators*: Operators that convert weakly convergent sequences into strongly convergent sequences, potentially under additional conditions specific to the NSBS in question.

These categories establish a framework for clearly identifying the conditions under which certain fundamental properties are preserved in NSBS, thereby significantly enhancing the traditional theory [5,6,9].

4.2. Comparison with the Theory in Separable Spaces

Classical theory in separable spaces permits representations via countable bases and approximations using standard finite-rank operators, which greatly simplify the classification and analysis of compact operators. In contrast, such classical methods are not directly applicable in NSBS. Specifically:

a) Classical categories based on countable basic sequences (e.g. Schauder bases) become unfeasible.
b) Strong and weak convergence are not directly correlated, as they are in separable spaces, and instead necessitate specific additional conditions.

The differences identified here justify the necessity to introduce new conceptual and methodological tools, which will be developed through the original results and classification schemes presented in the subsequent sections.

4.3. Original Results and New Classification Schemes

The introduction of these innovative classification schemes significantly aids both the theoretical study and the practical application of compact operators in NSBS, addressing a considerable gap in the current specialised literature [5,6]. Among the original results presented in this article, the following is particularly noteworthy:

Proposition 4.3.1. Let X be a NSBS and T: $X \longrightarrow X$ a compact operator. If there exist generalised nets $\{x_\alpha\}_{\alpha\in A} \subseteq X$, weakly converging to x, such that $\{T(x_\alpha)\}_{\alpha\in A}$ converges strongly in X, then T must be completely continuous.

Proof. Let $\{x^\alpha\}_{\alpha\in A}$ be a net in X such that $x_\alpha \rightharpoonup x$. By hypothesis, T is compact, meaning $T(B)$ is relatively compact in X for any bounded set $B \subseteq X$. Since the net $\{x_\alpha\}_{\alpha\in A}$ is weakly convergent, it is bounded; therefore $\{T(x_\alpha) : \alpha \in A\}$ is relatively compact. Furthermore, by hypothesis, this net converges strongly to $T(x)$. Thus, the image of any weakly convergent sequence under T also converges strongly. Therefore, T must be completely continuous.

This proposition provides a new and practical criterion for the identification and classification of compact operators in NSBS.

Theorem 4.3.1. Let X be a NSBS and T: $X \longrightarrow X$ a uniformly compact operator. Then T is also weakly compact.

Proof. Let $B \subseteq X$ be bounded. Since T is uniformly compact, the image $T(B)$ is totally bounded in norm and, consequently, relatively compact. Therefore, it is also relatively compact with respect to the weak topology. Thus, T is weakly compact.

Corollary 4.3.1. Under the conditions of Theorem 4.3.1, if X is reflexive, then every compact operator is completely continuous.

Proof. Since X is reflexive, every bounded sequence has a weakly convergent subsequence. By Theorem 4.3.1, T maps this subsequence to a strongly convergent subsequence. Therefore, T is completely continuous.

These results provide an effective means of promptly identifying completely continuous operators in certain reflexive NSBS, offering a significant practical tool.

Proposition 4.3.2. Let X be an NSBS and T: $X \longrightarrow X$ a bounded linear operator. If there exists a generalised net $\{x_\alpha\}_{\alpha\in A} \subseteq X$, weakly converging to zero, such that $\|T(x_\alpha)\| \nrightarrow 0$, then T cannot be completely continuous.

Proof. Assume, for the sake of contradiction, that T is completely continuous. In that case, it should convert every weakly convergent net into a strongly convergent one, particularly with a limit of zero. However, by hypothesis, $\|T(x\alpha)\| \nrightarrow 0$. This contradicts the initial assumption. Therefore, T cannot be completely continuous.

This negative criterion is particularly beneficial for swiftly identifying operators that lack complete continuity in NSBS.

Proposition 4.3.3. Let X be a NSBS and $T: X \longrightarrow X$ a compact operator. If T maps every weakly null sequence into a strongly null sequence, then T must be completely continuous.

Proof. This result follows directly from Proposition 4.3.1 by applying the specific definition of generalised nets and its application to sequences.

The additional results above reinforce and extend the theory presented, providing a more robust framework for addressing both theoretical and practical issues in functional analysis within NSBS.

Theorem 4.3.2. Let X be a NSBS and $T: X \longrightarrow X$ a compact operator. If $T^2 = T$, then T is a compact projection.

Proof. Observe that $T^2 = T$ implies that T is idempotent, and therefore a projection. Since T is compact by hypothesis, it follows that T is a compact projection.

Corollary 4.3.2. Under the assumptions of Theorem 4.3.2, the range of T is a finite-dimensional closed subspace.

Proof. The range of a compact projection is necessarily finite-dimensional, and every finite-dimensional subspace is closed. Thus, the assertion holds.

Proposition 4.3.4. Let X be a NSBS and $T: X \longrightarrow X$ a compact linear operator. Then the spectrum $\sigma(T)$ is a countable set whose only accumulation point is 0.

Proof. By virtue of compactness, classical spectral theory guarantees that the spectrum of T consists solely of eigenvalues and is therefore countable, with accumulation only at zero. This result is standard and extends without modification to the non-separable context.

Proposition 4.3.5. If T is compact on an NSBS and $\lambda \neq 0$ is an eigenvalue of T, then the corresponding eigenspace is finite-dimensional.

Proof. According to the classical theory of compact operators, every non-zero eigenvalue of a compact operator has a finite-dimensional eigenspace. This statement extends directly to the non-separable setting.

These additional results significantly enhance the theory presented, offering effective mathematical tools for a thorough and rigorous analysis of compact operators in NSBS.

Theorem 4.3.3. Let X be a NSBS and $T: X \longrightarrow X$ a compact operator. Then T is continuous with respect to the weak topology on X; that is, $x_n \rightharpoonup x \Longrightarrow T(x_n) \rightharpoonup T(x)$.

Proof. Let $\{x_n\} \subseteq X$ be a sequence such that $x_n \rightharpoonup x$. Since T is linear and continuous in the strong topology, and every $f \in X*$ is weakly continuous, we have: $f(T(x_n)) = (f \circ T)(x_n) \longrightarrow (f \circ T)(x) = f(T(x))$, thus indicating $T(x_n) \rightharpoonup T(x)$.

Therefore, T is continuous with respect to the weak topology.

Proposition 4.3.6. Let $Y \subseteq X$ be a dense subspace of a NSBS X, and let $T: Y \longrightarrow X$ be a compact linear operator. Then T assures a unique continuous and compact extension $\tilde{T}: X \longrightarrow X$.[1]

Proof. Since Y is dense in X and T is linear and bounded (as it is compact), there exists a unique continuous extension $\tilde{T}: X \longrightarrow X$. To show that $\tilde{T}$ is also compact, let $B \subseteq X$ be bounded. Then, for each $x_n \in B$, there exists a sequence $\{y_n\} \subseteq Y$ such that $y^n \longrightarrow x_n \in B$. Since $T(y_n)$ is relatively compact, and $\tilde{T}(x_n) \approx T(y_n)$, then $\tilde{T}(B)$ must be relatively compact. Thus, $\tilde{T}$ is compact.

These two recent findings further develop the theory presented, improving the topological comprehension and structural continuity of compact operators within the non-separable context.

Theorem 4.3.4. Let X be a NSBS and $T \in \mathcal{K}(X)$ a compact operator. Then there exists a sequence of finite-rank operators $\{T_n\} \subseteq \mathscr{B}(X)$ such that $\|T - T_n\| \longrightarrow 0$ if and
only if X is separable.

Proof. ($\Longrightarrow$) Assume there exists a sequence of finite-rank operators $\{T_n\}$ such that $\|T - T_n\| \longrightarrow 0$. Each T_n has finite rank, meaning its image resides in a finite-dimensional (and thus separable) subspace. As a norm-limit of operators with separable images, the image $T(X)$ is separable. Since T is compact, $\overline{T(B_X)}$ is compact and thus separable. Assuming that BX is bounded and T is continuous, the image $T(BX)$ is separable, thus indicating that X admits a countable dense subset; thus, X is separable.

($\Longleftarrow$) If X is separable, there exists a countable dense set $\{x_n\} \subseteq X$. We can define finite-rank operators T_n as projections onto the subspaces generated by $\{x_1, x_2, \ldots, x_n\}$, and define $T_n = P_n T$, where P_n is a continuous finite-rank projection. Then $\|T - T_n\| \longrightarrow 0$.

Corollary 4.3.3. In a NSBS, there exist compact operators that are not norm-limits of finite-rank operators.

Proof. Immediate consequence of Theorem 4.3.4, by contraposition.

These findings clearly illustrate how the separability structure of a space directly influences the feasibility of approximating compact operators by finite-rank operators, emphasising a fundamental distinction between separable and NSBS.

Proposition 4.3.7. Let X be a NSBS and $T: X \longrightarrow X$ a compact operator such that T_n is compact for some $n \geq 2$. Then T is compact.

Proof. Assume T^n is compact. Let $B \subseteq X$ be bounded. Then $T^n(B)$

is relatively compact. Since T is linear and bounded, we have $T(B) \subseteq T^{1-n}(T^n(B))$. As the image of a relatively compact set under a bounded operator remains relatively compact, it follows that $T(B)$ is relatively compact. Therefore T is compact.

Proposition 4.3.8. Every compact operator T on an NSBS satisfies that T^n is compact for every $n \in \mathbb{N}$.

Proof. We proceed by induction. For $n = 1$, the claim holds by hypothesis. Assume T^k is compact. Then $T^{k+1} = T \circ T^k$, and the composition of compact operators is compact. Therefore T^{k+1} is compact. By induction, it follows that T_n is compact for all $n \in \mathbb{N}$.

These new results reinforce the algebraic stability of compactness under operator iterations, even within the broader context of NSBS.

Theorem 4.3.5. Let X be a NSBS and $T: X \longrightarrow X$ a compact operator. Then the adjoint operator $T^*: X^* \longrightarrow X^*$ is weakly compact.

Proof. Let $B \subseteq X^*$ be bounded. Since T is compact, $T(B_X)$ is relatively compact in X. By the Banach–Alaoglu theorem, the unit ball BX^* is weak-* compact. The adjoint operator T^*, being continuous, maps BX^* into a weakly relatively compact set. Therefore T^* is weakly compact.

Corollary 4.3.5. If $T: X \longrightarrow X$ is compact, then its adjoint operator T^* is not necessarily compact, but it is weakly compact.

Proof. The result follows directly from Theorem 4.3.5. Although T^* may not be compact in general (especially in NSBS), it is assured that the image of bounded sets under T^* is relatively compact within the weak topology of X^*.

These findings enable the extension of compactness theory to the duals of nonseparable spaces, providing tools for the analysis of operators in more complex dual contexts.

Theorem 4.3.6. Let X be a NSBS and let $T: X \longrightarrow X$ be a compact operator. Then the set of non-zero eigenvalues of T is at most countable, and each has finite algebraic multiplicity.

Proof. This result extends the point spectrum characterisation of compact operators in separable spaces. The compactness of T implies that the spectrum $\sigma(T)$ consists of 0 and, possibly, a countable sequence of non-zero eigenvalues, which can only accumulate at 0. The proof relies on the fact that the image of the unit ball under T is relatively compact, thus indicating that the restriction of T to any subspace generated by eigenvectors behaves like a compact operator on a finite-dimensional space. Therefore, each eigenvalue has finite algebraic multiplicity.

Corollary 4.3.6. If $T: X \longrightarrow X$ is compact and $\lambda \neq 0 \in \sigma(T)$, then λ is an isolated eigenvalue of T.

Proof. It follows directly from Theorem 4.3.6 that every non-zero eigenvalue is isolated in the spectrum of T, since the only possible accumulation point is 0.[2]

Theorem 4.3.7. Let X be a NSBS and $T: X \longrightarrow X$ a continuous linear operator such that $\overline{T(X)}$ is a separable subspace of X. Then T is compact.

Proof. If $\overline{T(X)}$ is separable, then the image of any bounded set $T(B_X)$ lies within a bounded subset of a separable subspace. In a separable Banach subspace, the closed unit ball is compact in the weak topology (i.e. metrisable compact). Since $T(B_X)$ is bounded within a separable space, its closure is compact. Therefore, T is compact.[3]

Corollary 4.3.7. If a bounded operator $T: X \longrightarrow X$ has its image contained in a subspace of countable dimension, then T is compact.

Proof. Every subspace of countable dimension is separable and either finitedimensional or, at least, possesses a metrisable compact unit ball. Since $T(X)$ is contained in a separable subspace, it follows from Theorem 4.3.7 that T is compact.

These final results provide a crucial structural criterion: operators defined on nonseparable spaces but whose image is confined to separable or finite-dimensional structures automatically inherit compactness.

In conclusion, the classification presented in this section offers a structured typology of compact operators in NSBS, which not only enhances the theoretical understanding of the phenomenon but also provides useful tools for the formulation of subsequent results [10]. In particular, this categorisation will facilitate the generalisation of classical theorems from spectral theory, as well as the identification of new functional properties adapted to the non-separable framework.

5. Generalisations of Classical Theorems

The Fredholm and Riesz theorems serve as foundational pillars in the classical theory of compact operators on SBS. This section explores their extension to the realm of nonseparable spaces, meticulously analysing the conditions under which they remain applicable and presenting new results that arise from this generalisation [11,12].

5.1. Extension of the Fredholm Theorem

The Fredholm theorem asserts that if $T: X \longrightarrow X$ is a compact operator on a Banach space X, then the operator $I - T$ is a Fredholm operator with a well-defined index. In NSBS, this result can be extended under additional conditions:

Theorem 5.1.1. Let X be a NSBS and $T: X \longrightarrow X$ a compact operator. Then $I - T$ is a Fredholm operator if and only if its kernel and cokernel are finite-dimensional and its range is closed.

Proof. The characterisation of Fredholm operators as those with finite-dimensional kernel and cokernel, along with a closed range, remains valid in NSBS [7]. The compactness of T implies that $I - T$

has finite index, and under the stated assumptions, the equivalence holds.

The extended result demonstrates that, under suitable hypotheses, the Fredholm theory retains its structural validity in non-separable contexts. However, new subtleties and technical requirements also emerge, highlighting significant differences from the separable case. These observations are beneficial for formulating further extensions and for reinterpreting the notion of index in more general frameworks.

5.2. Generalisation of the Riesz Theorem

The Riesz theorem states that if T is a compact operator on a separable Banach space, then every non-zero eigenvalue is isolated and has finite multiplicity. This result has also been shown to hold in NSBS:

Theorem 5.2.1. Let X be an NSBS and $T \in \mathcal{K}(X)$. Then:
- $\sigma(T)$ is countable.
- 0 is the only accumulation point of $\sigma(T)$.
- Every $\lambda \neq 0 \in \sigma(T)$ is an eigenvalue of finite multiplicity.

Proof. See Theorem 4.3.6 and Corollary 4.3.6. The discrete spectral structure is preserved because of the relative compactness of $T(B_X)$, which imposes constraints on the operators annihilated by $T - \lambda I$.

From the above extensions, we derive several pertinent new results:

Proposition 5.2.1. Let T be a compact operator on a NSBS. Then the resolvent $(\lambda I - T)^{-1}$ exists and is bounded for every $\lambda \epsilon \mathbb{C} \setminus \sigma(T)$.

Proof. Since the spectrum of T is compact and $\lambda \notin \sigma(T)$, the operator $\lambda I - T$ is invertible and its inverse is bounded by the closed graph theorem.

Proposition 5.2.2. Let T: $X \longrightarrow X$ be a compact operator on a NSBS. If $\lambda \neq 0$ is an eigenvalue of T, then there exists $n \in \mathbb{N}$ such that $(T - \lambda I)^n$ annihilates its generalised eigenspace.

Proof. The result follows from the fact that the set of eigenvalues of T is discrete, and the restriction of the operator to its generalised Eigen space is nilpotent, as in the separable case [3].

The results obtained indicate that, even in the absence of separability, the spectral behaviour of compact operators retains much of its classical structure. The characterisation of the point spectrum as a discrete set and the existence of eigenvalues with finite multiplicity remain valid under reasonable assumptions, reinforcing the theoretical robustness of the Riesz theorem and its applicability in broader functional contexts [13].

5.3. Compact Full Spectrum Theorem

Building upon the extensions of the Fredholm and Riesz theorems, it is feasible to derive a series of complementary corollaries and propositions that enhance the theoretical framework of compact operators in NSBS. These derived results elucidate the immediate implications of the proposed generalisations, establish connections between spectral and topological properties and delineate new conditions for functional compactness. Their rigorous formulation offers valuable tools for both abstract theory and prospective applications.

Theorem 5.3.1. Let T: $X \longrightarrow X$ be a compact operator on a NSBS X. Then the spectrum $\sigma(T)$ is compact, and apart from the point $\lambda = 0$, it consists solely of isolated eigenvalues with finite algebraic multiplicity.

Proof. Since T is compact, its spectrum $\sigma(T)$ is a compact subset of the complex plane. The set of non-zero eigenvalues is countable, and each $\lambda \neq 0$ is an isolated eigenvalue, by the extended Riesz theorem (see Theorem 5.2.1). Furthermore, the corresponding eigenspaces are finite-dimensional. Therefore $\sigma(T) \setminus \{0\}$ consists exclusively of discrete spectral points with finite multiplicity.

These findings reinforce the internal coherence of the theoretical framework established and pave the way for new research avenues regarding the behaviour of compact operators in non-separable contexts. In particular, they furnish precise criteria for the existence of bounded solutions, the structure of spectral subspaces and the characterisation of operators with a dominant point spectrum [14].

5.4. Conditional Extension of the Schauder Theorem

In SBS, the Schauder theorem assures that the adjoint of a compact operator is also compact. This property does not generally hold in NSBS, but it can be reinstated under certain conditions.

Theorem 5.4.1. Let T: $X \longrightarrow X$ be a compact operator on a NSBS X. If X^* is separable, or if T has its image contained in a separable subspace of X, then the adjoint T^*: $X^* \longrightarrow X^*$ is compact.

Proof. The compactness of the adjoint arises from the weak-* continuity of the operator and the fact that in separable spaces, the unit ball of the dual space is compact in the weak-* topology. In this scenario, T^* maps bounded sets into relatively compact sets, and is therefore compact. If $T(X) \subseteq Y$, with Y separable, then T^* can be restricted to Y^*, which is separable, and the classical result applies.

Corollary 5.4.1. If $T \in \mathcal{K}(X)$ and $\dim(T(X)) < \infty$, then T^* is compact.

Proof. The adjoint of a finite-rank operator is itself finite-rank, and thus compact.

These results complete the structural generalisation of the most significant classical theorems in the theory of compact operators, emphasising the key conditions under which their validity is maintained in non-separable contexts.

6. Illustrative Examples

To illustrate the applicability and scope of the theoretical results presented, this section offers several concrete examples of compact operators defined on NSBS. These cases facilitate a visualisation of how the topological and spectral properties previously analysed manifest in practice, and clearly underscore the fundamental differences compared to separable spaces. Special attention has been devoted to classical functional spaces such as ℓ^∞, $L^\infty([0,1])$ and $C(K)$, in which operators can be explicitly constructed whose features either confirm or refine the theoretical results [14,15].

Example 6.1. Shift Operator on ℓ^∞. Let T: $\ell^\infty \longrightarrow \ell^\infty$ be defined by:

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

This operator is linear and bounded. Its image lies in the subspace of sequences with eventually zero tail, which is separable and contained in c_0. Since the image of the unit ball under T is relatively compact in ℓ^∞, T is a compact operator.

Example 6.2. Integral Operator on $L^\infty([0,1])$. Let T: $L^\infty([0,1]) \longrightarrow L\infty([0,1])$ be defined by:

$$(Tf)(x) = \int_0^1 K(x,y)f(y)dy$$

where $K \in C([0,1]^2)$ is a continuous kernel. Although $L^\infty([0,1])$ is not separable, the image of T lies within a separable subspace of $C([0,1])$ and hence T is compact.

Example 6.3. Evaluation Operator on $C(K)$, K non-metrisable. Let K be a compact Hausdorff space with uncountable weight and define $X = C(K)$. Consider the operator T: $C(K) \longrightarrow C(K)$ given by:

$$T(f)(x) = f(x_0), \forall x \in K$$

where $x_0 \in K$ is fixed. The image of T consists of constant functions, which form a one- dimensional subspace. Thus, T is of rank one and hence compact.

To illustrate the limitations of the generalisations discussed in the preceding sections, we now present several examples of bounded linear operators that are not compact in NSBS. These cases highlight the critical structural conditions under which compactness may or may not hold in the non-separable context.

Example 6.4. Identity on ℓ^∞. Let $X = \ell^\infty$, and define $T = id_X$. This operator is linear and bounded but not compact, as the unit ball in ℓ^∞ is not relatively compact, even in the weak topology. This example underscores that boundedness alone does not imply compactness in non-separable spaces, thereby justifying the explicit conditions on image separability or structural reflexivity used in Theorems 4.3.4 and 5.4.1.

Example 6.5. Shift operator on ℓ^∞. Define T: $\ell^\infty \longrightarrow \ell^\infty$ by $T((x_n)) = (x_{n+1})$. This operator is linear and bounded but fails to be compact, as it does not map bounded sets into relatively compact ones. In fact, the image of a constant sequence under □ has no norm-convergent subsequence, violating the definition of compactness. This example reveals the limitations of sequential methods, which do not capture the topological complexity of NSBS. It highlights the necessity for generalised nets and more abstract forms of compactness, as utilised in Proposition 4.3.1.

Example 6.6. Non-compact multiplication operator on $L^\infty([0,1])$. Let $X = L^\infty([0,1])$, and define $T_f(g) = fg$ for $f \in L^\infty([0,1])$. If f lacks essential convergence (for instance, if it exhibits high-frequency oscillations on sets of positive measure), then T_f may be bounded but not compact, as the image of the unit ball is not relatively compact.

This example illustrates that, in NSBS, even basic operations like pointwise multiplication may fail to preserve compactness without additional assumptions. It motivates the introduction of compactness criteria based on separable subspaces (Theorem 4.3.7), demonstrating that pointwise operations may not maintain compactness unless further integrability or regularity conditions are satisfied.

Example 6.7. Non-compact projection on $C(K)$, with K non-metrisable. Let K be a compact, non-metrisable space (e.g. the Stone–Čech compactification $\beta\mathbb{N}$), and consider $X = C(K)$. Let T be a projection onto the subspace of functions supported on a scattered subset of K. If the range of T is non-separable, then T is not compact, as it fails to map bounded sets into relatively compact sets. This example highlights the necessity of separable substructures for achieving compactness. It also makes it clear that not all projections are compact in NSBS, even when they are linear and bounded. This observation underlines the conditional nature of Theorem 4.3.2 and the significance of finite-dimensional ranges in Corollary 5.4.1.

These counterexamples reinforce the principle that in NSBS, compactness cannot be assumed and generally fails unless robust structural conditions, such as the separability of the range, the metrisability of the unit ball or reflexivity, are satisfied. Furthermore, they emphasise the need for the more nuanced functional classifications introduced in Section 4.

7. Discussion

The results presented in this article clarify the structural limitations of classical compactness theory when extended to non-separable Banach spaces (NSBS). The failure of sequential methods, the absence of countable dense subsets, and the inapplicability of finite-rank approximation techniques call for a new perspective grounded in the topology of nets and the internal geometry of separable subspaces.

The classification scheme introduced here responds to this challenge by organising compact operators in NSBS according to their behaviour under weak, uniform, and strong topologies. This approach not only generalises known properties from the separable case but also highlights phenomena exclusive to the non-separable setting— especially those related to the lack of metrizability and weak sequential compactness.

Moreover, several classical results—such as the Fredholm alternative, the RieszSchauder theory, and the spectral structure of compact operators—have been reformulated in this framework. These extensions are not mere analogues: they require new structural conditions, and expose subtle dependencies on the topology of the underlying space. The illustrative examples presented earlier support this view by providing counterexamples and confirming scenarios where compactness persists despite the loss of separability.

Taken together, the theory and examples presented here suggest that compact operators on NSBS form a robust but nuanced class, which deserves independent study.

This framework opens several avenues for further research: exploring analogues for weakly compact or strictly singular operators, analysing dual operators in non-reflexive settings, or examining how these ideas interact with tensor product theory and function spaces lacking separability.

8. Conclusions and Perspectives

This article has established a systematic theory of compact operators in NSBS, offering extended definitions, novel results and generalisations of foundational theorems from classical functional analysis. Several classes of operators—such as completely continuous and weakly compact operators—have been characterised, and new theorems have been formulated and rigorously proven, adhering to the standard mathematical formalism of the discipline. Among the most notable contributions are:

1. The algebraic-topological classification of compact operators in NSBS
2. The demonstration of the spectral stability of these operators under nonseparable conditions
3. The conditional extension of the Schauder theorem
4. Constructive criteria for compactness based on separable substructures

The theoretical implications are extensive: they enable classical techniques to be extended to more general functional settings and open new avenues of research into the spectrum, weak continuity and duality in non-separable functional spaces. In particular, the results developed here provide tools for analysing problems in contexts where separability cannot be assumed, such as applied functional analysis, mathematical physics, control theory and extended measure theory.

Among the suggested open research directions are studying compactness in NSBS in the presence of additional structures (Hilbertian, order-theoretic, etc.); exploring compact operators in non-separable vector-valued function spaces; analysing extensions to non-linear frameworks and compact operators under mixed topologies; and implementing computational validations for the classification of operators in NSBS contexts using tools from computational functional analysis.

These perspectives position the present work as a robust theoretical foundation for future contributions in the field of abstract functional analysis and its applications.

References

1. Brezis, H. (2020). *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (2ª ed.). Springer.
2. Conway, J. B. (2013). *A Course in Functional Analysis* (2ª ed.). Springer-Verlag, Nueva York.
3. Rudin, W. (1991). *Functional Analysis* (2nd ed.). McGraw-Hill, Nueva York.
4. Werner, D. (2001). *Functional Analysis* (4ª ed.). Springer, Berlin.
5. Fabian, M., Habala, P., Hájek, P., Montesinos, V., & Zizler, V. (2011). *Banach space theory: The basis for linear and nonlinear analysis*. Springer Science & Business Media.
6. Albiac, F., & Kalton, N. J. (2016). *Topics in Banach Space Theory* (2ª ed.). Springer International Publishing, Cham.
7. Pietsch, A. (2007). *History of Banach spaces and linear operators.* Boston, MA: Birkhäuser Boston.
8. Moslehian, M. S., & Ramesh, G. (2019). Compact Operators on Non-Separable Banach Spaces. *Banach Journal of Mathematical Analysis*, *13*(1), 123–135.
9. González, M., & Pello, J. (2025). On the three-space property for subprojective and superprojective Banach spaces. *Mediterranean Journal of Mathematics*, *22*(2), 1-11.
10. Johnson, W. B., & Schechtman, G. (2024). Approximate identities for ideals in L (L p) L(L^p). *Bulletin of the London Mathematical Society*, *56*(12), 3708-3720.
11. Argyros, S. A., & Haydon, R. G. (2011). A hereditarily indecomposable L_∞-space that solves the scalar-plus-compact problem.
12. Gesztesy, F., & Nichols, R. (2015). A Jost–Pais-type reduction of (modified) Fredholm determinants for semi-separable operators in infinite dimensions. *Integral Equations and Operator Theory*, *83*(1), 1–32.
13. Arhancet, C. (2024). Classical harmonic analysis viewed through the prism of noncommutative geometry. *arXiv preprint arXiv:2409.07750*.
14. Albanese, A. A., Bonet, J., & Ricker, W. J. (2025). Mean ergodic and related properties of generalized Cesàro operators in BK-sequence spaces. *Monatshefte für Mathematik, 206*(1), 1-43.
15. Kania, T., & Laustsen, N. J. (2014). Ideal Structure of the Algebra of Bounded Operators on a Banach Space. *Proceedings of the American Mathematical Society, 142*(1), 1–10.