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Research Article

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Classified Foster and Cauer Circuits for the Choice of a Model

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Abstract

Approximations of constant phase elements (CPE) or Warburg impedance are frequently made by equivalent electrical circuits. However, these approximating circuits generally do not take into account the boundary conditions involved by making such approximations. In this paper, we assess various models or possible topologies and classify them according to their asymptotic limits. We show that circuit topologies having the same number and the same types of elements has the same impedance (thus the same frequency behaviour). Having identical factorized impedance proves the equivalence between several proposed models (serial Foster, parallel Foster, serial Cauer and parallel Cauer). We conclude that the choice of a model for parameter identification needs to take into account the specific boundary conditions.

Keywords: Foster Models, Cauer Models, Non-Integer Order, Transfer Functions, Model Identification, Model Topology and Boundary Conditions

Introduction

In the beginning of the XXth century, electrical engineers have developed different filter structures realizing various transfer functions. Among them, we particularly focus on Foster and Cauer topologies [1, 2]. These structures can be developed up to a chosen order by adding stages of RC or RL elements Figure 1 to Figure 3. O. Brune gave in 1931 a table of different LC circuits including four circuit solutions (two of Foster type and two of Cauer type) "*to find the necessary and sufficient conditions to be satisfied by the impedance function of a finite passive network and to construct a network corresponding to any function satisfying these conditions [1].*" Since then, modeling by equivalent electrical networks has been increasingly used in other domains such as electrochemistry, thermal science or linear viscoelastic mechanics [3-5]. Despite of their wide adoption, these kinds of models are often misused due to wrong structure choice.

Attempts were made to extend the approach to the case of infinite-order or non-integer order transfer functions, such as Warburg impedance or Constant Phase Elements (CPE). Both Warburg impedance and CPE are theoretical behavioural models, but real electrochemical systems do not correspond necessarily to Warburg and CPE. Therefore, we propose another method to model these systems. The objective of this paper is to present various Foster and Cauer structures classified according to their boundary conditions at zero and infinity. This study also highlights the equivalences of different models having the same impedance.

The present work does not address the case of multiport systems or meshed networks in power systems [6-8]. Instead, we consider one-port linear systems only characterized by their impedances as functions of frequency. RLC circuits were also studied, but we limit the study to RC or RL structures, with just an extension to the case of LC structures [9-12]. We do not either focus on CPE which are treated in several papers [13-18]. The actual realization of CPE by various electrochemical means is also the object of numerous researches and realizations [17, 19-23]. However, we shall look more how to obtain a well-behaving model from measurements in electrochemical impedance spectroscopy. The accuracy inside the frequency range [F_{min}, F_{max}] is important, but topology choice must also be constrained by the boundary conditions for frequencies $F = 0$ and $F = \infty$.

Sections 2 and 3 of the paper shows how Foster and Cauer structures naturally occur when making mathematical approximations of infinite-order transfer functions under simplifying assumptions and regular geometry. Section 4 is an exhaustive study of the different network topologies with a focus on their asymptotic behaviours. As a matter of example, section 5 presents a use case of choosing a well-behaving model to an electrochemical cell.

Infinite Order Impedances

A large class of physical phenomena are defined as "diffusive" or "propagative". They have in common the same local mathematical formalism coupling two quantities. If we limit the problem to the linear case, we can describe the system of coupled variables (u, i) by the means of electrical analogies and the Laplace transform:

$$\begin{cases}\nabla u = -zi \\ \nabla i = -yu\end{cases} \tag{1}$$

Where z and y can be seen respectively as an impedance and an admittance per unit of length (in the unidimensional case), representing the distributed parameters.

When the geometry of the system is simple enough to explicitly integrate the local equations, we can obtain exact analytical models composed of formal impedances. These impedances are in particular expressed with hyperbolic trigonometry functions or Bessel functions and can be considered as infinite order transfer functions. For example, with Z = zx and Y = yx, the impedance becomes:

$$\boldsymbol{Z} = \sqrt{\frac{z}{y}}\,tanh\left(\sqrt{zy}x\right) = \sqrt{\frac{Z}{Y}}\,tanh\left(\sqrt{ZY}\right) \tag{2}$$

Represents the transfer function between the temperature and the flux at the end of a thermal wall of length x, the other end being maintained at a constant temperature. However, it is also the impedance of a transmission line with short-circuited far-end.

Unfortunately, this infinite order models are sometimes inefficient, particularly for real-time applications. Thus, we present three methods in order to obtain models of reduced orders, represented by electric networks (compact models).

Reduced Order Models

The first two methods are based on the Mittag-Lefler theorem whereas the third relies on the continued fraction theory [24]. Let us consider the hyperbolic tangent that appeared previously in the expression of a particular infinite order impedance. This function can be developed thanks to the Mittag-Lefler theorem.

$$tanh(x) = \sum_{n=1}^{\infty} \frac{2x}{x^2+\left(n\pi-\frac{\pi}{2}\right)^2} \tag{3}$$

As well as its inverse,

$$coth(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2+(n\pi)^2} \tag{4}$$

On the other hand, the same function admits the continued fraction representation [25, 26].

$$tanh(x) = \cfrac{1}{\cfrac{1}{x}+\cfrac{1}{\cfrac{3}{x}+\cfrac{1}{\cfrac{5}{x}+\ddots}}} \tag{5}$$

Continued fractions are well known in number theory to provide a kind of best rational approximation. In the case of the hyperbolic tangent, the convergents of the continued fraction are the Padé approximants of the function. That is to say the optimal approximations when x tends to zero.

$$tanh(x) \approx \frac{1}{\frac{1}{x}} \quad , \quad tanh(x) \approx \cfrac{1}{\cfrac{1}{x}+\cfrac{1}{\cfrac{3}{x}}} = \frac{3x}{3+x^2}$$

$$tanh(x) \approx \cfrac{1}{\cfrac{1}{x}+\cfrac{1}{\cfrac{3}{x}+\cfrac{1}{\cfrac{5}{x}}}} = \frac{15x+x^3}{15+6x^2} \tag{6}$$

Using the three different developments, we can obtain three kinds of approximate models for the impedance Z (eq.7). With the first development, we can write:

$$Z = \sqrt{\frac{Z}{Y}} \sum_{n=1}^{\infty} \frac{2\sqrt{ZY}}{ZY+x\left(n\pi-\frac{\pi}{2}\right)^2} = \sum_{n=1}^{\infty} \frac{1}{\frac{Y}{2}+\frac{\left(n\pi-\frac{\pi}{2}\right)^2}{2Z}} \tag{7}$$

Keeping only the first 2N poles of the impedance brings:

$$Z \approx \sum_{n=1}^{N} \frac{1}{\frac{Y}{2}+\frac{\left(n\pi-\frac{\pi}{2}\right)^2}{2Z}} \tag{8}$$

That is the impedance of the network of the Figure 1, provided we have:

$$Z_n = \frac{2Z}{\left(n\pi-\frac{\pi}{2}\right)^2} \qquad Y_n = \frac{Y}{2} \tag{9}$$

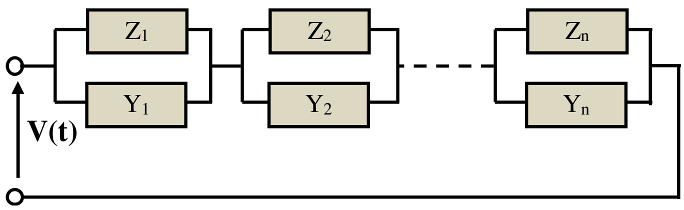


Figure 1: Reduced Model of the First Kind

With the second development (eq.7), the inverse of Z becomes:

$$\frac{1}{\boldsymbol{Z}} = \sqrt{\frac{Y}{Z}}\,coth\left(\sqrt{ZY}\right) = \frac{1}{Z} + \sum_{n=1}^{\infty} \frac{1}{\frac{Z}{2}+\frac{(n\pi)^2}{2Y}} \tag{10}$$

This time we keep the first 2N zeros of the impedance.

$$\frac{1}{\boldsymbol{Z}} \approx \frac{1}{Z} + \sum_{n=1}^{N} \frac{1}{\frac{Z}{2}+\frac{(n\pi)^2}{2Y}} \tag{11}$$

Thus, we can identify the network of the Figure 2 with the following values:

$$Z_1 = Z \quad Y_1 = \infty \tag{12}$$

$$Z_n = \frac{Z}{2} \quad Y_n = \frac{2Y}{\left((n-1)\pi\right)^2} \quad \text{For n > 1} \tag{13}$$

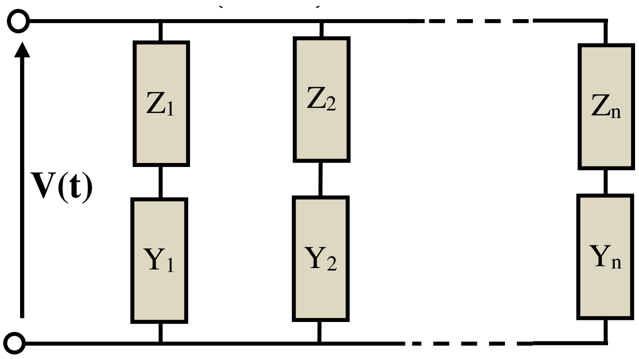


Figure 2: Reduced Model of the Second Kind

The last development (3.5) allows us to express the impedance with a continued fraction:

$$Z=\sqrt{\frac{Z}{Y}}\frac{1}{\frac{1}{\sqrt{ZY}}+\frac{1}{\frac{3}{\sqrt{ZY}}+\frac{1}{\frac{5}{\sqrt{ZY}}+\ddots}}}=\frac{1}{\frac{1}{Z}+\frac{1}{\frac{3}{Y}+\frac{1}{\frac{5}{Z}+\ddots}}} \quad (14)$$

Whose N-th convergent are the impedance of the network Figure 3 composed of elements.

$$Y_1=\infty \quad Z_n=\frac{Z}{4n-3} \quad Y_n=\frac{Y}{4n-5} \quad for\ n\geq 2 \quad (15)$$

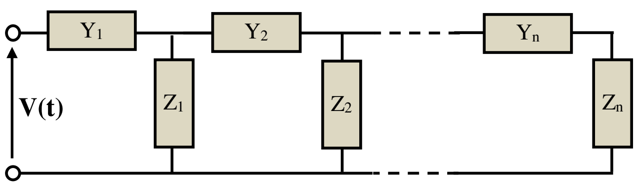


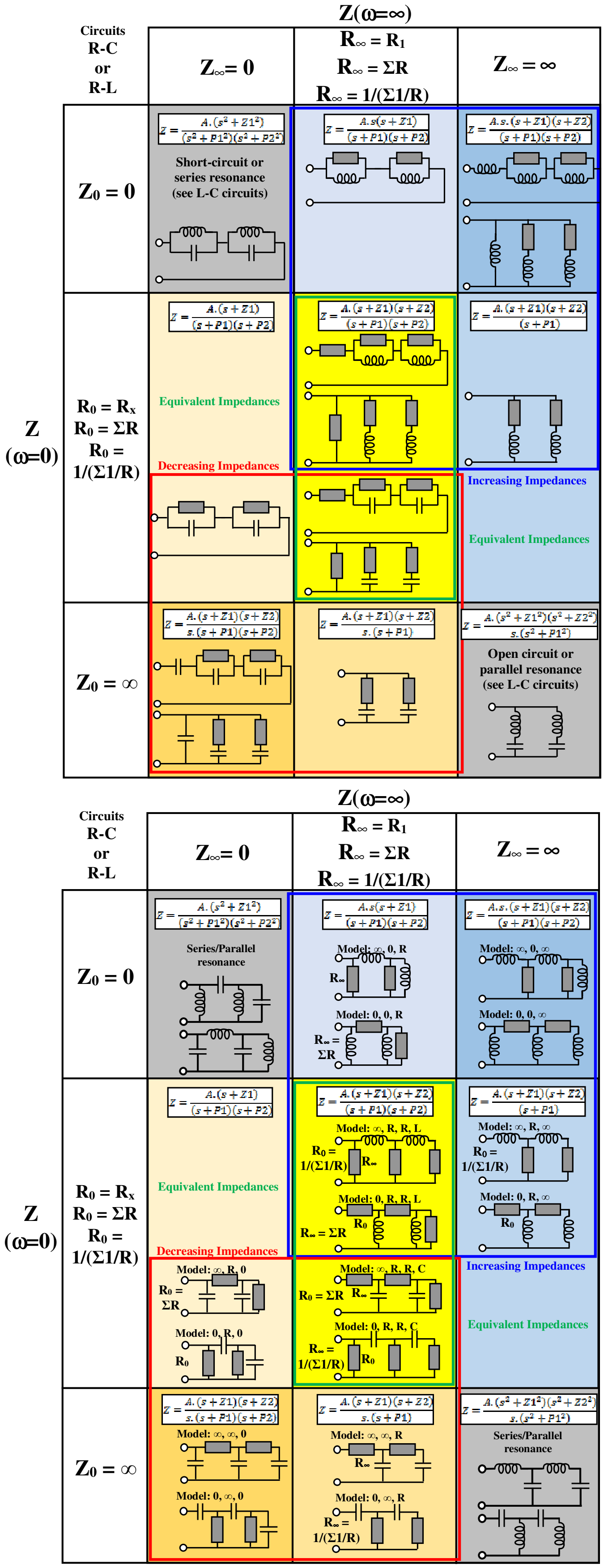
Figure 3: Reduced Model of the Third kind

Then, the Mittag-Lefler theorem and the continued fraction theory ensure the convergence of the approximate models to the infinite-order model while N → ∞. These techniques can be seen as an extension of the Foster and Cauer syntheses from rational functions to the meromorphic ones [27, 28].

Table of Different Foster and Cauer Circuits Classified with their Asymptotic Limits

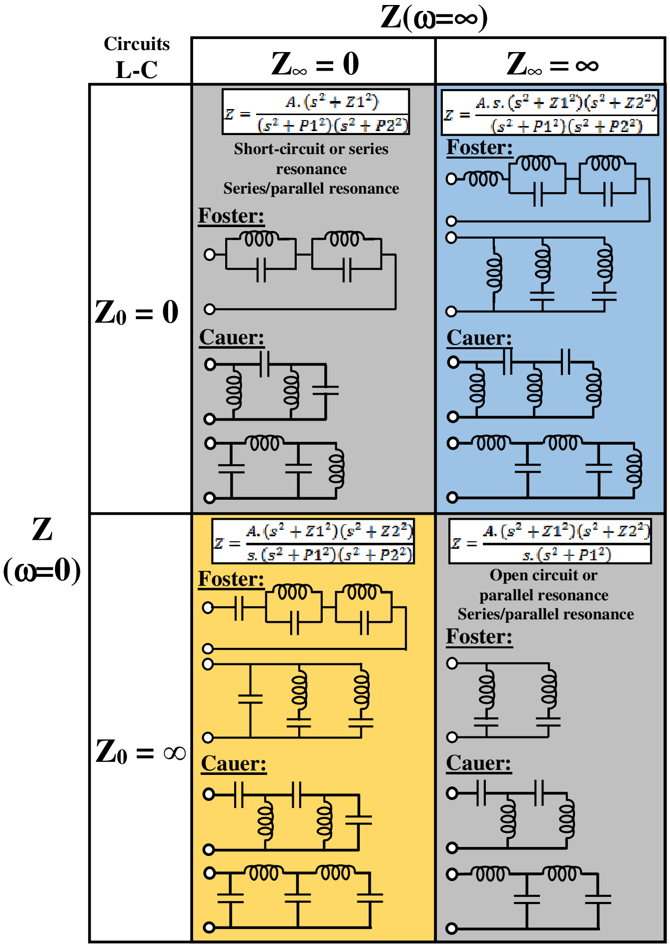
Analysis of Foster and Cauer structures according to their respective boundary conditions lead to nine categories. The classification has been done for the two kinds of models. Various RC and RL Foster circuits can be compiled such as in Table 1. Circuits in each of the nine cells correspond to the same type of factorised impedance and are thus equivalent. A set of parameters for different circuits can give the same impedance with same frequency behaviour. We can identify 14 different Foster-type topologies. Similarly, Table 2 compiles 20 RC and RL circuits with Cauer-type topologies.

Table 1: Foster's RC and RL Circuits for Different Boundary Conditions



For the sake of completeness, Table 3 give the same classification for resonant LC Foster and Cauer structures. The 14 topologies fall only into four categories regarding their asymptotic behaviours.

Table 2: Foster and Cauer LC Circuits for Different Boundary Conditions



We can put in evidence several differences between Foster and Cauer models:

- We can find series and parallel Cauer models for all type of asymptotic behaviour (Table 2). However, this is not the case for series and parallel Foster models. Therefore, Cauer models appears to be more versatile than Foster models.
- In Foster models, the different RC stages are associated either in series or in parallel. Therefore, permutations between positions of RC stages do not affect the value of the impedance. We can say that the structures of these circuits are not ordered. As a consequence, we can have different sets of parameters [R0, C1, R1, …, CN, RN] corresponding to the same circuit. Therefore, to compare two sets of parameters we must order them first.
- In Cauer models, instead, we cannot exchange the elements without affecting the impedance. We can say that these structures of circuits are ordered. Therefore, to compare sets of parameters we do not need to order them. The latter property of Cauer structures is a significant advantage over Foster ones when performing parameter identification. Actually, the possible number of permutations increases with the order N as N! (Factorial N). If there exists an optimum solution, then it also exists (N!-1) other solutions identical by permutations in the case of Foster models. That may reveal problematic for optimization algorithms because of the density of local optima. This is not the case with Cauer models. The optimum or local optima remain in this case unique.
- Series Cauer circuits are a developed from zero frequency and the model improve for high frequency by adding elements at the end RN Figure 4. That is very coherent with the Table 2 if we change boundary conditions. The presence of the resistance R0 changes the boundary condition in high frequency and the presence or not of the resistance RN change the boundary condition in low frequency.

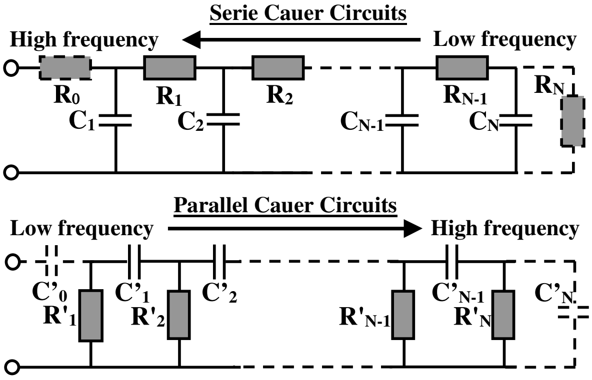


Figure 4: Development from Frequency Infinite of Serie Cauer and Development from Frequency Zero of Parallel Cauer

- Parallel Cauer circuits are developed from low frequencies Figure 4. The presence of the capacitor C'0 change the boundary condition in low frequency and the presence of the capacitor C'N change the boundary condition in high frequency. That is coherent with the Table 2.

Use Case: Identification in Electrochemical Impedance Spectroscopy (EIS)

The first thing is to choose a model with the help of tables in section 4. The choice must take into account boundary conditions of the model for F = 0 and F = ∞. In the same cell of the tables, we have equivalent models which give equivalent solutions. Therefore, the choice between equivalent models is a false problem.

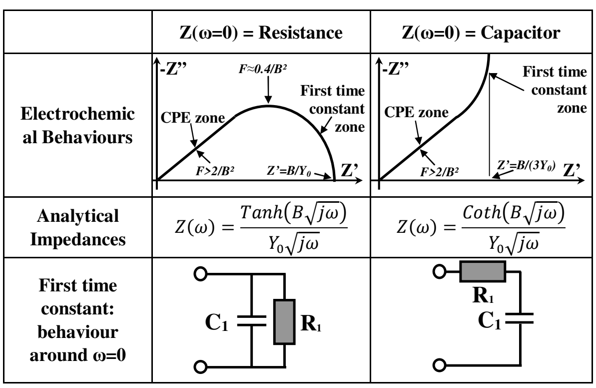


Figure 5: The Behaviours of Warburg Impedances and Analytical Expressions

Let's give a name to each of the four model families: "Tanh", "R+Tanh", "Coth", "R+Coth", which correspond to a transfer function and a boundary conditions:

Tanh: Z(ω=0)=R Z(ω=α)=0

R+Tanh: Z(ω=0)=R Z(ω=α)=R

Coth: Z(ω=0)=α Z(ω=α)=0

R+ Coth: Z(ω=0)=α Z(ω=α)=R

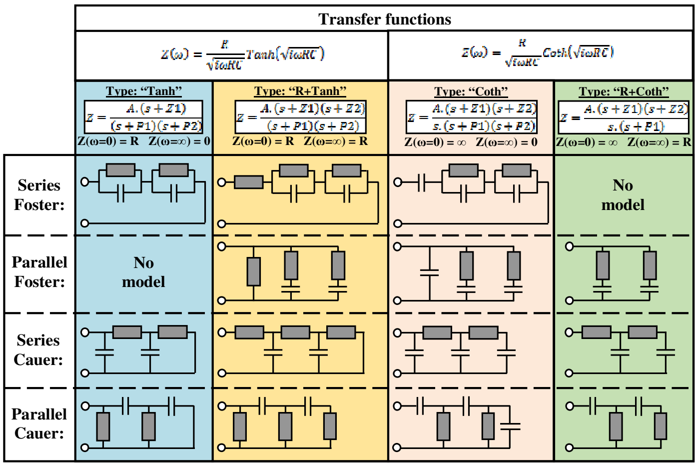


Figure 6: Some Examples of Equivalent Foster and Cauer Network

The principle of parameter optimization is presented Figure 7. An optimization algorithm tries to minimize the distance between the measures and the model by adjusting the parameters. The "fminsearch" function of the Matlab optimization toolbox was used in this example. Optimization of parameters can be applied to all structures, and to the factorised impedance.

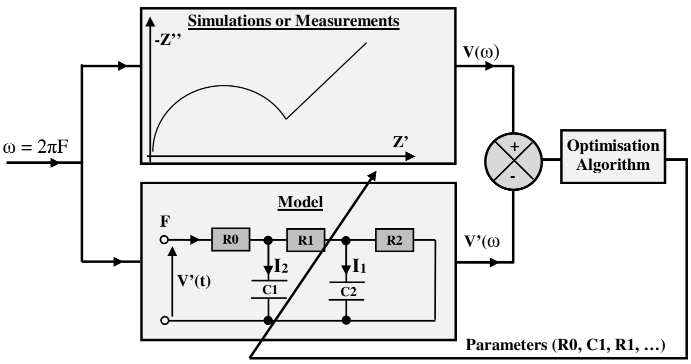


Figure 7: Parameters Identification of Electrochemical Impedance Spectroscopy (EIS)

The interest is to use results found at the order N to initialize the search at the order N+1 that allows being close to the solution and a better convergence. With this method we can go up to order N = 12 with 24 or 25 parameters. However, a model of order N = 4 to N = 6 is in practice enough.

For very high frequencies, an inductance may be added to the model in a second time. Nevertheless, the coil takes into account the length of the wires between the instrument and the electrodes and not electrochemical phenomena.

Impedance Spectroscopy Measurements

Measures of impedance spectroscopy were made with a Solartron 1260 available in our laboratory. The scanning in frequency was set between 0,001 Hz and 10 kHz. The currents of discharge and charge for this type of battery element is being of the order of 1A, the magnitude of the current of measure will be fixed in 100mA to be for the small signals.

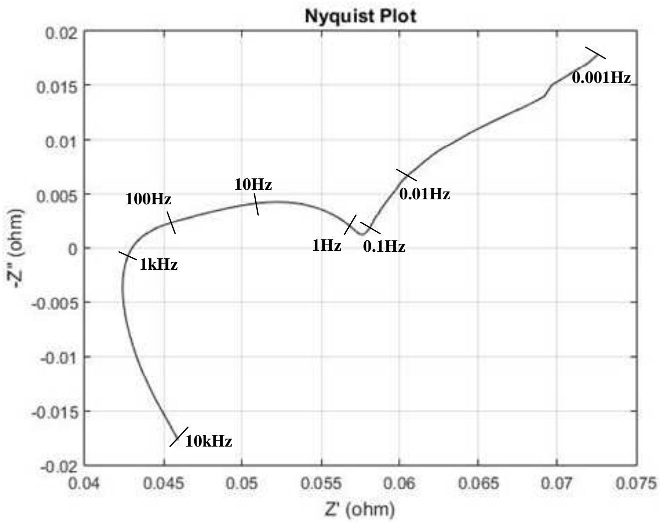


Figure 8: Measures Presented in Nyquist Plot

We have 121 test points (frequency, Z', Z"). Z' being the real part of the impedance and Z" the imaginary part of the impedance. The measures of the impedance made are presented on Figure 8 in the Nyquist plot (polar representation).

We thus obtain excellent measures with a very low noise level. We can notice that the system becomes inductive from 1000 Hz. Derivating a Warburg or CPE function from this curve is clearly not direct.

We chose to fit this experimental curve with 4 models having different transfer functions of and thus different boundary conditions as Figure 9. We shall take the models of the serial Cauer family, which exist for 4 different transfer functions.

The model of series Cauer is a development from high frequencies. If the first condenser C1 becomes negative, it is to reflect the inductive behaviour of the studied system.

Model's Convergence

For 4 various models of serial Cauer of the Figure 6, we identified the various parameters. The presented curves Figure 9 are all for N=6 that is 12 or 13 parameters.

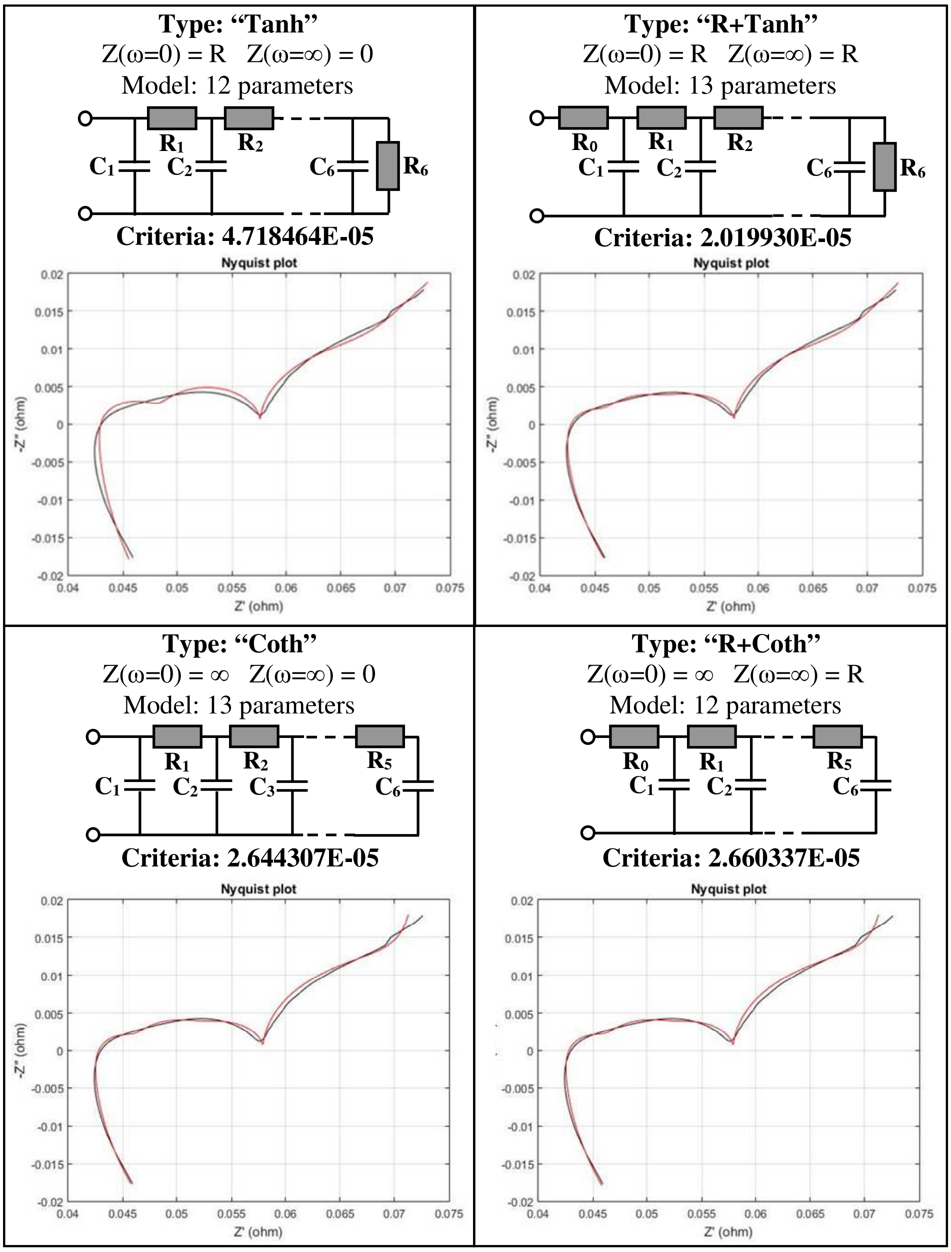


Figure 9: Comparison Measures and Identified Models for N=6 in the Nyquist Plot for 4 Models Studied

For these four various models we have a good convergence. The numbers of parameters are not equivalent either 12 or 13 parameters. The obtained convergence criteria are of the same order and we cannot base ourselves on the criteria to determine that the best model is.

Synthesis and model extrapolations

Although these four models have different boundary conditions, they coincide very well with the measures on a range of 7 decades (0,001 Hz-10k Hz) after parametric optimization.

However, to have another vision of these optimised models, we can wonder how they behave if we extrapolate their behaviours on two decades in low and high frequency (10µ Hz–1M Hz).

The Figure 10 gives the results of the extrapolations obtained in the Nyquist plan for 4 established models. The obtained results are in agreement with the section 4, where the models were classified by their boundary conditions in low frequency $\omega = 0$ and in high frequency $\omega = \infty$.

Only the model of type **"R+Tanh"** with $Z(\omega=0) = R$ $Z(\omega=\infty) = R$ seems to have a reasonable behaviours at the level of the extrapolations in HF and in LF which corresponds well to the behaviours of a battery.

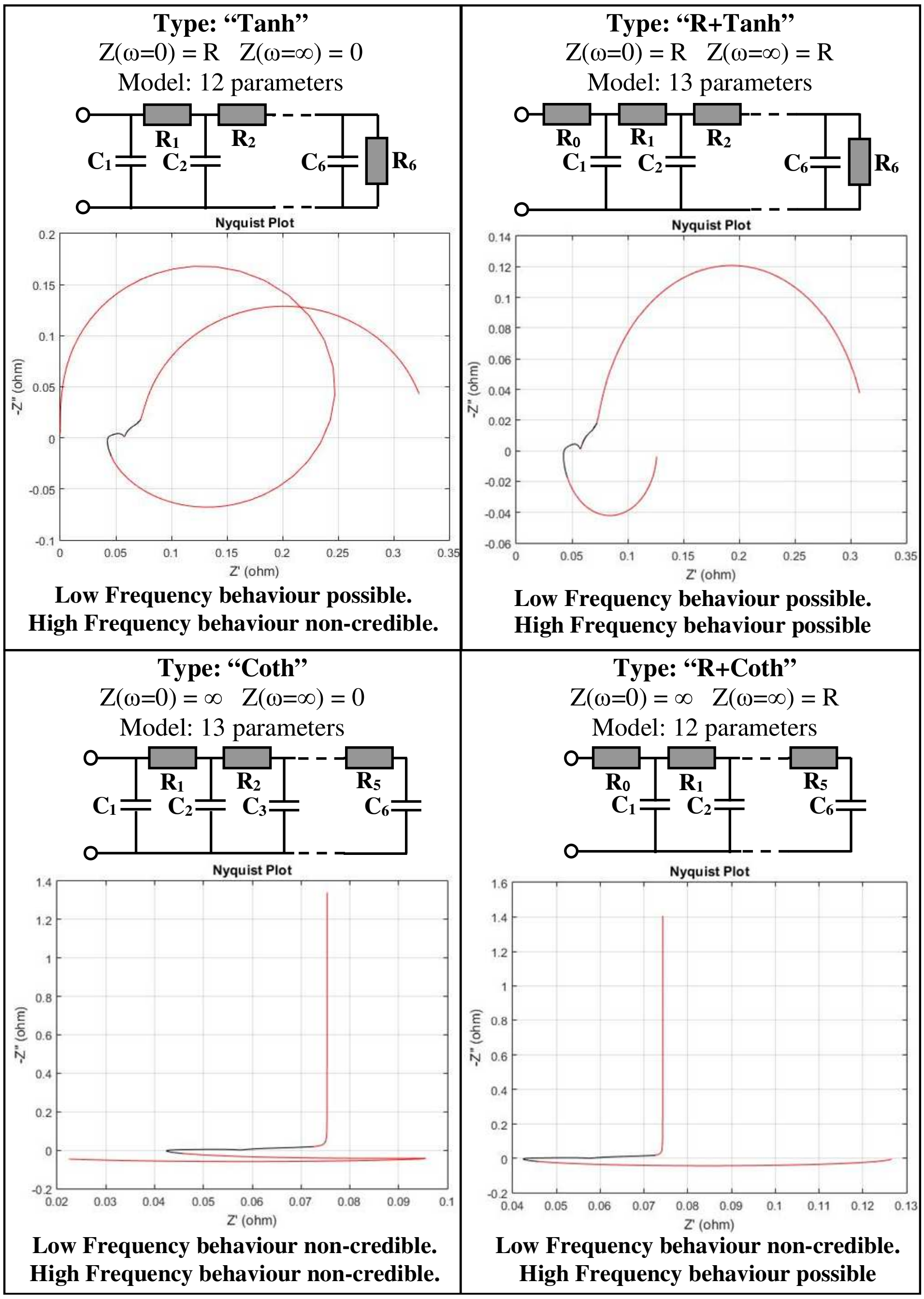


Figure 10: Comparison of Extrapolations in the Nyquist Plot for 4 Models Studied

Conclusion

Many papers give models using elements of Warburg and CPE. We have to underline here that CPE is only a part of the behaviour of Warburg impedance. Both Warburg and CPE are theoretical behaviours models, but the behaviours of real systems do not correspond necessarily to Warburg and CPE. Imposing that the model behaves as a Warburg impedance is restrictive, and an artificial constraint not necessarily justified.

On another hand, the analytical development of functions in Tanh and Coth in the form of continuous fractions leads to the structures of the circuits of Foster and Cauer. Identification of the free parameters of these circuits or their impedances allows obtaining good approximations of experimental measurements, without to impose of constraints. With a sufficient number of parameters, we can obtain a good approximation of any function.

The major result in this paper is the classification of structures regarding the boundary conditions allows to clearly exhibiting the more suitable model between the various topologies, together with their equivalences.

The second important result is the equivalence of circuits, which have the same factorized impedance.

However, good approximations of experimental measurements can hide a bad model if we do not take into account that the model must have the same boundary conditions as the physical system. We made the connection between the various circuits and the boundary conditions, which are fundamental to have a good modelling. A difficulty is that the presence or not of an element at the beginning and at the end of circuit is enough for changing the boundary conditions

A model as a ladder circuit give directly access to the resistance in DC, the resistance in high frequency, and the total capacitance of the system.

Finally, to choose a model needs to choose boundary conditions, which exceed widely the interval of frequencies on which are identified the parameters of the model. Thus, the model contains intrinsically information wider than the bandwidth being of use to its identification, but which are only right if the boundary conditions are well chosen.

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