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Classical Poincaré Conjecture via 4D Topology

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Abstract
The classical Poincaré conjecture that every homotopy 3-sphere is diffeomorphic to the 3-sphere is confirmed by Perelman in arXiv papers solving Thurston's program on geometrizations of 3-manifolds. A new confirmation of this conjecture is given by a method of 4D topology. For this proof, the spun torus-knot of every knot in every homotopy 3-sphere is observed to be a ribbon torus-knot in the 4-sphere, where Smooth 4D Poincaré Conjecture and Ribbonness of a sphere-link with (not necessarily meridian-based) free fundamental group are used. By examining a disk-chord system of a ribbon solid torus bounded by the spun torus-knot, it is proved that the knot belongs to a 3-ball in the homotopy 3-sphere. Then by Bing's result, it is confirmed that the homotopy 3-sphere is diffeomorphic to the 3-sphere.

Keywords: Homotopy 3-Sphere, Spun Torus-Knot, Ribbon solid torus, Chord diagram.

1. Introduction

A *homotopy 3-sphere* is a smooth 3-manifold M homotopy equivalent to the 3-sphere S^3. The following Poincaré Conjecture is positively shown by Perelman in arXiv papers solving positively Thurston's program on geometrizations of 3-manifolds [1-6].

Poincaré Conjecture

Every homotopy 3-sphere M is diffeomorphic to S^3.

A new confirmation of this result is presented here by combining Smooth 4D Poincaré Conjecture and Free Ribbon Lemma for an S^2-link in the 4-sphere S^4 with R. H. Bing's result on Poincaré Conjecture [7,8]. A *homotopy 4-sphere* is a smooth 4-manifold X homotopy equivalent to the 4-sphere S^4. The following conjecture was a folklore conjecture.

Smooth 4D Poincaré Conjecture

Every smooth homotopy 4-sphere X is diffeomorphic to S^4.

The positive proof of this conjecture is shown in [9]. A *surface-link* in S^4 is a surface L smoothly embedded in S^4. When L is connected, it is a *surface-knot.* If all components of L are 2-spheres, then it is an S^2*-link*. A surface-link L in S^4 is *trivial* if L bounds disjoint handlebodies in S^4, and a *ribbon surface-link* if L is equivalent to a surface-link obtained from a trivial S^2-link O by surgery along disjointedly embedded 1-handles on O in S^4. The following *Free Ribbon Lemma is* used in Section 3 [10].

Free Ribbon Lemma

Any S^2-link L in S^4 with free fundamental group $\pi_1(S^4 \setminus L b,)$ is a ribbon S^2-link in S^4.

The proof of this lemma is moved from this preprint version to the paper [10] (for completeness of the argument), which is done by using Smooth 4D Poincaré Conjecture and Smooth Unknotting Conjecture explained as follows:

Smooth Unknotting Conjecture

Every smooth surface-link L in S^4 with a meridian-based free fundamental group $\pi_1(S^4 \setminus L, b)$ is a trivial surface-link.

The proof of this conjecture is shown by [11-13]. Artin's spinning construction of a knot k in S^3 to construct the spun S^2-knot $K(k)$ in the 4-sphere S^4 allows us to generalize to a connected graph γ in every homotopy 3-sphere M to construct the spun S^2-link $K(\gamma)$ in a homotopy 4-sphere $X(M)$ which is diffeomorphic to S^4 by Smooth 4D Poincaré Conjecture, so that $X(M)$ is identified with S^4 [9]. This construction is applied to a Heegaard graph γ of M (associated to a Heegaard splitting of M). Then the spun S^2-link $K(\gamma)$ is an S^2-link in $X(M)$ with free fundamental group (not always meridian-based free group). By Free Ribbon Lemma, the spun S^2- link $K(\gamma)$ is a ribbon S^2-link in $X(M)$. It is observed that for every knot k in every homotopy 3-sphere M, there is a Heegaard graph γ of M such that k is contained in the loop system of $\ell(\gamma)$ of γ. This means that the spun S^2-knot $K(k)$ of every knot k in every homotopy 3-sphere M is a ribbon S^2-knot in $X(M)$. Then, by definition, the spun torus-knot $T(k)$ of every knot k in every homotopy 3-sphere M is a ribbon torus-knot in $X(M)$. Thus, the spun torus-knot $T(k)$ always bounds a ribbon solid torus V_R in $X(M)$. By an argument of a disk-chord system of V_R bounded by the spun torus-knot $T(k)$ in $X(M)$, the following result is shown.

Theorem 1.1

Every knot k in every homotopy 3-sphere M belongs to a 3-ball D^3 in M.

By combining Theorem 1.1 with the following result of Bing in it is proved that every homotopy 3-sphere M is diffeomorphic to S^3 [7,8]. Thus, the proof of Poincaré conjecture is completed.

Bing's Theorem

A homotopy 3-sphere M is diffeomorphic to S^3 if every knot k in M belongs to a 3-ball in M.

Outline of the proof of Poincaré Conjecture is as follows:

(1st Step) By using Smooth 4D Poincaré Conjecture, show that Artin's spinning construction of every Heegaard graph γ of every homotopy 3-sphere M gives a spun S^2-link $K(\gamma)$ in S^4 with free fundamental group (not always meridian-based free group).
(2nd Step) By Free Ribbon Lemma, the spun S^2-link $K(\gamma)$ is a ribbon S^2-link in S^4.
(3rd Step) Show that every knot k in M is contained in a loop system $\ell(\gamma)$ of a Heegaard graph γ of M, so that the spun S^2-knot $K(k)$ of k is a ribbon S^2-knot in S^4.
(4th Step) By definition of a ribbon surface-knot, show that the spun torus-knot $T(k)$ of k in M is a ribbon torus-knot in S^4.
(5th Step) By using a ribbon solid torus V_R bounded by the spun torus-knot $T(k)$ in S^4 and a disk-chord system of V_R, show that K belongs to a 3-ball D^3 in M.
(6th Step) By Bing's theorem, M is diffeomorphic to S^3.

In Section 2, Artin's spinning construction of a connected graph in a homotopy 3-sphere is explained. In Section 3, an argument of a disk-chord system of a ribbon solid torus bounded by a ribbon torus-knot is explained. In Section 4, the proof of Theorem 1.1 is done.

2. Artin's Spinning Construction of a Connected Graph in a Homotopy 3- Sphere

Throughout this section, M denotes a homotopy 3-sphere unless otherwise mentioned. For a homotopy 3-sphere M, let $M^{(o)}$ be the compact once-punctured manifold $\mathrm{cl}(M \setminus B)$ of M for a 3-ball B in M. Let

$$S = \partial B = \partial M^{(o)}$$

be the boundary 2-sphere of $M^{(o)}$. The closed smooth 4-manifold $X(M)$ defined by

$$X(M) = M^{(o)} \times S^1 \cup S \times D^2$$

is called the *spun manifold* of M with axis 4-submanifold $S \times D^2$. As a convention, the 3-submanifold $M^{(o)} \times 1$ of the product $M^{(o)} \times S^1$ is identified with $M^{(o)}$. In particular, a point $(q, 1) \in M^{(o)} \times 1$ is identified with the point $q \in M^{(o)}$. This 4-manifold $X(M)$ is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument and hence $X(M)$ is diffeomorphic to the 4-sphere S^4 by Smooth 4D Poincaré Conjecture. *From now on, the identification $X(M) = S^4$ is fixed. A legged loop with base point* v is the union $k \cup \omega$ of a loop k and an arc ω joining the base point v with a point of k. The arc ω is called a *leg*. A legged loop system with base point v is the union

$$\gamma = \cup_{i=1}^{n} \ell_i \cup \omega_i$$

of n legged loops $\ell_i \cup \omega_i$ $(i = 1, 2, \ldots, n)$ meeting only at the same base point v. Let $\ell(\gamma) = \cup_{i=1}^{n} \ell_i = \ell_*$ denote be the loop system of the legged loop system γ. Let $\omega_* = \cup_{i=1}^{n} \omega_i$ and $v_* = \ell_* \cap \omega_*$. A regular neighborhood B of $\omega*$ in M is taken as a 3-ball B used for the compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of M. Deform the subgraph $\gamma \cap B$ of γ so that

$$\omega_* \subset B, \quad \omega_* \cap S = v_* \quad \text{and} \quad \ell_* \cap B = \ell_* \cap S = a'_*$$

for a regular neighborhood arc system $a'*$ of v_* in ℓ_*. Let

$$a(\gamma) = \cup_{i=1}^{n} a_i = a_*$$

for a proper arc $a_i = \mathrm{cl}(\ell_i \setminus a'_i)$ $(i = 1, 2, \ldots, n)$ in $M^{(o)}$. Let

$$\dot{a}(\gamma) = \partial a_* = \partial a'_*$$

be the set of $2n$ points in the boundary 2-sphere S of $M^{(o)}$. The *spun* S^2-link of the graph γ is the S^2-link $K(\gamma)$ in the 4-sphere $X(M)$ defined by

$$K(\gamma) = a(\gamma) \times S^1 \cup \dot{a}(\gamma) \times D^2.$$

Lemma 2.1

The inclusion $M^{(o)} \setminus a(\gamma) \subset X(M) \setminus K(\gamma)$ induces an isomorphism

$$\sigma : \pi_1(M \setminus \gamma, v^+) \to \pi_1(X(M) \setminus K(\gamma), v^+)$$

sending a meridian system of the proper arc system $a(\gamma)$ in $M^{(o)}$ to a meridian system of $K(\gamma)$, where the base point v^+ is taken in $S \setminus a'_*$

Proof of Lemma 2.1

Note that there is a canonical isomorphism

$$\pi_1(M^{(o)} \setminus a(\gamma), v^+) \cong \pi_1(M \setminus \gamma, v^+).$$

Then the desired isomorphism σ is obtained by applying the van Kampen theorem between $(M^{(o)} \setminus a(\gamma)) \times S^1$ *and* $(S \setminus \dot{a}(\gamma)) \times D^2$. This completes the proof of Lemma 2.1.

Here is a note on Lemma 2.1.

Note 2.2

Ageneral connected graph γ with Euler characteristic $\chi(\gamma) = 1 - n$ in M is deformed into a legged loop system γ in M by choosing a maximal tree to shrink to a base point v. Note that there are only finitely many maximal trees of γ such that the loop systems $\ell(\gamma)$ of the resulting legged loop systems γ are distinct as links. By Lemma 2.1, we can obtain finitely many distinct spun S^2-links in S^4 with isomorphic fundamental groups obtained by taking different maximal trees of the connected graph γ. This is a detailed explanation on the spun S^2-link of a connected graph associated with a maximal tree in [23, p.204] when $M = S^3$.

When a homotopy 3-sphere M is given by a Heegaard spitting $V \cup V'$ pasting along a Heegaard surface $F = \partial V = \partial V'$ of genus n, a legged loop system γ with loop system $\ell(\gamma)$ of $2n$ loops is constructed as follows. A *spine* of a handlebody V of genus n is a legged loop system γV in $F = \partial V$ with base point v such that the inclusion map $\gamma V \to V$ induces an isomorphism $\pi_1(\gamma, v) \to \pi_1(V, v)$. A regular neighborhood $\dot{V}$ of γV in F is a planar surface in F. By [22, Theorem 10.2], there is a diffeomorphism $(V \times [0, 1], \dot{V} \times 0) \to (V, \dot{V})$ sending every point $(x, 0) \in \dot{V} \times 0$ to $x \in \dot{V}$. The surface V is called a *spine surface* of V. Let γ_V and γ_V' be spines of the handlebodies V and V' in F with the same base point v, respectively. A *Heegaard graph* of M is a legged loop system $\gamma = \gamma M$ in M with base point v which is the union of legged loop systems γ^+_V and $\gamma^+_{V'}$ obtained from γ_V *and* $\gamma_{V'}$, by pushing $\gamma V \setminus v$ and $\gamma_{V'} \setminus v$ into the interiors Int$V$ and IntV', respectively. The following lemma is obtained.

Lemma 2.3

For every Heegaard graph γ of every homotopy 3-sphere M , the fundamental group $\pi_1(X(M) \setminus K(\gamma), v^+)$ of the spun S^2-link $K(\gamma)$ in the 4-sphere $X(M)$ is a free group of rank $2n$.

Proof of Lemma 2.3

The closed complement cl$(M \setminus N(\gamma))$ for a regular neighborhood $N(\gamma)$ of γ in M is diffeomorphic to the handlebody $F^{(o)} \times [-1, 1]$ for the once-punctured surface $F^{(o)}$ of F. Since the fundamental group $\pi_1(F^{(o)} \times [0, 1], v^+)$ with base point v^+ taken in $(\partial F^{(o)}) \times [0, 1]$ is a free group of rank $2n$, the desired result is obtained from Lemma 2.1.

It is noted that this free group in Lemma 2.3 is not necessarily a meridian-based free group. Here is an example.

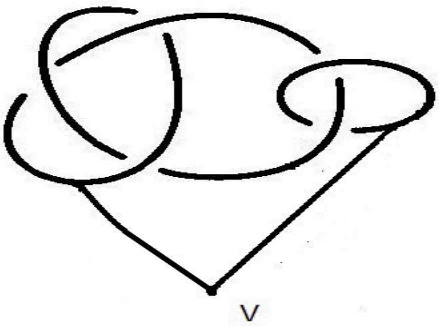

Figure 1: A legged loop system γ in S^3 with free fundamental group of rank 2

Example 2.4

Let γ be a legged loop system with base point v in $M = S^3$ illustrated in Figure 1 with $\pi_1(M \setminus \gamma, v^+)$ a free group of rank 2. In fact, a trivial legged loop system is obtained by sliding an edge along another edge, so that $\pi_1(M \setminus \ell(\gamma), v^+)$ is a free group of rank 2. A regular neighborhood V of γ in M and the closed complement $V' = $ cl$(M \setminus V)$ constitute a genus 2 Heegaard splitting $V \cup V'$ of M by noting that the 3-manifold V' is a handlebody of genus 2 by the loop theorem and the Alexander theorem (cf. e.g., [23]). Thus, the union $V \cup V'$ is a genus 2 Heegaard splitting of M . Since the legged loop system γ with base point v is a spine of V by sliding the base point v into ∂V, there is a Heegaard graph γ_M of M with γ as γ^+_V . By Lemma 2.3, the spun S^2-link $K(\gamma_M)$ in the 4-sphere $X(M) = S^4$ has the free fundamental group $\pi_1(X(M) \setminus K(\gamma_M), v^+)$ of rank 4, which does not admit any meridian basis because the spun S^2-link $K(\gamma M)$ in S^4 contains, as a component, the spun trefoil S^2-knot whose fundamental group is known to be not infinite cyclic.

Given a proper arc system a_* in $M^{(o)}$, there is a legged loop system γ in M with the proper arc system $a(\gamma) = a_*$ in $M^{(o)}$. The spun S^2-link $K(\gamma)$ in $X(M)$ is uniquely determined by the arc system a_* and thus denoted by $S(a_*)$. The following lemma is used toward the final step of the proof of Poincaré conjecture.

Lemma 2.5

Let a_* be a proper arc system in a compact once-punctured manifold $M^{(o)} = $ cl$(M \setminus B)$ of a homotopy 3-sphere M. If the spun S^2-link $S(a_*)$ in the 4- sphere $X(M)$ is a trivial S^2-link, then the proper arc system a_* is in a boundary-collar $S \times [0, 1]$ of $M^{(o)}$.

Proof of Lemma 2.5

By Lemma 2.1, the fundamental group $\pi_1(M^{(o)} \setminus a(\gamma), v^+)$ is a meridian-based free group. Consider the 2-sphere S as the boundary

$$\partial(d \times [0, 1]) = d \times 0 \cup (\partial d) \times [0, 1] \cup d \times 1$$

of the product $d \times [0, 1]$ for a disk d so that $d \times 0$ contains one end of the proper arc system a_* and $d\times 1$ contains the other end of the proper arc system a_*. Let $(E; E_0, E_1)$ be the triplet obtained from $(M^{(o)}, d \times 0, d \times 1)$ by removing a tubular neighborhood of a_* in $M^{(o)}$. For $v^+ \in E_0$, the inclusion $E_0 \subset E$ induces an isomorphism

$$\pi_1(E_0, v^+) \to \pi_1(E, v^+).$$

By [22, Theorem 10.2], E is diffeomorphic to the connected sum of the product $E_0 \times [0, 1]$ and a homotopy 3-sphere. This means that the proper arc system a_* is in a boundary-collar $S \times [0, 1]$. This completes the proof of Lemma 2.5.

3. A Ribbon Surface-Link and a Disk-Chord System of A Ribbon Handle Body System

By combining Lemmas 2.3 with Free Ribbon Lemma in Section 1, the following lemma is obtained.

Lemma 3.1

The spun S^2-link $K(\gamma)$ of every Heegaard link γ of every homotopy 3-sphere M is a ribbon S^2 link in $X(M)$.

The following lemma makes a connection between a knot in M and a Heegaard graph of M.

Lemma 3.2

For every knot k in every homotopy 3-sphere M, there is a Heegaard graph γ of M such that the knot k is equivalent to a component of the loop system $\ell(\gamma)$ of γ in M.

Proof of Lemma 3.2

By considering k as a polygonal loop in M, there is a triangulation T of M whose 1-skeleton $T^{(1)}$ contains the knot k. The graph $T^{(1)}$ is deformed into a legged loop system γ in M so that k is a component of the loop system $k(\gamma')$. Let V' be a regular neighborhood of γ' in M which is a handlebody. The legged loop system γ' is deformed into a spine $\gamma_{V'}$ of the handlebody V'. The closed complement $V = \mathrm{cl}(M \setminus V')$ is also a handlebody, so that there is a Heegaard splitting $V \cup V'$ of M. Hence there is a Heegaard graph γ of M obtained from γ_V and $\gamma_{V'}$ such that k is equivalent to a component of the loop system $\ell(\gamma)$.

By Lemma 3.2, there is a Heegaard graph γ of M whose loop system contains the knot k. By Lemma 3.1, the spun S^2-link $K(\gamma)$ is a ribbon S^2-link in $X(M)$, so that the spun S^2-knot $K(k)$ is a ribbon S^2-knot in $X(M)$ because any component of a ribbon S^2-link in S^4 is a ribbon S^2-knot in S^4 by definition. Thus, the following result is obtained.

Lemma 3.3

For every knot k in every homotopy 3-sphere M, the spun S^2-knot $K(k)$ is a ribbon S^2-knot in $X(M)$.

For a knot k in the interior of $M^{(0)} = \mathrm{cl}(M \setminus B)$ for a 3-ball B, the *spun torus-knot* of k is a torus-knot $T(k)$ in $X(M)$ given by the inclusions $T(k) = k \times S^1 \subset M^{(o)} \times S^1 \subset M^{(o)} \times S^1 \cup S \times D^2 = X(M)$. The spun torus-knot $T(k)$ in $X(M)$ is uniquely constructed up to choices of a 3-ball B. The following lemma is important to our purpose.

Lemma 3.4

For every knot k in every homotopy 3-sphere M, the spun torus-knot $T(k)$ is a ribbon torus-knot in $X(M)$.

Proof of Lemma 3.4

From construction, the spun S^2-knot $K(k)$ in $X(M)$ is obtained from $T(k)$ by the unique 2-handle surgery, so that the spun torus-knot $T(k)$ is obtained from the spun S^2-knot $K(k)$ by the converse 1-handle surgery. By definition, the spun torus-knot $T(k)$ is a ribbon torus-knot, completing the proof.

Assume that a ribbon surface-link L is obtained from a trivial oriented S^2-link O by surgery along a 1-handle system $h*$ of disjointedly embedded oriented 1-handles h_j ($j = 1, 2, \ldots, s$) (for some s) on O in S^4. A ribbon handlebody system bounded by a ribbon surface-link is discussed here [15]. Let B_* be a system of disjoint 3-balls B_i ($i = 1, 2, \ldots, m$) in S^4 bounded by O.

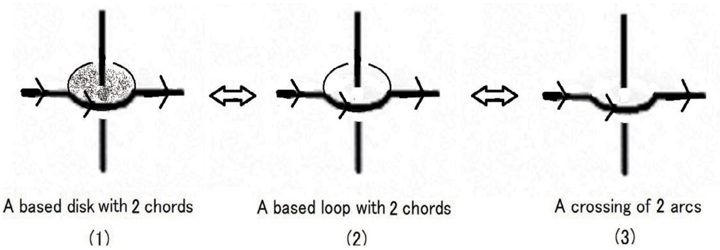

Figure 2: Two arcs of k near a disk d_i drawn as thick lines

The intersection $h_j \cap O$ *consists of two disks, called the attaching disks* of h_j to O. A *meridian disk* of the 1-handle h_j is a proper disk in h_j parallel to any one of the attaching disks. By an isotopic deformation of the 1-handle system h_*, the intersection $h_* \cap \mathrm{Int}B_i$ can be assumed to be a meridian disk system (possible empty) in h_*, whose number of meridian disks is called the *ribbon index* of h_* in B_i. A *ribbon handlebody system* of a ribbon surface-link L is the union $V_R = B_* \cup h_*$, which is an immersed handlebody system bounded by L in S^4. The *ribbon index* of V_R is the total number of the ribbon indexes of $h*$ in B_i for all i. The *disk-chord system* of a ribbon surface-link L is the pair (d_*, α_*) of a disk system d_*, called a *based disk system*, and an arc system α_*, called a *chord system*, in S^4 obtained from the ribbon handlebody system $V_R = B_* \cup h_*$ by shrinking the 3-ball B_i into a disk d_i for every i and then shrinking the 1-handle h_j into a core arc α_j of h_j spanning the loop system $o_* = \partial d_*$, called a *based loop system,* for every j. See Figure 2 (1) for a situation around a disk in a based disk system. From construction, the ribbon index of h_* in B_i is equal to the number of the transverse intersection points $\alpha_* \cap \mathrm{Int}d_i$, called the *chord index* of α_* in d_i. The *chord index* of the disk-chord system (d_*, α_*) is the total number of the chord indexes of α_* in d_i for all i. By the orientations of L and S^4, the based disk system d_* can be uniquely oriented, and the ribbon handlebody system V_R and the ribbon surface-link L are uniquely recovered from the disk-chord system (d_*, α_*) by thickening the chord system α_* and the based disk system d_*, where an argument is needed for uniqueness of the embedded 1-handle system [16]. Let $\Delta^2 \subset \Delta^3 \subset \Delta^4$ be the inclusions such that Δ^4 is a 4-ball in S^4, Δ^3 is a proper 3-ball of Δ^4 and Δ^2 is a proper disk of Δ^3. A disk-chord system (d_*, α_*) of L in S^4 can be moved into $\mathrm{Int}\Delta^3$ isotopically by first moving a neighborhood of the based disk system d_* into $\mathrm{Int}\Delta^3$ and then moving the remaining part of the arc system α_* into $\mathrm{Int}\Delta^3$ [15]. So, assume that a disk-chord system (d_*, α_*) of L is in $\mathrm{Int}\Delta^3$. The ribbon handlebody system V_R and the ribbon surface-link L are uniquely realized from a disk-chord system (d_*, α_*) of L in $\mathrm{Int}\Delta^4$. A *chord graph* of L is the graph $o_* \cup \alpha_*$ in $\mathrm{Int}\Delta^3$ obtained from a disk-chord system (d_*, α_*) in $\mathrm{Int}\Delta^3$ by taking $o_* = \partial d_*$. A *chord diagram* of L is a diagram $C(o_*, \alpha_*)$ in $\mathrm{Int}\Delta^2$ for a chord graph $o_* \cup \alpha_*$ of L in $\mathrm{Int}\Delta^3$. A ribbon surface-link L in S^4 is uniquely realized in $\mathrm{Int}\Delta^4$ from a chord graph $o_* \cup \alpha_*$ of L in $\mathrm{Int}\Delta^3$ and also from a chord diagram $C(o_*, \alpha_*)$ of L in $\mathrm{Int}\Delta^2$, because the based loop system o_* in $\mathrm{Int}\Delta^3$ constructs uniquely the trivial S^2-link O by the Horibe-Yanagawa lemma in [15]. On the other hand, a ribbon handlebody system V_R of L cannot be uniquely recovered because in general a disjoint disk system d_* in the interior of Δ^3 with $\partial d_* = o_*$ is not unique [15]. So, to fix a ribbon handlebody system V_R of L, every loop of the based loop system o_* should be fixed as it is shown in Figure. 2 (2). The following observation is obtained from the above argument.

Observation 3.5

A ribbon surface-link L and a ribbon handlebody system V_R in S^4 are uniquely realized in IntΔ^4 from a disk-chord system (d_*, α_*) in IntΔ^3, and also from a chord graph $o_* \cup \alpha_*$ in IntΔ^3 or a chord diagram $C(o_*, \alpha_*)$ in IntΔ^2 by fixing every loop of the based loop system o_* as it is shown in Figure 2 (2).

A chord diagram has the advantage of being easy to handle. For example, the moves on chord diagrams for equivalent ribbon surface-links are known in [18-21]. A ribbon handlebody V_R bounded by a ribbon torus-knot T is called a *ribbon solid torus*. The following lemma is an easy exercise of the moves on chord diagrams in [18] and used in Section 4.

Lemma 3.6

Every ribbon solid torus of ribbon index n bounded by a ribbon torus- knot T in IntΔ^4 is deformed into a ribbon solid torus V_R with $\partial V_R = T$ which is realized by a disk-chord system (d_*, α_*) in IntΔ^3 of IntΔ^4 where

$$d_* = \{d_i | \, i = 1, 2, \ldots, n\}, \alpha_* = \{\alpha_i | \, i = 1, 2, \ldots, n\} \quad \text{and} \quad o_* = \partial d_*$$

such that

(1) the chord α_i connects o_i to o_{i+1} for every i ($i = 1, 2, \ldots, n$) with $o_{n+1} = o_1$, and

(2) the chord index of α_* to d_i is equal to 1 for every i.

The disk-chord system (d_*, α_*) in Lemma 3.6 is *called a circular primitive disk-chord system* or briefly a *CP disk-chord system* (see Figure 3 (1), (2) for examples). The *spine* of a disk-chord system (d_*, α_*) is a graph Γ obtained from $d_* \cup \alpha_*$ by shrinking every disk d_i into a vertex v_i for every i. A *regular maximal tree* of Γ is a tree τ^+ in Γ obtained from a maximal tree τ of Γ by taking a regular neighborhood of τ in Γ. *A regular maximal tree* of a disk-chord system (d_*, α_*) is a disk-chord system $\tau^+(d_*, \alpha_*)$ obtained from a regular maximal tree τ^+ of the spine Γ by making every vertex v_i in τ^+ back to the original disk d_i for every i. Let $\partial\tau^+(d_*, \alpha_*) = \partial \tau^+$ be the set of all the degree 1 vertexes of τ^+. The arc system $e_* = \mathrm{cl}(\Gamma \setminus \tau^+) = \mathrm{cl}((d_* \cup \alpha_*) \setminus \tau^+(d_*, \alpha_*))$ is called the *complementary arc system* of a regular maximal tree $\tau^+(d_*, \alpha_*)$ in a disk-chord system (d_*, α_*).

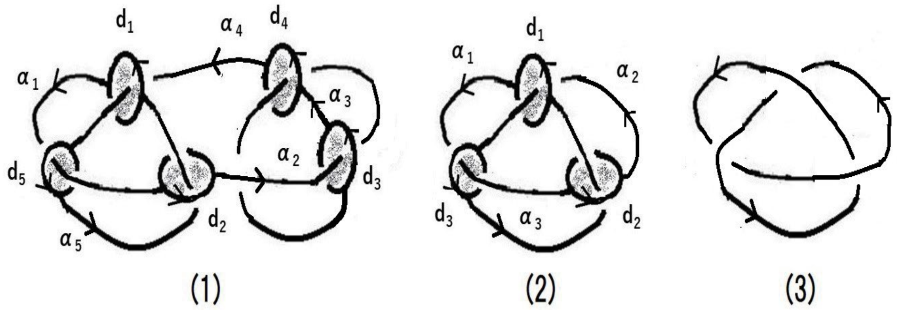

Figure 3: CP disk-chord systems of ribbon solid tori (1), (2) bounded by the spun torus-kot of the trefoil knot (3)

4. Main Result: Proof of Theorem 1.1

Throughout this section, the proof of Theorem 1.1 is done. Let k be a knot in a homotopy 3-sphere M. If k is a trivial knot in M, then the knot k belongs to a 3-ball D^3 in M. So, assume that k is a non-trivial oriented knot in M. Since the spun torus-knot $T(k)$ is a ribbon torus-knot in $X(M)$ by Lemma 3.4, there is a ribbon solid torus V_R of some ribbon index n with $\partial V_R = T(k)$ in IntΔ^4 which is realized by a CP disk-chord system (d_*, α_*) of chord index n in IntΔ^3 and a chord diagram $C(d_*, \alpha_*)$ in IntΔ^2 by Observation 3.5. Since there is a meridian-preserving isomorphism $\pi_1(M \setminus k, v^+) \rightarrow \pi_1(X(M) \setminus T(k), v^+)$ by the van Kampen theorem, the longitude of k in M represents an infinite order element in the fundamental group $\pi_1(X(M) \setminus T(k), v^+)$. This implies that an oriented meridian loop of V_R is a uniquely determined loop in $T(k)$ up to isotopies of $T(k)$, and the CP disk-chord system (d_*, α_*) is assumed that k meets d_i with just one boundary arc and just one interior point transversely for every i, as in Figure 2 (1) (see also Figure 3 (1), (2) for examples). Assume that k is in Int$M^{(o)}$. The following lemma is obtained.

Sublemma 4.1

The disk system d_i ($i = 1, 2, \ldots, n$) is deformed into Int$M^{(o)}$ by an isotopy of $X(M)$ keeping the knot k fixed.

Proof of Sublemma 4.1

For every i, let c_i be a simple arc in d_i connecting the point $k \cap \mathrm{Int}(d_i)$ to a point in the arc $k \cap \partial d_i$. The arc system c_i ($i = 1, 2, \ldots, n$) is deformed into a bi-collar neighborhood $M^{(o)} \times [-1, 1]$ of $M^{(o)}$ with $M^{(o)} \times 0 = M^{(o)}$ in $X(M)$ by an isotopy keeping $M^{(o)}$ fixed. Then the arc system c_i ($i = 1, 2, \ldots, n$) is projected into $M^{(o)}$ by a general position argument. A deformed disk system d_i ($i = 1, 2, \ldots, n$) in $M^{(o)}$ is obtained from the arc system c_i($i = 1, 2, \ldots, n$) in $M^{(o)}$ by widening them as a small disk system, completing the proof of Sublemma 4.1.

By Sublemma 4.1, consider that the CP disk-chord system $(d_*, \alpha*)$ of V_R is in $M^{(o)}$. The spine Γ of (d_*, α_*) is a degree 4 graph in $M^{(o)}$. For every regular maximal tree τ^+ of Γ, there is a disk δ^2 in $M^{(o)}$ with $\partial \tau^+ = \tau^+ \cap \partial\delta^2$ such that a neighborhood of every degree 4 vertex of τ^+ in δ^2 gives Figure 2 (1) in $\tau^+(d_*, \alpha_*)$. The disk δ^2 is called a *regular support disk* for $\tau^+(d_*, \alpha_*)$. This disk δ^2 is moved into the 2-sphere $S = \partial M^{(o)}$. Let $\delta^3 = \delta^2 \times [0, 1]$ be a collar of δ^2 in $M^{(o)}$ which is a 3-ball with $\delta^3 \cap S = \delta^2 \times 0 = \delta^2$. Let e_* be the complementary arc system of $\tau^+(d_*, \alpha_*)$ in (d_*, α_*) consisting of arcs e_i ($i = 1, 2, \ldots, n+1$), where n is the chord index of the CP disk-chord system (d_*, α_*) which is determined by the Euler characteristics $\chi(\Gamma) = -n$. The knot k in $M^{(o)}$ is deformed in $M^{(o)}$ so that the intersection $t = k \cap \delta^3$ is a tangle in δ^3 whose projection image under the canonical projection

$$\delta^3 = \delta^2 \times [0, 1] \rightarrow \delta^2$$

is the regular maximal tree τ^+ in the regular support disk δ^2 by pushing $\tau^+(d_*, \alpha_*) \setminus \partial\tau^+(d_*, \alpha_*)$ into $\delta^2 \times (0, 1)$ and then by creating a crossing point by the move from (1) to (3) in Figure 2. Then the regular maximal tree τ^+ in δ^2 can be regarded as a tangle diagram of t in δ^2. Let $[t, \tau^+]$ be the disk union between the tangle t and the graph τ^+ in the preimage of τ^+ under the canonical projection $\delta^3 \rightarrow \delta^2$. The following sublemma is essentially observed for an inbound arc diagram [17].

Sublemma 4.2 The spun S^2-link $T(t)$ of a tangle t in δ^3 in the 4-disk

$$U^4 = \delta^3 \times [0,1] \times S^1 \cup \delta^2 \times D^2 \subset M^{(o)} \times S^1 \cup S \times D^2 = X(M)$$

bounds a ribbon 3-ball system

$$V'_R = [t, \tau^+] \times S^1 \cup \tau^+ \times D^2$$

which extends to a ribbon solid torus V_R of the spun torus-knot $T(k)$ such that the compact complement $\mathrm{cl}(V_R \setminus V'_R)$ is a disjoint 3-ball system bounded by the spun S^2-link $S(e_*)$ in $X(M)$.

Proof of Sublemma 4.2

If t is a 1-string tangle with τ^+ a simple arc, then $V'_R = [t, \tau +] \times S^1 \cup \tau^+ \times D^2$ is a 1-handle thickening t, that is a ribbon 3-ball with ribbon index 0. If t is a 2-string tangle with τ^+ just one degree 4 vertex graph, then t is the 2-tangle in Figure 2 (3) and V_R' is a ribbon 3-ball system with ribbon index 1 giving the disk chord system of Figure 2 (1). In the general case of t and τ^+, as a combination result of these two observations, V'_R is a ribbon 3-ball system giving a disk-chord system $\tau^U(d_*, \alpha_*)$ in the 4-disk U^4 such that $\tau^U(d_*, \alpha_*)$ is diffeomorphic to the regular maximal tree $\tau^+(d_*, \alpha_*)$ of (d_*, α_*) in δ^3. Let δ^4 be a 4-ball in U with δ^3 as a proper 3-ball. The following sublemma is needed.

Sublemma 4.3

There is an orientation-preserving diffeomorphism of $X(M)$ sending $(U^4, \tau^U(d_*, \alpha_*))$ to $(\delta^4, \tau^+(d_*, \alpha_*))$.

Proof of Sublemma 4.3 For the regular maximal tree τ^+ in the regular support disk δ^2, find a 2-disk $\delta^2_0 \subset \mathrm{Int}\delta$ such that $\tau' = \delta^2_0 \cap \tau^+$ has $\mathrm{cl}(\tau^+ \setminus \tau') \cong (\partial\tau^+) \times [0,1]$ and construct a 4-ball $\delta^4_0 \subset \mathrm{Int}U^4$ with δ^2_0 as a trivial proper disk. Then construct a proper 3-ball $\delta^3_0 \subset \delta^4_0$ with δ^2_0 as a proper disk. Note that there is an orientation-preserving diffeomorphism of S^4 sending the triad $(\delta^4_0, \delta^3_0, \delta^2_0)$ to the triad $(\delta^4, \delta^3, \delta^2)$ and the regular maximal tree $\tau'(d_*, \alpha_*)$ of (d_*, α_*) given by τ' in δ^3_0 to $\tau^+(d_*, \alpha_*)$ in δ^3. Since $\mathrm{cl}(U^4 \setminus \delta^4_0)$ is diffeomorphic to $S^3 \times [0, 1]$ (see [9]), there is an orientation-preserving diffeomorphism

$$(\mathrm{cl}(U^4 \setminus \delta^4_0), \mathrm{cl}(U^4 \setminus \delta^4_0) \cap \tau^+) \rightarrow (S^3, \partial\tau^+) \times [0, 1].$$

Then there is a triad (U^4, U^3, U^2) with U^3 a proper 3-ball in U^4 and U^2 a proper 2-disk in U^3 such that there is an orientation-preserving diffeomorphism of S^4 sending the triad (U^4, U^3, U^2) to the triad $(\delta^4_0, \delta^3_0, \delta^2_0)$ and $\tau^U(d_*, \alpha_*)$ in U^3 to $\tau'(d_*, \alpha_*)$ in δ^3_0. Thus, there is an orientation-preserving diffeomorphism of S^4 sending the triad (U^4, U^3, U^2) to the triad $(\delta^4, \delta^3, \delta^2)$ and $\tau^U(d_*, \alpha_*)$ in U^3 to $\tau^+(d_*, \alpha_*)$ in δ^3. This completes the proof of Sublemma 4.3.

By Sublemma 4.3, the ribbon 3-ball system V'_R realizing $\tau^U(d_*, \alpha_*)$ in U^4 extends to a ribbon solid torus V_R in S^4. This means that the spun S_2-link $S(e_*)$ in $X(M)$ bounds the disjoint 3-ball system $\mathrm{cl}(V_R \setminus V'_R)$. This completes the proof of Sublemma 4.2.

By Lemma 2.5 and Sublemma 4.2, the proper arc system e_* and hence k are in a 3-ball D^3 which is a regular neighborhood of $\delta^2 \times [0, 1]$ in $M^{(o)}$. This completes the proof of Theorem 1.1.

5. Conclusion

A general problem arising from this paper is how any given ribbon solid torus bounded by the spun torus-knot $T(k)$ of a knot k relates to a knot diagram $D(k)$ of k. For example, the CP disk-chord system (d_*, α_*) in Figure 3 (1) is seen to represent a ribbon solid torus bounded by the spun torus-knot $T(k)$ of the trefoil knot k in Figure 3 (3). In fact, the ribbon torus-knot given by Figure 3 (1) is equivalent to the ribbon torus-knot given by Figure 3 (2) by moves on chord diagrams and by Sublemma 4.2 the CP disk-chord system of Figure 3 (2) is the CP disk-chord system of the spun ribbon solid torus of the trefoil knot diagram $D(k)$ shown in Figure 3 (3) [18-21]. It would be interesting to point out that the CP disk-chord system (d_*, α_*) in Figure 3 (1) is not the CP disk-chord system of the spun ribbon solid torus of any knot diagram $D'(k)$ of the trfoil knot k. To see this, the cross-index is used [24]. If (d_*, α_*) is obtained from the spun ribbon solid torus of a trefoil knot diagram $D'(k)$, then the complementary arc system e_* of any regular maximal tree $\tau^+(d_*, \alpha_*)$ in (d_*, α_*) in a regular support disk δ^2 must have the cross-index 0 in the annulus A given by any extended disk δ^+ such that $\mathrm{Int}\delta^+ \supset \delta^2$ and e_* is an immersed arc system in the annulus $A = \mathrm{cl}(\delta^+ \setminus \delta^2)$. However, the coss-index of e_* in an annulus A is 1 for the diagram given in Figure 3 (1). This means that the CP disk-chord system (d_*, α_*) in Figure 3 (1) is not the CP disk-chord system of the spun ribbon solid torus of any trefoil knot diagram $D'(k)$.

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