

# Approximate Spectral Triples and Non-commutative Geometric Structures on Non-Separable Banach Spaces

Juan Alberto Molina García\* *Independent researcher, Spain***\*Corresponding Author**

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This article develops a geometric extension of the approximate  $C^*$ - and  $W^*$ -algebra framework introduced in earlier work, showing that NSBS naturally give rise to noncommutative geometric structures in the sense of Connes. While classical noncommutative geometry is traditionally based on separable  $C^*$ -algebras and Hilbert spaces, many analytically relevant settings—such as  $\ell^\infty$ ,  $L^\infty(\mu)$  for non- $\sigma$ -finite measures, or  $C(\beta\mathbb{N})$ —lack separability and fall outside the scope of the usual spectral triple framework. The present work overcomes these limitations by developing a theory of approximate spectral triples, constructed as inductive limits of local spectral triples on separable components.

Given a non-separable Banach space  $X$ , its approximate operator algebra  $A^{\text{approx}}(X)$  is defined as an inductive limit of separable  $C^*$ -algebras  $A_F$ . We show that this structure canonically admits a family of densely defined local Dirac-type operators  $D_F$  acting on separable Hilbert modules  $H_F$  satisfying compatibility conditions that allow the construction of a global approximate Dirac operator  $D^{\text{approx}}$ . The pair  $(A^{\text{approx}}, D^{\text{approx}})$  behaves like a geometric object: it induces a variational metric on the state space, determines a quantum differential calculus, and provides a platform for an approximate version of Connes' distance formula.

We prove that approximate spectral triples endow  $A^{\text{approx}}$  with the structure of a weakly differentiable noncommutative manifold, with the following key features: 1) an approximate spectral metric  $d^{\text{approx}}$  defined on approximate states; 2) an approximate Calderón–Zygmund decomposition on separable subspaces; 3) a differential calculus generated by commutators  $[D_F, a_F]$ , compatible through the inductive system; 4) an associated approximate  $K$ -homology and  $K$ -theory enabling topological classification; 5) explicit geometric models on benchmark EBNS:  $\ell^\infty$ ,  $C(\beta\mathbb{N})$ , and  $L^\infty(\mu)$  with non- $\sigma$ -finite  $\mu$ .

This shows that non-separable Banach spaces carry a noncommutative geometric structure, expressible through approximate spectral data, and exhibiting differential regularity only locally. This yields a rigorous notion of non-separable quantum manifold: a geometric space whose differential, metric, and spectral structures are visible only through separable windows.

To our knowledge, this approach is entirely new and opens a path for applying noncommutative geometry to non-separable functional analysis, mathematical physics, and higher operator theory.

**Keywords:** Noncommutative Geometry; Spectral Triple, Approximate  $C^*$ -Algebra, Non-Separable Banach Space, Dirac Operator; Approximate Metric,  $K$ -Homology; Functional Analysis, Operator Algebras, Geometric Analysis**1. Introduction**

Noncommutative geometry, in the sense of Connes, identifies geometric structures with purely analytic data encoded in a  $C^*$ -algebra  $A$ , a Hilbert space  $H$ , and an unbounded self-adjoint operator  $D$ . The triple  $(A, H, D)$  captures the full differential, metric, and topological content of a space, including curvature, dimension, and  $K$ -theoretical invariants [1]. Its power lies in its ability to treat geometric objects

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(manifolds, graphs, fractals, quantum spaces) through functional analytic tools. However, the classical theory rests heavily on one assumption: all objects involved are separable.

This includes:

- Separable  $C^*$ -algebras,
- Hilbert spaces with countable orthonormal basis,
- Compact resolvent conditions that implicitly require separability.

This excludes some fundamental analytic spaces:

$$\ell^\infty, L^\infty(\mu) \text{ when } \mu \text{ is not } \sigma\text{-finite, } C(\beta\mathbb{N}),$$

as well as many Banach spaces arising in PDE theory, probability, and topological measure theory. In these contexts, spectral triples cannot exist in the classical sense.

In recent work, we introduced the general concept of an approximate  $C^*$ -algebra,  $A^{approx}(X)$ , associated with a Banach space  $X$ . This algebra is constructed as the inductive limit [2-4]:

$$A^{approx}(X) = \lim_{F \in \mathcal{F}(X)} \rightarrow A_F$$

where each  $A_F$  is a separable  $C^*$ -subalgebra acting on the separable subspace  $X_F \subseteq X$  generated by  $F$ . The family  $(A_F)$  provides separable windows through which the behaviour of the full space  $X$  can be partially observed.

The main goal of this article is to extend this operator-algebraic viewpoint to geometry. More precisely, we prove that every approximate  $C^*$ -algebra admits an associated approximate spectral triple, which behaves like a noncommutative differentiable structure on a non-separable object. This is achieved by:

- i. Constructing compatible Dirac-type operators  $D_F$  on each  $A_F$ .
- ii. Showing that the commutators  $[D_F, a_F]$  stabilise through the inductive system.
- iii. Defining a global operator

$$D^{approx} = [D_F]$$

in the approximate sense.

- iv. Introducing an approximate Connes' metric on the approximate state space.
- v. Proving that this object possesses differential calculus, metric properties, and  $K$ -homological interpretation.

From the geometric viewpoint, the central message is the following: NSBS generate noncommutative manifolds that possess only local (separable) differential structure, encoded by approximate spectral triples.

This yields a new class of geometric spaces:

- i. Not topological manifolds.
- ii. Not classical spectral triples,
- iii. But weak noncommutative manifolds whose geometry is visible only through separable approximations.

This framework unifies and extends the operator-algebraic foundations developed earlier and builds a conceptual bridge between functional analysis on NSNS and the geometric machinery of Connes.

## 2. Background

This section reviews the analytic and geometric structures that will be generalized in the non-separable setting. We begin with the classical notion of spectral triple in the sense of Connes, then recall the operator-algebraic framework underlying noncommutative geometry. Finally, we summarize the theory of approximate  $C^*$ - and  $W^*$ -algebras introduced in earlier work, which provides the foundation for constructing approximate spectral triples. Throughout this section, all Hilbert spaces are assumed to be complex, all  $C^*$ -algebras are involutive, and operator norms are denoted by  $\|\cdot\|$ .

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## 2.1. Classical Spectral Triples

In Connes' formulation of noncommutative geometry, a spectral triple is the triple  $(A, H, D)$ , [1,5] where:

- i.  $A$  is a unital  $C^*$ -algebra represented faithfully on a separable Hilbert space  $H$ .
- ii.  $D$  is an unbounded, self-adjoint operator on  $H$  with compact resolvent

$$(1 + D^2)^{-1/2} \in \mathcal{K}(H);$$

- iii. The commutators  $[D, a]$  are bounded for all  $a$  in a dense  $*$ -subalgebra  $A \subseteq A$ .

These data encode the metric, differential, and analytic structure of a noncommutative manifold. Key features include:

- a) Metric structure. The Connes–Kantorovich metric on the state space  $\mathcal{S}(A)$  is given by

$$d(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

- b) Differential calculus. Derivations are encoded in commutators:

$$\delta(a) = [D, a], a \in \mathcal{A}$$

- c) Dimension and regularity. Summability or heat kernel estimates on  $D$  provide geometric information.
- d)  $K$ -homology class. The bounded transform

$$F = D(1 + D^2)^{-1/2}$$

defines a cycle representing an element in the analytic  $K$ -homology group of  $A$ .

All these aspects rely crucially on the separability of  $H$ . In non-separable settings, compactness of the resolvent fails dramatically; even defining commutators becomes delicate.

## 2.2. Operator Algebras in the Separable Framework

We summarise the classical properties of  $C^*$ - and  $W^*$ -algebras that will later be modified to accommodate non-separability.

A  $C^*$ -algebra  $A$  is a Banach  $*$ -algebra satisfying the  $C^*$ -identity:

$$\|a^*a\| = \|a\|^2$$

Important examples include: i)  $C_0(X)$  for locally compact Hausdorff spaces  $X$ ; ii)  $\mathcal{B}(H)$ , the bounded operators on a Hilbert space; iii)  $\mathcal{K}(H)$ , compact operators.

The Gelfand–Naimark theorem shows: I) Commutative  $C^*$ -algebras are isometric to continuous functions on locally compact spaces; II) Noncommutative  $C^*$ -algebras generalise operator algebras; III)  $W^*$ -algebras (von Neumann algebras); IV) A  $W^*$ -algebra  $M$  is a  $C^*$ -algebra that is weak $*$ -closed in  $\mathcal{B}(H)$  for some representation [4,6-8].

Equivalent characterisations: A)  $M = M''$  (double commutant theorem); B)  $M$  is monotone complete and possesses a pre-dual; C) Examples include  $L^\infty(I)$ , group von Neumann algebras, and  $W^*$ -algebras generated by spectral measures.

## 2.3. Noncommutative Geometric Structures

In Connes' framework, geometry is recast through analytic data. A spectral triple  $(A, H, D)$  gives rise to: 1) A first-order differential calculus generated by bounded commutators  $[D, a]$ ; 2) A spectral metric; 3) Invariants in  $K$ -homology and cyclic cohomology; 4) Index pairings of Fredholm modules; 5) Local geometric quantities (dimension, volume form, curvature) encoded in heat kernel asymptotic.

All of these depend on separability. When separability is abandoned, major obstacles arise: a)  $\mathcal{K}(H)$  is no longer the closure of finite-rank operators; b) Resolvents may fail to be compact or even measurable; c) The Dixmier trace becomes problematic; d) The metric may fail to distinguish states.

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Thus, we need a fundamentally different geometric mechanism: one based not on global separability but on local separable windows.

## 2.4. Approximate $C^*$ - and $C^*$ - Algebras

Given a non-separable Banach space  $X$ , let  $\mathcal{F}(X)$  denote the directed family of all separable, finite-generated subspaces. Each  $F \in \mathcal{F}(X)$  generates a separable Banach subspace  $X_F$ , and we consider a family of separable  $C^*$ -subalgebras:

$$A_F \subseteq \mathcal{B}(X_F), F \in \mathcal{F}(X)$$

such that:

1.  $F \subseteq G \Rightarrow X_F \subseteq X_G$  and  $A_F \subseteq A_G$ .
2.  $\bigcup_F A_F$  is norm-dense in  $A^{approx}(X)$ .

The approximate  $C^*$ -algebra defined by  $X$  is the inductive limit:

$$A^{approx}(X) = \lim_{F \in \mathcal{F}(X)} \rightarrow A_F$$

with norm

$$\| [a_F] \| = \sup_F \| a_F \|$$

Analogously, approximate  $W^*$ -algebras arise from inductive systems of separable  $W^*$ -subalgebras with compatibility of pre-duals. This structure reflects the intuition: a non-separable Banach space behaves geometrically like a union of separable pieces, each of which carries classical analytic and geometric structures.

Approximate positivity, approximate states, and approximate GNS constructions have been established in earlier articles.

## 2.5. Motivation for Approximate Spectral Triples

In classical NCG, the object's geometry is fully encoded by the spectral triple  $(A, H, D)$ . In the non-separable setting, this fails globally, but we can still construct local spectral triples:

$$(A_F, H_F, D_F), F \in \mathcal{F}(X)$$

each satisfying the usual axioms on a separable Hilbert space  $H_F$ . If these local triples satisfy suitable *compatibility conditions*, they induce:

- i) A global approximate Dirac operator  $D^{approx}$ .
- ii) Commutators  $[D_F, a_F]$  agreeing across the inductive system.
- iii) Local spectral metrics compatible with an approximate global metric;
- iv) Approximate differential calculus and approximate  $K$ -homology.

Thus, the correct geometric object associated with a non-separable Banach space is not a spectral triple but an approximate spectral triple.

## 2.6. Goal of the Paper

The objective is to construct a full analogue of Connes' geometric machinery in the approximate setting. We will show that:

- a. Approximate spectral triples exist on any approximate  $C^*$ -algebra;
- b. They define a meaningful spectral metric;
- c. They support a differential calculus and  $K$ -theoretical invariants;
- d. Concrete EBNS models admit computable approximate geometric structures.

This culminates in the idea that NSBS carry weak noncommutative manifolds, whose geometry is visible only locally, through separable subspaces.

## 3. Approximate Spectral Triples

In this section we construct the analytic core of the approximate noncommutative geometry framework on non-separable Banach spaces. The aim is to generalise the notion of spectral triple—introduced by Connes as the foundation of noncommutative geometry—to settings

where separability fails, by assembling compatible local spectral triples defined on separable subspaces [1].

Our approach relies on three classical pillars:

1. The theory of spectral triples [1,9],
2. Inductive-limit techniques in  $C^*$ -algebras and operator theory [2,4,6,10],
3. Perturbation and closure results for unbounded operators [11,12].

These ingredients allow us to construct a well-behaved approximate Dirac operator  $D^{approx}$ , its bounded transform  $F^{approx}$ , and the associated approximate differential calculus.

### 3.1. Motivation

Let  $X$  be a non-separable Banach space, and let

$$A^{approx}(X) = \lim_{F \in \mathcal{F}(X)} \rightarrow A_F$$

be its approximate  $C^*$ -algebra. Each algebra  $A_F$  acts naturally on a separable Hilbert space  $H_F$ , and the classical theory guarantees the existence of geometric data on such separable components: i) a faithful representation  $\pi_F: A_F \rightarrow \mathcal{B}(H_F)$ ; ii) an unbounded self-adjoint operator  $D_F$  with compact resolvent; iii) bounded commutators  $[D_F, a_F]$  for  $a_F$  in a dense  $*$ -subalgebra [2,4,6,10].

These local triples  $(A_F, H_F, D_F)$ , in the sense of Connes, are the geometric microscopes through which we analyse the global structure of  $X$ . The challenge is to glue these into a single object. This requires a notion of compatibility and a coherent way to treat families of Dirac operators and commutators. The mechanism is provided by inductive limits and approximate equivalence relations developed earlier [1].

### 3.2. Local Spectral Triples and Compatibility

For each  $F \in \mathcal{F}(X)$ , we assume given a separable spectral triple

$$(A_F, H_F, D_F)$$

*Definition 3.2.1 (Compatibility System).* A family  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  is compatible if:

1. Module compatibility. If  $F \subseteq G$ , there exists an isometric embedding  $\iota_{FG}^H: H_F \hookrightarrow H_G$  such that  $\iota_{FG}^H(\pi_F(a_F)\xi) = \pi_G(a_G)\iota_{FG}^H(\xi)$  for all  $\xi \in H_F$ , where  $a_G$  is the image of  $a_F$  under the inductive system.
2. Dirac compatibility. The operators satisfy  $D_G \circ \iota_{FG}^H = \iota_{FG}^H \circ D_F$  on the domain of  $D_F$ .
3. Commutator compatibility. For every  $a_F \in A_F$ ,  $[D_F, a_F]$  and  $[D_G, a_G]$  agree on the embedded copy of  $H_F$ .
4. Norm control.  $\sup_{F \in \mathcal{F}(X)} \|(1 + D_F^2)^{-\frac{1}{2}}\| < \infty$ .

These conditions ensure that the family behaves as a directed inductive system of geometric objects.

### 3.3. Approximate Dirac Operators

Given a compatible family  $\{D_F\}$ , we wish to construct a global object  $D^{approx}$ .

Since the global Hilbert space  $H^{approx}$  may be non-separable, we define it as the inductive limit:

$$H^{approx} = \lim_{F \in \mathcal{F}(X)} \rightarrow H_F$$

The embeddings  $\iota_{FG}^H$  induce canonical identifications of vectors in this limit space.

*Definition 3.3.1 (Approximate Dirac Operator).* The approximate Dirac operator is the equivalence class

$$D^{approx} = [D_F]_{F \in \mathcal{F}(X)}$$

acting on  $H^{approx}$  by  $D^{approx}(\iota_F(\xi)) := \iota_F(D_F\xi)$ ,  $\xi \in \text{Dom}(D_F)$ . Well-definedness follows from compatibility [11-13].

*Proposition 3.3.1.* Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples in the sense of definition 3.2.1, and let

$$H^{approx} = \lim_{F \in \mathcal{F}(X)} \rightarrow H_F$$

be the inductive limit Hilbert space, constructed via the isometric embeddings  $\iota_{FG}^H: H_F \hookrightarrow H_G$  for  $F \subseteq G$ . Define the approximate Dirac operator  $D^{approx}$  on  $H^{approx}$  by

$$Dom(D^{approx}) = \bigcup_{F \in \mathcal{F}(X)} \iota_F(Dom(D_F))$$

and

$$D^{approx}(\iota_F(\xi)) = \iota_F(D_F \xi), \xi \in Dom(D_F)$$

where  $\iota_F: H_F \rightarrow H^{approx}$  denotes the canonical isometric embedding. Then,  $D^{approx}$  is densely defined, symmetric, and closed.

*Proof.* We split the proof into four parts: well-definedness, density of the domain, symmetry, and closedness [4,11].

First, we must show that the above definition of  $D^{approx}$  does not depend on the choice of representative. Let  $\xi \in Dom(D_F)$  and suppose that, for some  $G \in \mathcal{F}(X)$  with  $F \subseteq G$ , we also view  $\xi$  as an element of  $H_G$  via the embedding  $\iota_{FG}^H: H_F \hookrightarrow H_G$ .

By the construction of the inductive limit,  $\iota_F(\xi) = \iota_G(\iota_{FG}^H(\xi)) \in H^{approx}$ . By Dirac compatibility (definition 3.2.1, item 2), we have  $D_G \circ \iota_{FG}^H = \iota_{FG}^H \circ D_F$  on  $Dom(D_F)$ . Hence,  $\iota_G(D_G \iota_{FG}^H(\xi)) = \iota_G(\iota_{FG}^H(D_F \xi)) = \iota_F(D_F \xi)$ .

Thus, whether we compute  $D^{approx}$  using the representative  $(F, \xi)$  or  $(G, \iota_{FG}^H \xi)$ , we obtain the same vector in  $H^{approx}$ . Therefore  $D^{approx}$  is well-defined on  $Dom(D^{approx})$ .

Second, by construction of the inductive limit,

$$H^{approx} = \overline{\bigcup_{F \in \mathcal{F}(X)} \iota_F(H_F)}$$

For each  $F$ , the operator  $D_F$  is densely defined on  $H_F$  (since we are in the usual spectral triple framework), i.e.

$$\overline{Dom(D_F)} = H_F$$

Applying the isometric embedding  $\iota_F$ , we get  $\overline{\iota_F(Dom(D_F))} = \iota_F(H_F)$ , for every  $F$ . Now,

$$Dom(D^{approx}) = \bigcup_F \iota_F(Dom(D_F))$$

and therefore,

$$\overline{Dom(D^{approx})} \supseteq \overline{\iota_F(Dom(D_F))} = \iota_F(H_F), \forall F$$

Taking the closure over all  $F$ ,

$$\overline{\text{Dom}(D^{approx})} \supseteq \bigcup_F \overline{\iota_F(H_F)} = H^{approx}$$

Hence  $\text{Dom}(D^{approx})$  is dense in  $H^{approx}$ .

Third, let  $x, y \in \text{Dom}(D^{approx})$ . By definition, there exist  $F, G \in \mathcal{F}(X)$  and vectors  $\xi \in \text{Dom}(D_F), \eta \in \text{Dom}(D_G)$ , such that  $x = \iota_F(\xi), y = \iota_G(\eta)$ .

Since  $\mathcal{F}(X)$  is directed, there exists  $H \in \mathcal{F}(X)$  with  $F \subseteq H$  and  $G \subseteq H$ . Using the embeddings  $\iota_{FH}^H: H_F \rightarrow H_H$  and  $\iota_{GH}^H: H_G \rightarrow H_H$ , and the canonical inclusion  $\iota_H: H_H \rightarrow H^{approx}$ , we can regard both  $x$  and  $y$  as elements of  $\iota_H(H_H)$ :

$$x = \iota_H(\iota_{FH}^H(\xi)), y = \iota_H(\iota_{GH}^H(\eta))$$

For simplicity of notation, write  $\xi_H := \iota_{FH}^H(\xi), \eta_H := \iota_{GH}^H(\eta)$ , so that

$$x = \iota_H(\xi_H), y = \iota_H(\eta_H)$$

Now,  $D^{approx}x = D^{approx}(\iota_H(\xi_H)) = \iota_H(D_H\xi_H)$ , and similarly,  $D^{approx}y = \iota_H(D_H\eta_H)$ . The scalar product on  $H^{approx}$  is compatible with the embeddings:  $\langle \iota_H(u), \iota_H(v) \rangle_{H^{approx}} = \langle u, v \rangle_{H_H}, u, v \in H_H$ . Therefore,

$$\langle D^{approx}x, y \rangle_{H^{approx}} = \langle \iota_H(D_H\xi_H), \iota_H(\eta_H) \rangle = \langle D_H\xi_H, \eta_H \rangle_{H_H}$$

Since  $D_H$  is symmetric on  $H_H$ ,  $\langle D_H\xi_H, \eta_H \rangle_{H_H} = \langle \xi_H, D_H\eta_H \rangle_{H_H}$ . Thus,  $\langle D^{approx}x, y \rangle = \langle \xi_H, D_H\eta_H \rangle_{H_H} = \langle \iota_H(\xi_H), \iota_H(D_H\eta_H) \rangle = \langle x, D^{approx}y \rangle$ . Hence,  $D^{approx}$  is symmetric on its domain.

Forth, to prove that  $D^{approx}$  is closed, we use the characterisation in terms of its graph. Recall that the graph of  $D^{approx}$  is

$$\mathcal{G}(D^{approx}) = \{(x, D^{approx}x) : x \in \text{Dom}(D^{approx})\} \subseteq H^{approx} \times H^{approx}$$

We must show that  $\mathcal{G}(D^{approx})$  is closed in the Hilbert space  $H^{approx} \times H^{approx}$ . For each  $F \in \mathcal{F}(X)$ , consider the graph of  $D_F$ ,

$$\mathcal{G}(D_F) = \{(\xi, D_F\xi) : \xi \in \text{Dom}(D_F)\} \subseteq H_F \times H_F$$

Since  $D_F$  is closed,  $\mathcal{G}(D_F)$  is a closed subspace of  $H_F \times H_F$ . Now consider the inductive system of Hilbert spaces

$$H_F \times H_F \xrightarrow{(\iota_{FG}^H, \iota_{FG}^H)} H_G \times H_G, F \subseteq G$$

These are isometric embeddings, and the inductive limit Hilbert space is canonically isomorphic to  $H^{approx} \times H^{approx} = \lim_F (H_F \times H_F)$ .

The graphs  $\mathcal{G}(D_F)$  form a compatible family of closed subspaces under these embeddings, because, if  $\xi \in \text{Dom}(D_F)$ , then  $(\iota_{FG}^H\xi, \iota_{FG}^H D_F\xi) = (\iota_{FG}^H\xi, D_G\iota_{FG}^H\xi) \in \mathcal{G}(D_G)$ , by Dirac compatibility  $D_G \circ \iota_{FG}^H = \iota_{FG}^H \circ D_F$ . Therefore, the family  $\{\mathcal{G}(D_F)\}_F$  defines an inductive system of closed subspaces.

Let  $\mathcal{G}^{approx} = \lim_F \rightarrow \mathcal{G}(D_F) \subseteq H^{approx} \times H^{approx}$  denote the inductive limit of these graphs under the embeddings  $(\iota_{FG}^H, \iota_{FG}^H)$ . By the very definition of  $D^{approx}$ , we have  $\mathcal{G}(D^{approx}) = \mathcal{G}^{approx}$ , since each element  $x \in \text{Dom}(D^{approx})$  belongs to some  $\iota_F(\text{Dom}(D_F))$ , and then  $(x, D^{approx}x) = (\iota_F(\xi), \iota_F(D_F\xi)) \in \iota_F(\mathcal{G}(D_F))$ .

It remains to show that  $\mathcal{G}^{approx}$  is closed in  $H^{approx} \times H^{approx}$ , because:

- i) Each  $\mathcal{G}(D_F)$  is closed in  $H_F \times H_F$ .
- ii) The embeddings  $H_F \times H_F \rightarrow H_G \times H_G$  are isometries.
- iii) The inductive limit of a directed system of Hilbert spaces with isometric embeddings carries the final (inductive) Hilbert topology.

It follows that the inductive limit of a directed system of closed subspaces is closed in the inductive limit space. Concretely, if a sequence  $(x_n, y_n) \in \mathcal{G}^{approx}$  converges in  $H^{approx} \times H^{approx}$  to  $(x, y)$ , then there exists an  $F_0$  and a tail of the sequence contained in  $\iota_{F_0}(\mathcal{G}(D_{F_0}))$ . Since  $\mathcal{G}(D_{F_0})$  is closed in  $H_{F_0} \times H_{F_0}$ , the limit  $(x, y)$  lies in  $\iota_{F_0}(\mathcal{G}(D_{F_0})) \subseteq \mathcal{G}^{approx}$ . Thus  $\mathcal{G}^{approx}$  is closed. Therefore  $\mathcal{G}(D^{approx})$  is closed, and  $D^{approx}$  is a closed operator on  $H^{approx}$ .

Combining all the former considerations (first to forth), we conclude that  $D^{approx}$  is well-defined, densely defined, symmetric, and closed, as claimed.  $\square$

*Proposition 3.3.2 (Approximate Compactness of the Resolvent).* Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples in the sense of definition 3.2.1. And let  $H^{approx} = \lim_{F \in \mathcal{F}(X)} \rightarrow H_F$ , and  $D^{approx}$  be the approximate Dirac operator constructed in definition 3.3.1 and proposition 3.3.1.

Define the bounded operator  $R^{approx} := (1 + (D^{approx})^2)^{-1/2}$  by the continuous functional calculus. Then for each  $F \in \mathcal{F}(X)$ ,  $R^{approx} |_{H_F} = (1 + D_F^2)^{-1/2}$

and in particular every such restriction is compact on  $H_F$ . Thus  $R^{approx}$  is approximately compact.

*Proof.* Fix  $F \in \mathcal{F}(X)$ . By assumption,  $(A_F, H_F, D_F)$  is a classical spectral triple on the separable Hilbert space  $H_F$ . In particular, by the standard axiom of spectral triples, the resolvent of  $D_F$  is compact [1]:

$$R_F := (1 + D_F^2)^{-1/2} \in \mathcal{K}(H_F)$$

Equivalently, the bounded transform  $F_F := D_F(1 + D_F^2)^{-1/2}$  is a Fredholm operator, and  $R_F$  belongs to the  $C^*$ -algebra of compact operators on  $H_F$ . Thus, each local Dirac operator  $D_F$  has compact resolvent.

We now show that the family  $\{R_F\}_{F \in \mathcal{F}(X)}$  is compatible with the inductive system. Let  $F, G \in \mathcal{F}(X)$  with  $F \subseteq G$ . By definition 3.2.1 (Dirac compatibility), the embeddings satisfy  $D_G \circ \iota_{FG}^H = \iota_{FG}^H \circ D_F$  on  $\text{Dom}(D_F)$ .

We wish to deduce a corresponding relation for the resolvents. This follows from the functional calculus for self-adjoint operators: if  $T$  and  $S$  are self-adjoint operators on Hilbert spaces  $H_1$  and  $H_2$ , and  $U: H_1 \rightarrow H_2$  is an isometry such that  $SU = UT$  on  $\text{Dom}(T)$ , then for

every bounded Borel function  $f$  on  $\mathbb{R}, f(S)U = Uf(T)$ .

In our setting, take  $T = D_F, S = D_G, U = \iota_{FG}^H$ . Then, for any bounded Borel function  $f$ , we obtain  $f(D_G) \circ \iota_{FG}^H = \iota_{FG}^H \circ f(D_F)$ . Take specifically the function  $f(\lambda) = (1 + \lambda^2)^{-1/2}$ . This function is continuous and bounded on  $\mathbb{R}$ , so the functional calculus applies. Hence,  $(1 + D_G^2)^{-1/2} \circ \iota_{FG}^H = \iota_{FG}^H \circ (1 + D_F^2)^{-1/2}$ . In other words,

$$R_G \circ \iota_{FG}^H = \iota_{FG}^H \circ R_F$$

This shows that the family  $\{R_F\}_F$  is compatible with the inductive system of Hilbert spaces  $\{H_F\}_F$  and embeddings  $\iota_{FG}^H$ .

Now, by compatibility, the family  $\{R_F\}_F$  defines a unique bounded operator on the inductive limit Hilbert space  $H^{approx}$ , which we denote by  $R^{loc} := [R_F]_{F \in \mathcal{F}(X)}$ . Concretely, for any vector  $x \in H^{approx}$ , there exists  $F \in \mathcal{F}(X)$  and  $\xi \in H_F$  such that  $x = \iota_F(\xi)$ , and we define

$$R^{loc}(x) = R^{loc}(\iota_F(\xi)) := \iota_F(R_F \xi) = \iota_F((1 + D_F^2)^{-1/2} \xi)$$

Well-definedness follows from the same compatibility argument as in the proof of the definition of  $D^{approx}$ : if we represent  $x$  via a larger index  $G \supseteq F$ , the relation

$$R_G \circ \iota_{FG}^H = \iota_{FG}^H \circ R_F$$

ensures that the value of  $R^{loc}(x)$  does not depend on the chosen representative.

Furthermore, since  $\|R_F\| \leq 1$  for all  $F$  (as  $\|(1 + D_F^2)^{-1/2}\| \leq 1$ ), we have

$$\|R^{loc}\| = \sup_F \|R_F\| \leq 1$$

so  $R^{loc}$  is a bounded operator on  $H^{approx}$ .

We now show that  $R^{loc}$  coincides with the operator  $(1 + (D^{approx})^2)^{-1/2}$ , obtained via the continuous functional calculus applied to  $D^{approx}$ .

Let  $f(\lambda) = (1 + \lambda^2)^{-1/2}$ . By proposition 3.3.1,  $D^{approx}$  is self-adjoint (indeed closed and symmetric with appropriate domain assumptions; one can either assume essential self-adjointness or consider the closure, which we again denote by  $D^{approx}$ ). Therefore, the continuous functional calculus yields a bounded operator

$$f(D^{approx}) = (1 + (D^{approx})^2)^{-1/2}$$

We claim that

$$f(D^{approx}) = R^{loc}$$

To see this, take  $\xi \in \text{Dom}(D_F)$  and consider its image  $x = \iota_F(\xi) \in H^{approx}$ . Using the same functional calculus compatibility argument as before, but now at the level of the inductive limit, we obtain

$$f(D^{approx}) \iota_F(\xi) = \iota_F(f(D_F)\xi) = \iota_F((1 + D_F^2)^{-1/2} \xi) = R^{loc}(\iota_F(\xi))$$

Since the set of such vectors  $\iota_F(\xi)$  (with  $\xi \in \text{Dom}(D_F)$  and  $F \in \mathcal{F}(X)$ ) is dense in  $H^{approx}$ , and both  $f(D^{approx})$  and  $R^{loc}$  are bounded operators, the equality on a dense subset implies

$$f(D^{\text{approx}}) = R^{\text{loc}}$$

Thus, we may identify

$$R^{\text{approx}} := (1 + (D^{\text{approx}})^2)^{-1/2} = R^{\text{loc}}$$

Finally, by definition, an operator  $T \in \mathcal{B}(H^{\text{approx}})$  is approximately compact if, for every separable component  $H_F \subseteq H^{\text{approx}}$ , its restriction  $T|_{H_F}$  is compact on  $H_F$ . In our case, take  $T = R^{\text{approx}}$ . For each  $F$ , we have shown that

$$R^{\text{approx}}|_{H_F} = R_F = (1 + D_F^2)^{-1/2}$$

And also, we know that  $R_F \in \mathcal{K}(H_F)$ . Therefore, for every  $F \in \mathcal{F}(X)$ , the restriction of  $R^{\text{approx}}$  to  $H_F$  is compact. Hence  $R^{\text{approx}}$  is approximately compact.  $\square$

*Proposition 3.3.3 (Bounded Transform of the Approximate Dirac Operator).* Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples, and let

$$H^{\text{approx}} = \lim_{F \in \mathcal{F}(X)} \rightarrow H_F$$

and  $D^{\text{approx}}$  be the associated approximate Dirac operator, as in definition 3.3.1 and proposition 3.3.1. Define the bounded transform of  $D^{\text{approx}}$  by

$$F^{\text{approx}} := D^{\text{approx}}(1 + (D^{\text{approx}})^2)^{-1/2}$$

Then:

- (1)  $F^{\text{approx}}$  is a bounded self-adjoint operator on  $H^{\text{approx}}$  with  $\|F^{\text{approx}}\| \leq 1$ .
- (2) For every  $F \in \mathcal{F}(X)$ , the restriction of  $F^{\text{approx}}$  to  $H_F$  coincides with the classical bounded transform  $F_F := D_F(1 + D_F^2)^{-1/2}$ , i.e.

$$F^{\text{approx}}|_{H_F} = F_F$$

In particular,  $F^{\text{approx}}$  is an approximately Fredholm operator: its restriction to each separable component  $H_F$  is a Fredholm operator in the usual sense [13,14].

*Proof.* We use the results already established in propositions 3.3.1 and 3.3.2. Let  $D^{\text{approx}}$  be the approximate Dirac operator constructed in definition 3.3.1. By proposition 3.3.1,  $D^{\text{approx}}$  is densely defined, symmetric and closed; we regard it as self-adjoint (or replace it by its unique self-adjoint closure, which we still denote by  $D^{\text{approx}}$ ). Consider the function

$$f(\lambda) = \frac{\lambda}{\sqrt{1 + \lambda^2}}, \lambda \in \mathbb{R}$$

This is a bounded, continuous, real-valued function on  $\mathbb{R}$ , with

$$|f(\lambda)| \leq 1, \forall \lambda \in \mathbb{R}$$

and  $\lim_{|\lambda| \rightarrow \infty} f(\lambda) = \text{sgn}(\lambda)$ . By the continuous functional calculus for self-adjoint operators, we may define  $F^{\text{approx}} := f(D^{\text{approx}})$ , and it is immediate that:

- i)  $F^{\text{approx}}$  is a bounded operator on  $H^{\text{approx}}$ ;
- ii)  $F^{\text{approx}}$  is self-adjoint, since  $f$  is real-valued;

iii) its norm satisfies

$$\|F^{approx}\| = \|f(D^{approx})\| \leq \sup_{\lambda \in \mathbb{R}} |f(\lambda)| = 1$$

On the other hand, by the functional calculus identities,

$$f(D^{approx}) = D^{approx}(1 + (D^{approx})^2)^{-1/2}$$

Thus, the definition of  $F^{approx}$  in the statement matches exactly this functional calculus expression. This proves item (1).

Now, fix  $F \in \mathcal{F}(X)$ . On the separable Hilbert space  $H_F$ , the operator  $D_F$  is the Dirac operator of a classical spectral triple. The same function  $f(\lambda) = \lambda(1 + \lambda^2)^{-1/2}$  defines its bounded transform:  $F_F = f(D_F) = D_F(1 + D_F^2)^{-1/2}$ , which is a bounded self-adjoint operator with  $\|F_F\| \leq 1$ .

Since  $(A_F, H_F, D_F)$  is a spectral triple,  $F_F$  is a Fredholm operator on  $H_F$ . We want to show that  $F^{approx}$  restricts to  $F_F$  on  $H_F$ .

Let  $F, G \in \mathcal{F}(X)$  with  $F \subseteq G$ . Recall from definition 3.2.1 that the Dirac operators and embeddings satisfy  $D_G \circ \iota_{FG}^H = \iota_{FG}^H \circ D_F$  on  $\text{Dom}(D_F)$ , where  $\iota_{FG}^H: H_F \rightarrow H_G$  is an isometric embedding.

For self-adjoint operators and bounded Borel functions, the functional calculus is functorial with respect to such intertwining relations: if  $U: H_F \rightarrow H_G$  is an isometry and

$$D_G U = U D_F \text{ on } \text{Dom}(D_F)$$

then for every bounded Borel function  $g$ , one has  $g(D_G)U = Ug(D_F)$ . Apply this with:

a)  $U = \iota_{FG}^H$ ,

b)  $g = f$ , with  $f(\lambda) = \lambda(1 + \lambda^2)^{-1/2}$ .

We obtain  $f(D_G) \circ \iota_{FG}^H = \iota_{FG}^H \circ f(D_F)$ . That is,  $F_G \iota_{FG}^H = \iota_{FG}^H F_F$ , where

$$F_G := f(D_G) = D_G(1 + D_G^2)^{-1/2}$$

Thus, the family  $\{F_F\}_F$  is compatible with the inductive system of Hilbert spaces and embeddings.

By the compatibility established above, the family of bounded operators  $\{F_F\}$  defines a unique bounded operator  $F^{loc} = [F_F]_F$  on the inductive limit Hilbert space  $H^{approx}$ , given by  $F^{loc}(\iota_F(\xi)) := \iota_F(F_F \xi)$ ,  $\xi \in H_F$ .

We claim that  $F^{approx} = f(D^{approx}) = F^{loc}$ . Take  $\xi \in \text{Dom}(D_F) \subseteq H_F$  and consider  $x = \iota_F(\xi) \in H^{approx}$ . By the construction of the approximate Dirac operator and again by functoriality of functional calculus, we have:

$$f(D^{approx}) \iota_F(\xi) = \iota_F(f(D_F)\xi) = \iota_F(F_F \xi) = F^{loc}(\iota_F(\xi))$$

Thus  $f(D^{approx})$  and  $F^{loc}$  agree on the dense subspace

$$\bigcup_{F \in \mathcal{F}(X)} \iota_F(\text{Dom}(D_F)) \subseteq H^{approx}$$

Since both operators are bounded on  $H^{approx}$ , the equality on a dense subspace implies  $F^{approx} = f(D^{approx}) = F^{loc}$ . In particular, for each fixed  $F$ ,

$$F^{approx} |_{H_F} = F_F$$

This establishes item (2).

Finally, since for each  $F$ ,  $F_F$  is the bounded transform of a spectral triple, it is a Fredholm operator on the separable Hilbert space  $H_F$ . Hence, in the sense of the approximate framework,  $F^{approx}$  is approximately Fredholm: all its local restrictions to separable components are Fredholm operators, and its Fredholm behaviour is entirely dictated by the family  $\{F_F\}$ .

We conclude that: (I)  $F^{approx}$  is bounded, self-adjoint, and contractive; (II) its restriction to each separable component  $H_F$  coincides with  $F_F$ , the classical bounded transform of  $D_F$ ; (III) it is approximately Fredholm. This proves proposition 3.3.3.  $\square$

### 3.4. Approximate Commutators

Given  $a = [a_F] \in A^{approx}$ , define the approximate commutator [1,15]

$$[D^{approx}, a] := [[D_F, a_F]_F]$$

*Proposition 3.4.1 (Boundedness of the Approximate Commutator).* Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples in the sense of definition 3.2.1, and let

$$A^{approx} = \lim_F \rightarrow A_F, H^{approx} = \lim_F \rightarrow H_F, D^{approx} = [D_F]$$

be the associated approximate  $C^*$ -algebra, Hilbert space, and approximate Dirac operator. Let  $a \in A^{approx}$  be represented by a compatible family  $(a_F)_{F \in \mathcal{F}(X)}$  with  $a_F \in A_F$ . Assume that the commutators  $[D_F, a_F]$  are uniformly bounded:

$$M(a) := \sup_{F \in \mathcal{F}(X)} \| [D_F, a_F] \| < \infty$$

Then, there exists a unique bounded operator  $[D^{approx}, a] \in \mathcal{B}(H^{approx})$  such that, for every  $F \in \mathcal{F}(X)$ ,  $[D^{approx}, a] |_{H_F} = [D_F, a_F]$ , and  $\| [D^{approx}, a] \| \leq M(a)$ . In particular, the approximate commutator  $[D^{approx}, a]$  is well-defined and bounded on  $H^{approx}$ .

*Proof.* We proceed in three steps: construction on a dense subspace, well-definedness and compatibility, and bounded extension.

Recall that

$$H^{approx} = \lim_{F \in \mathcal{F}(X)} \rightarrow H_F$$

is the inductive limit Hilbert space obtained from the directed system  $\{H_F, \iota_{FG}^H\}$ , where each  $\iota_{FG}^H: H_F \hookrightarrow H_G$  is an isometric embedding for  $F \subseteq G$ , and the canonical embeddings  $\iota_F: H_F \rightarrow H^{approx}$  satisfy  $\iota_G \circ \iota_{FG}^H = \iota_F$  whenever  $F \subseteq G$ . The algebra  $A^{approx}$  acts on  $H^{approx}$  compatibly: if  $a = [a_F]$ , then  $a \iota_F(\xi) = \iota_F(a_F \xi)$ ,  $\xi \in H_F$ .

We first define an operator

$$T_0: \bigcup_{F \in \mathcal{F}(X)} \iota_F(H_F) \rightarrow H^{approx}$$

on the algebraic inductive limit (which is dense in  $H^{approx}$ ) by setting

$$T_0(\iota_F(\xi)) := \iota_F([D_F, a_F]\xi), \xi \in H_F$$

We will show that  $T_0$  is well-defined and bounded, and hence extends uniquely to a bounded operator on the completion  $H^{approx}$ . So that, we must check that the definition of  $T_0$  does not depend on the particular representative of a vector in  $H^{approx}$ .

Suppose that  $x \in \bigcup_F \iota_F(H_F)$  admits two representations:  $x = \iota_F(\xi) = \iota_G(\eta)$ , for some  $F, G \in \mathcal{F}(X)$ ,  $\xi \in H_F, \eta \in H_G$ . Since  $\mathcal{F}(X)$  is directed, there exists  $H \in \mathcal{F}(X)$  such that  $F \subseteq H$  and  $G \subseteq H$ . Using the embeddings  $\iota_{FH}^H: H_F \hookrightarrow H_H$  and  $\iota_{GH}^H: H_G \hookrightarrow H_H$ , we may regard both  $\xi$  and  $\eta$  as vectors in  $H_H$ . Let

$$\xi_H := \iota_{FH}^H(\xi) \in H_H, \eta_H := \iota_{GH}^H(\eta) \in H_H$$

By construction of the inductive limit,  $\iota_F(\xi) = \iota_H(\xi_H), \iota_G(\eta) = \iota_H(\eta_H)$ , and the equality  $\iota_F(\xi) = \iota_G(\eta)$  implies  $\iota_H(\xi_H) = \iota_H(\eta_H)$ . Since  $\iota_H$  is isometric and hence injective, we obtain  $\xi_H = \eta_H$  in  $H_H$ .

Now we use the commutator compatibility assumption (definition 3.2.1, item 3): for  $F \subseteq H$ ,  $[D_H, a_H] \circ \iota_{FH}^H = \iota_{FH}^H \circ [D_F, a_F]$ , and similarly for  $G \subseteq H$ ,

$$[D_H, a_H] \circ \iota_{GH}^H = \iota_{GH}^H \circ [D_G, a_G]$$

Thus,  $[D_H, a_H]\xi_H = [D_H, a_H](\iota_{FH}^H(\xi)) = \iota_{FH}^H([D_F, a_F]\xi)$ , and  $[D_H, a_H]\eta_H = [D_H, a_H](\iota_{GH}^H(\eta)) = \iota_{GH}^H([D_G, a_G]\eta)$ . But we have already established that  $\xi_H = \eta_H$ , so  $[D_H, a_H]\xi_H = [D_H, a_H]\eta_H$ . Applying  $\iota_H$  to both sides, we get

$$\iota_H([D_H, a_H]\xi_H) = \iota_H([D_H, a_H]\eta_H).$$

On the other hand,  $\iota_H([D_H, a_H]\xi_H) = \iota_F([D_F, a_F]\xi)$ , and  $\iota_H([D_H, a_H]\eta_H) = \iota_G([D_G, a_G]\eta)$ , by the expressions above. Therefore,  $\iota_F([D_F, a_F]\xi) = \iota_G([D_G, a_G]\eta)$ . That is,  $T_0(\iota_F(\xi)) = T_0(\iota_G(\eta))$ . Hence  $T_0$  is well-defined on the algebraic inductive limit  $\bigcup_F \iota_F(H_F)$ . Now, for any  $F \in \mathcal{F}(X)$  and any  $\xi \in H_F$ , we have, since  $\iota_F$  is isometric

$$\|T_0(\iota_F(\xi))\| = \|\iota_F([D_F, a_F]\xi)\| = \|[D_F, a_F]\xi\|$$

By the uniform boundedness hypothesis,

$$\|[D_F, a_F]\xi\| \leq \|[D_F, a_F]\| \cdot \|\xi\| \leq M(a) \cdot \|\xi\|$$

Therefore,

$$\|T_0(\iota_F(\xi))\| \leq M(a) \cdot \|\xi\| = M(a) \cdot \|\iota_F(\xi)\|$$

It follows that  $T_0$  is bounded on the dense subspace  $\bigcup_F \iota_F(H_F)$ , with  $\|T_0\| \leq M(a)$ . Hence  $T_0$  extends uniquely by continuity to a bounded linear operator  $[D^{approx}, a] \in \mathcal{B}(H^{approx})$  on the whole Hilbert space  $H^{approx}$ , with

$$\|[D^{approx}, a]\| = \|T_0\| \leq M(a)$$

Moreover, by construction  $[D^{approx}, a]|_{H_F} = [D_F, a_F], \forall F \in \mathcal{F}(X)$ , since  $[D^{approx}, a](\iota_F(\xi)) = \iota_F([D_F, a_F]\xi)$ . This proves the

existence, boundedness, and locality of the approximate commutator  $[D^{approx}, a]$ , and establishes the statement of the proposition.  $\square$

*Proposition 3.4.2 (Localisation Property).* Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples, and let

$$A^{approx} = \lim_F \rightarrow A_F, H^{approx} = \lim_F \rightarrow H_F, D^{approx} = [D_F]$$

be the associated approximate spectral triple. Let  $a \in A^{approx}$  be represented by a compatible family  $(a_F)_{F \in \mathcal{F}(X)}$ , and assume, as in proposition 3.4.1, that

$$M(a) := \sup_{F \in \mathcal{F}(X)} \| [D_F, a_F] \| < \infty$$

so that the approximate commutator  $[D^{approx}, a]$  defines a bounded operator on  $H^{approx}$ . Then, for each  $F \in \mathcal{F}(X)$ , the restriction of  $[D^{approx}, a]$  to the separable component  $H_F$  coincides with the local commutator:  $[D^{approx}, a] |_{H_F} = [D_F, a_F]$ .

Equivalently, the approximate derivation  $\delta^{approx}(a) := [D^{approx}, a]$  is completely determined by the family of local derivations  $\delta_F(a_F) := [D_F, a_F]$ .

*Proof.* The claim is essentially the localisation of the construction carried out in proposition 3.4.1, but we give a self-contained argument.

From proposition 3.4.1, we know that the approximate commutator  $[D^{approx}, a]$  is constructed as the bounded extension of the operator

$$T_0: \bigcup_{F \in \mathcal{F}(X)} \iota_F(H_F) \rightarrow H^{approx}$$

defined on the algebraic inductive limit by

$$T_0(\iota_F(\xi)) := \iota_F([D_F, a_F]\xi), \xi \in H_F$$

where  $\iota_F: H_F \rightarrow H^{approx}$  denotes the canonical isometric embedding. The operator  $[D^{approx}, a]$  is then the unique bounded linear extension of  $T_0$  to the whole Hilbert space  $H^{approx}$ .

Fix any  $F_0 \in \mathcal{F}(X)$ . We want to understand the restriction of  $[D^{approx}, a]$  to the subspace  $\iota_{F_0}(H_{F_0}) \subseteq H^{approx}$ . Let  $\xi \in H_{F_0}$ . By definition of the approximate commutator as the extension of  $T_0$ , we have

$$[D^{approx}, a](\iota_{F_0}(\xi)) = T_0(\iota_{F_0}(\xi)) = \iota_{F_0}([D_{F_0}, a_{F_0}]\xi)$$

Now interpret this identity in two ways:

- i) On the left-hand side,  $[D^{approx}, a](\iota_{F_0}(\xi))$  is simply the action of the global commutator on the vector  $\iota_{F_0}(\xi) \in H^{approx}$ .
- ii) On the right-hand side,  $\iota_{F_0}([D_{F_0}, a_{F_0}]\xi)$  is the image in  $H^{approx}$  of the local commutator  $[D_{F_0}, a_{F_0}]\xi \in H_{F_0}$ .

Thus, as elements of  $H^{approx}$ ,

$$[D^{\text{approx}}, a](\iota_{F_0}(\xi)) = \iota_{F_0}([D_{F_0}, a_{F_0}]\xi)$$

This identity holds for every  $\xi \in H_{F_0}$ .

By definition, the restriction of  $[D^{\text{approx}}, a]$  to the component  $H_{F_0}$  (identified with its image  $\iota_{F_0}(H_{F_0})$  in  $H^{\text{approx}}$ ) is the map  $[D^{\text{approx}}, a] \upharpoonright_{H_{F_0}} : H_{F_0} \rightarrow H_{F_0}$ , such that  $\iota_{F_0} \circ ([D^{\text{approx}}, a] \upharpoonright_{H_{F_0}}) = [D^{\text{approx}}, a] \circ \iota_{F_0}$ .

But from the identity above, we have

$$[D^{\text{approx}}, a] \circ \iota_{F_0}(\xi) = \iota_{F_0}([D_{F_0}, a_{F_0}]\xi) = \iota_{F_0} \circ [D_{F_0}, a_{F_0}](\xi)$$

for all  $\xi \in H_{F_0}$ . Since  $\iota_{F_0}$  is injective and isometric, we deduce that

$$[D^{\text{approx}}, a] \upharpoonright_{H_{F_0}} = [D_{F_0}, a_{F_0}] \text{ as operators on } H_{F_0}$$

This is precisely the localisation statement of the proposition.

The above relation shows that the approximate derivation  $\delta^{\text{approx}}(a) := [D^{\text{approx}}, a]$  is uniquely determined by the family of local derivations  $\delta_F(a_F) := [D_F, a_F], F \in \mathcal{F}(X)$ , in the sense that  $\delta^{\text{approx}}(a) \upharpoonright_{H_F} = \delta_F(a_F) \forall F$ .

Conversely, any family of local derivations  $\{\delta_F\}$  satisfying the compatibility conditions of proposition 3.4.1 yields a unique bounded global operator on  $H^{\text{approx}}$ . This completes the proof.  $\square$

### 3.5 Approximate Spectral Triples: Definition

We are ready to give the main definition.

*Definition 3.5.1 (Approximate Spectral Triple).* An approximate spectral triple is a triple  $(A^{\text{approx}}, H^{\text{approx}}, D^{\text{approx}})$  obtained from a compatible family  $(A_F, H_F, D_F)$  such that:

- i.  $A^{\text{approx}}$  acts non-degenerately on  $H^{\text{approx}}$ .
- ii. The commutator  $[D^{\text{approx}}, a]$  is bounded for all  $a$  in a dense  $*$ -subalgebra  $\mathcal{A}^{\text{approx}} \subseteq A^{\text{approx}}$ .
- iii. Resolvents of  $D^{\text{approx}}$  are approximate compact.
- iv. The family  $\{D_F\}$  is uniformly locally summable (e.g.,  $p$ -summable on each  $H_F$ ).

This definition captures the essential components of classical spectral triples while allowing for the non-separable setting.

### 3.6. Fundamental Properties

*Theorem 3.6.1 (Local-to-Global Principle).* Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples in the sense of definition 3.2.1, where  $\mathcal{F}(X)$  is a directed family of separable components. Let

$$A^{\text{approx}} = \lim_{F \in \mathcal{F}(X)} A_F, H^{\text{approx}} = \lim_{F \in \mathcal{F}(X)} H_F, D^{\text{approx}} = [D_F]$$

denote the approximate  $C^*$ -algebra, Hilbert space, and approximate Dirac operator constructed in sections 2–3. Then:

- i. For every  $F \in \mathcal{F}(X)$ , the canonical embeddings identify:  $A_F \subseteq A^{\text{approx}}, H_F \subseteq H^{\text{approx}}$ , and  $D^{\text{approx}} \upharpoonright_{H_F} = D_F$ . In particular, the approximate spectral triple  $(A^{\text{approx}}, H^{\text{approx}}, D^{\text{approx}})$  is completely determined by the family of local triples  $(A_F, H_F, D_F)$ .

- ii. Conversely, if  $(\tilde{A}^{approx}, \tilde{H}^{approx}, \tilde{D}^{approx})$  is any other approximate spectral triple built from the same compatible family  $\{(A_F, H_F, D_F)\}_F$  (via inductive limits), then there exists a unique unitary  $U: H^{approx} \rightarrow \tilde{H}^{approx}$  and a unique \*-isomorphism  $\Phi: A^{approx} \rightarrow \tilde{A}^{approx}$ , such that

$$U D^{approx} = \tilde{D}^{approx} U, U a = \Phi(a) U, \forall a \in A^{approx}$$

Thus, the approximate spectral triple is unique up to canonical unitary equivalence.

Proof. We split the proof into two parts, corresponding to the two claims.

Part (1): The approximate triple is determined by its local pieces

(a) Local subspaces and embeddings. By definition of the inductive limit Hilbert space,  $H^{approx} = \lim_F \rightarrow H_F$ , there exist isometric embeddings  $\iota_F: H_F \hookrightarrow H^{approx}$ , such that:

(a.1) If  $F \subseteq G$ , then  $\iota_G \circ \iota_{FG}^H = \iota_F$ , where  $\iota_{FG}^H: H_F \rightarrow H_G$  is the isometric embedding of the directed system.

(a.2) The union  $\bigcup_F \iota_F(H_F)$  is dense in  $H^{approx}$ .

We usually identify  $H_F$  with its image  $\iota_F(H_F)$ , i.e. think of  $H_F \subseteq H^{approx}$ .

Similarly, for the approximate  $C^*$ -algebra,  $A^{approx} = \lim_F \rightarrow A_F$ , we have \*-homomorphisms  $J_F: A_F \hookrightarrow A^{approx}$ , such that for  $F \subseteq G$ ,

$J_G \circ \iota_{FG}^A = J_F$ , where  $\iota_{FG}^A: A_F \rightarrow A_G$  is the connecting \*-homomorphism in the inductive system, and  $\bigcup_F J_F(A_F)$  is dense in  $A^{approx}$ .

Again, we identify  $A_F$  with its image in  $A^{approx}$ .

Thus, for every  $F$ , we can legitimately write  $A_F \subseteq A^{approx}, H_F \subseteq H^{approx}$ , in the sense of identifiable subalgebra and subspace.

(b) Restriction of  $D^{approx}$  to  $H_F$ . By definition 3.3.1, the approximate Dirac operator is given by

$$Dom(D^{approx}) = \bigcup_F \iota_F(Dom(D_F)), D^{approx}(\iota_F(\xi)) = \iota_F(D_F \xi), \xi \in Dom(D_F)$$

Let  $F_0 \in \mathcal{F}(X)$ , and let  $\xi \in Dom(D_{F_0}) \subseteq H_{F_0}$ . By the above formula,

$$D^{approx}(\iota_{F_0}(\xi)) = \iota_{F_0}(D_{F_0} \xi)$$

Therefore, on the subspace  $\iota_{F_0}(Dom(D_{F_0})) \subseteq H^{approx}$ ,

$$D^{approx} \upharpoonright_{\iota_{F_0}(Dom(D_{F_0}))} = \iota_{F_0} \circ D_{F_0} \circ \iota_{F_0}^{-1}$$

where  $\iota_{F_0}^{-1}$  denotes the inverse of  $\iota_{F_0}$  restricted to  $\iota_{F_0}(H_{F_0})$ . Thus, in the natural identification  $H_{F_0} \cong \iota_{F_0}(H_{F_0})$ , we may write simply  $D^{approx} \upharpoonright_{H_{F_0}} = D_{F_0}$ . Since each  $D_{F_0}$  is densely defined in  $H_{F_0}$ , and the domain of  $D^{approx}$  is the union of these embedded domains, this completely specifies the restriction of  $D^{approx}$  to each local Hilbert space.

(c) Determination by local data. The previous points show:

(c.1) The Hilbert space  $H^{approx}$  is generated (and topologically completed) by the directed family of inclusions of the  $H_F$ 's.

(c.2) The algebra  $A^{approx}$  is generated (and completed) by the directed family of inclusions of the  $A_F$ 's.

(c.3) The operator  $D^{approx}$  is defined by its values on each  $H_F$  and there it coincides with  $D_F$ .

Therefore, the triple  $(A^{approx}, H^{approx}, D^{approx})$  is entirely determined by the local family  $\{(A_F, H_F, D_F)\}_F$ . This proves statement (1).

Part (2): Uniqueness up to canonical unitary equivalence. Now suppose we are given another approximate spectral triple

$(\tilde{A}^{approx}, \tilde{H}^{approx}, \tilde{D}^{approx})$  constructed from the same compatible local data  $\{(A_F, H_F, D_F)\}_F$ . This means that:

i)  $\tilde{H}^{approx} = \lim \rightarrow_F H_F$  as a Hilbert space, via a family of isometric embeddings  $\tilde{\iota}_F: H_F \rightarrow \tilde{H}^{approx}$ .

ii)  $\tilde{A}^{approx} = \lim \rightarrow_F A_F$  as a  $C^*$ -algebra, via embeddings  $\tilde{j}_F: A_F \rightarrow \tilde{A}^{approx}$ .

iii)  $\tilde{D}^{approx}$  is defined in the same way as  $D^{approx}$ , i.e.

$$Dom(\tilde{D}^{approx}) = \bigcup_F \tilde{\iota}_F(Dom(D_F)), \tilde{D}^{approx}(\tilde{\iota}_F(\xi)) = \tilde{\iota}_F(D_F\xi)$$

We will use the universal property of inductive limits (both for Hilbert spaces and for  $C^*$ -algebras).

(a) Unitary equivalence of the Hilbert spaces There exist canonical isometric maps from each  $H_F$  into  $H^{approx}$  and into  $\tilde{H}^{approx}$ :

$$\iota_F: H_F \rightarrow H^{approx}, \tilde{\iota}_F: H_F \rightarrow \tilde{H}^{approx}$$

By the universal property of the inductive limit in the category of Hilbert spaces, there exists a unique isometry  $U: H^{approx} \rightarrow \tilde{H}^{approx}$ , such that  $U \circ \iota_F = \tilde{\iota}_F, \forall F \in \mathcal{F}(X)$ .

Moreover, since both inductive limits are generated by the images of the  $H_F$ 's and these images are dense in each Hilbert space, the isometry  $U$  has dense range. Being an isometric map with dense range,  $U$  is surjective; hence  $U$  is a unitary operator. So, we have a canonical unitary equivalence of the two Hilbert space inductive limits.

(b) \*-isomorphism of the approximate algebras. The same universal argument applies to the  $C^*$ -algebra inductive limits. There exist \*-homomorphisms

$$j_F: A_F \rightarrow A^{approx}, \tilde{j}_F: A_F \rightarrow \tilde{A}^{approx}$$

$$\text{with } j_G \circ \iota_{FG}^A = j_F, \tilde{j}_G \circ \tilde{\iota}_{FG}^A = \tilde{j}_F.$$

By the uniqueness of inductive limits in the category of  $C^*$ -algebras, there exists a unique \*-homomorphism  $\Phi: A^{approx} \rightarrow \tilde{A}^{approx}$ , such that  $\Phi \circ j_F = \tilde{j}_F, \forall F$ .

Since both  $A^{approx}$  and  $\tilde{A}^{approx}$  are generated (and completed) by the images of the  $A_F$ 's, the range of  $\Phi$  is dense;  $\Phi$  is isometric (by  $C^*$ -algebra theory and the way the norm on the inductive limit is defined), hence injective, and consequently surjective. Thus,  $\Phi$  is a \*-isomorphism.

(c) Intertwining of the Dirac operators and the representation. We must check that:

$$(c.1) UD^{approx} = \tilde{D}^{approx}U.$$

$$(c.2) Ua = \Phi(a)U, \forall a \in A^{approx}.$$

Take  $\xi \in Dom(D_F) \subseteq H_F$  and consider  $x = \iota_F(\xi) \in H^{approx}$ . Then,  $D^{approx}x = D^{approx}(\iota_F(\xi)) = \iota_F(D_F\xi)$ . Applying  $U$  and using  $U \circ \iota_F = \tilde{\iota}_F$ , we obtain

$$UD^{approx}x = U(\iota_F(D_F\xi)) = \tilde{\iota}_F(D_F\xi)$$

On the other hand,  $Ux = U(\iota_F(\xi)) = \tilde{\iota}_F(\xi)$ , and then,  $\tilde{D}^{approx}Ux = \tilde{D}^{approx}(\tilde{\iota}_F(\xi)) = \tilde{\iota}_F(D_F\xi)$  Thus, on the dense subspace  $\bigcup_F \iota_F(\text{Dom}(D_F))$ ,

$$UD^{approx} = \tilde{D}^{approx}U$$

By continuity (closures), this holds on the entire domain, and we obtain the operator equality.

For the algebra representation, let  $a = [a_F] \in A^{approx}$ , and  $\xi \in H_F$ . Then,  $a \iota_F(\xi) = \iota_F(a_F\xi)$ , so  $U(a \iota_F(\xi)) = U(\iota_F(a_F\xi)) = \tilde{\iota}_F(a_F\xi)$ .

On the other hand,  $\Phi(a) U(\iota_F(\xi)) = \Phi(a) \tilde{\iota}_F(\xi)$ . By definition of the inductive limit representation in  $\tilde{A}^{approx}$ ,  $\Phi(a)$  acts on  $\tilde{\iota}_F(\xi)$  as  $\tilde{\iota}_F(a_F\xi)$ . Hence,  $\Phi(a) U(\iota_F(\xi)) = \tilde{\iota}_F(a_F\xi) = U(a \iota_F(\xi))$ .

Therefore, on a dense subspace of  $H^{approx}$ ,  $Ua = \Phi(a)U$ , and the equality extends by continuity to all of  $H^{approx}$ . This shows that  $U$  and  $\Phi$  implement a unitary equivalence of the approximate spectral triples:

$$UD^{approx} = \tilde{D}^{approx}U, Ua = \Phi(a)U, \forall a \in A^{approx}$$

Combining Parts (1) and (2), the theorem is proved.  $\square$

*Definition 3.6.1 (Approximate p-summability).* Let  $(A^{approx}, H^{approx}, D^{approx})$  be an approximate spectral triple obtained from a compatible family  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$ . We say that the approximate spectral triple is approximately p-summable (for some  $p > 0$ ) if:

- For every  $F \in \mathcal{F}(X)$ , the operator  $(1 + D_F^2)^{-p/2}$  is trace-class on the separable Hilbert space  $H_F$ .
- The traces are uniformly bounded:

$$\sup_{F \in \mathcal{F}(X)} \text{Tr}((1 + D_F^2)^{-p/2}) < \infty$$

In that case, we define the approximate spectral dimension  $p$  (or approximate metric dimension) of the triple.

*Theorem 3.6.2 (Local summability implies approximate summability).* Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples, and let  $(A^{approx}, H^{approx}, D^{approx})$  be the associated approximate spectral triple. Assume that there exists  $p > 0$  such that:

- For every  $F \in \mathcal{F}(X)$ , the local spectral triple  $(A_F, H_F, D_F)$  is  $p$ -summable, i.e.

$$(1 + D_F^2)^{-p/2} \in \mathcal{L}^1(H_F)$$

where  $\mathcal{L}^1(H_F)$  denotes the ideal of trace-class operators on  $H_F$ ; and

- The traces are uniformly bounded:

$$\sup_{F \in \mathcal{F}(X)} \text{Tr}((1 + D_F^2)^{-p/2}) < \infty$$

Then, the approximate spectral triple  $(A^{approx}, H^{approx}, D^{approx})$  is approximately  $p$ -summable in the sense of definition 3.6.1. In particular, its metric and geometric behaviour is controlled uniformly by the family of local spectral dimensions.

Proof. The statement is essentially a precise formalisation of the idea that “summability on each separable window with uniform control implies global approximate summability”.

By hypothesis (1), for each  $F \in \mathcal{F}(X)$ , the triple  $(A_F, H_F, D_F)$  is a classical  $p$ -summable spectral triple. This means:

- a) Each  $D_F$  is self-adjoint on  $H_F$ .
- b) The operator  $T_F := (1 + D_F^2)^{-\frac{p}{2}}$  is trace-class, i.e.  $T_F \in \mathcal{L}^1(H_F)$ .

Recall that for a trace-class operator  $T_F$ , its trace is defined by

$$\text{Tr}(T_F) = \sum_{n=1}^{\infty} \langle T_F e_n, e_n \rangle$$

for any orthonormal basis  $(e_n)$  of  $H_F$ ; the result is independent of the chosen basis and finite. Thus, for each  $F$ , the spectral decay of  $D_F$  is sufficiently fast so that  $(1 + D_F^2)^{-p/2}$  is summable in the Hilbert–Schmidt sense and indeed trace-class.

By hypothesis (2), there is a global constant

$$C_p := \sup_{F \in \mathcal{F}(X)} \text{Tr}((1 + D_F^2)^{-p/2}) < \infty$$

This uniform bound means that, although the family of separable components  $\{H_F\}$  may be infinite and directed, the total spectral mass of each Dirac operator (as measured by the trace of  $(1 + D_F^2)^{-p/2}$ ) is uniformly controlled. This is crucial: without such a bound, we could have blow-up of dimension or volume as we move in the directed system.

We now verify explicitly that the approximate spectral triple satisfies the conditions of definition 3.6.1. By construction, the approximate spectral triple  $(A^{approx}, H^{approx}, D^{approx})$  is obtained from the local data via inductive limits, and, crucially, the local restrictions (see Christensen & Ivan, 2016) of  $D^{approx}$  coincide with the  $D_F$ ’s. More precisely, as shown in theorem 3.6.1 (local-to-global principle), for every  $F \in \mathcal{F}(X)$ ,  $D^{approx} |_{H_F} = D_F$  (with the natural identification  $H_F \subseteq H^{approx}$ ).

Consider the operator  $(1 + (D^{approx})^2)^{-p/2} = f_p(D^{approx})$ , where  $f_p(\lambda) = (1 + \lambda^2)^{-p/2}$ ,  $\lambda \in \mathbb{R}$  is a bounded Borel function. By the functional calculus for self-adjoint operators, this defines a bounded operator on  $H^{approx}$ . By the same functoriality of the functional calculus used in earlier propositions (3.3.2 and 3.3.3), we know that for each  $F$ ,

$$(1 + (D^{approx})^2)^{-p/2} |_{H_F} = (1 + D_F^2)^{-p/2}$$

Thus, the restriction of the global  $p$ -summability operator to each local component is exactly the local trace-class operator  $T_F$ . Now, by hypothesis (1), each  $T_F$  is trace-class on  $H_F$ ; by hypothesis (2), the traces satisfy

$$\sup_F \text{Tr}(T_F) = \sup_F \text{Tr}((1 + D_F^2)^{-p/2}) = C_p < \infty$$

This is exactly the content of definition 3.6.1: i) local trace-class property (one for each  $F$ ), ii) together with uniform control of the trace across the directed system. Therefore, by definition 3.6.1, the approximate spectral triple  $(A^{approx}, H^{approx}, D^{approx})$  is approximately  $p$ -summable.

The conclusion has an important geometric meaning: local spectral triples  $(A_F, H_F, D_F)$  encode a dimension parameter  $p$  on each separable window  $H_F$ ; the uniform bound  $C_p$  ensures that this dimension does not explode when one ranges over all separable components. Thus, the approximate spectral triple carries a well-defined approximate dimension  $p$ , and one can consistently define approximate versions of spectral invariants (zeta functions, heat traces, etc.) using the family  $\{T_F\}_F$  with uniform control. This completes the proof.  $\square$

*Theorem 3.6.3 (Local Differential Calculus).* We recall that for a (classical) spectral triple  $(A, H, D)$ , the associated first-order differential calculus is defined by

$$\Omega_D^1(A) = \text{span}\{a[D, b]: a, b \in \mathcal{A}\} \subseteq \mathcal{B}(H)$$

where  $\mathcal{A}$  is a dense \*-subalgebra of  $A$  on which all commutators  $[D, b]$  are bounded?

In the approximate setting, we adopt the following analogue. Let  $\{(A_F, H_F, D_F)\}_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples and let  $(A^{\text{approx}}, H^{\text{approx}}, D^{\text{approx}})$  be the associated approximate spectral triple. Let  $\mathcal{A}^{\text{approx}} \subset A^{\text{approx}}$  be a dense \*-subalgebra consisting of elements  $a = [a_F]$  such that the family of local commutators  $[D_F, a_F]$  is uniformly bounded (as in proposition 3.4.1).

Define

$$\delta^{\text{approx}}(a) := [D^{\text{approx}}, a], a \in \mathcal{A}^{\text{approx}}$$

and set

$$\Omega_{D^{\text{approx}}}^1(A^{\text{approx}}) = \text{span}\{a \delta^{\text{approx}}(b): a, b \in \mathcal{A}^{\text{approx}}\} \subseteq \mathcal{B}(H^{\text{approx}})$$

For each  $F$ , denote by  $\mathcal{A}_F \subseteq A_F$  the corresponding dense \*-subalgebra and

$$\delta_F(b_F) := [D_F, b_F], \Omega_{D_F}^1(A_F) = \text{span}\{a_F [D_F, b_F]: a_F, b_F \in \mathcal{A}_F\} \subseteq \mathcal{B}(H_F)$$

Under the assumptions above:

(1) For every  $a \in \mathcal{A}^{\text{approx}}$  represented by a compatible family  $(a_F)_F$ , the global derivation  $\delta^{\text{approx}}(a) = [D^{\text{approx}}, a]$  is completely determined by its local restrictions:

$$\delta^{\text{approx}}(a) |_{H_F} = [D_F, a_F] = \delta_F(a_F), \forall F \in \mathcal{F}(X)$$

(2) The global first-order calculus  $\Omega_{D^{\text{approx}}}^1(A^{\text{approx}})$  is the inductive limit of the local calculi:

$$\Omega_{D^{\text{approx}}}^1(A^{\text{approx}}) = \lim_{F \in \mathcal{F}(X)} \Omega_{D_F}^1(A_F) \subseteq \mathcal{B}(H^{\text{approx}})$$

in the sense that each element of  $\Omega_{D^{\text{approx}}}^1(A^{\text{approx}})$  is the norm limit of finite sums of operators of the form

$$\iota_F(a_F [D_F, b_F]) \iota_F^{-1}, a_F, b_F \in \mathcal{A}_F$$

and conversely, the image of every local form  $a_F [D_F, b_F]$  in  $\mathcal{B}(H^{\text{approx}})$  lies in  $\Omega_{D^{\text{approx}}}^1(A^{\text{approx}})$ .

*Proof.* We split the proof in two parts corresponding to items (1) and (2).

Part (1): The derivation is determined by local restrictions. This is essentially a global reformulation of proposition 3.4.2, but we give a direct argument. Let  $a \in \mathcal{A}^{\text{approx}}$ , represented by a compatible family  $(a_F)_{F \in \mathcal{F}(X)}$  with  $a_F \in \mathcal{A}_F$ . Assume

$$M(a) := \sup_F \| [D_F, a_F] \| < \infty$$

so that, by proposition 3.4.1, the commutator  $\delta^{approx}(a) := [D^{approx}, a]$  defines a bounded operator on  $H^{approx}$ .

Fix  $F_0 \in \mathcal{F}(X)$ . Let  $\xi \in H_{F_0}$ , and view  $\xi$  as a vector in  $H^{approx}$  via the canonical embedding  $\iota_{F_0}: H_{F_0} \rightarrow H^{approx}$ . On the one hand, by the definition of the approximate commutator on the algebraic inductive limit (proposition 3.4.1), we have

$$\delta^{approx}(a)(\iota_{F_0}(\xi)) = \iota_{F_0}([D_{F_0}, a_{F_0}] \xi)$$

On the other hand, the restriction  $\delta^{approx}(a) |_{H_{F_0}}$  is defined by

$$\iota_{F_0} \circ (\delta^{approx}(a) |_{H_{F_0}}) = \delta^{approx}(a) \circ \iota_{F_0}$$

Thus,

$$\iota_{F_0}(\delta^{approx}(a) |_{H_{F_0}}(\xi)) = \delta^{approx}(a)(\iota_{F_0}(\xi)) = \iota_{F_0}([D_{F_0}, a_{F_0}] \xi)$$

Since  $\iota_{F_0}$  is isometric and injective, we can cancel it and obtain  $\delta^{approx}(a) |_{H_{F_0}}(\xi) = [D_{F_0}, a_{F_0}] \xi, \forall \xi \in H_{F_0}$ . Therefore,

$\delta^{approx}(a) |_{H_{F_0}} = [D_{F_0}, a_{F_0}] = \delta_{F_0}(a_{F_0})$ , as operators on  $H_{F_0}$ . Since  $F_0$  was arbitrary, this proves that  $\delta^{approx}(a)$  is completely determined by the family of local derivations  $\{\delta_F(a_F) = [D_F, a_F]\}_F$ . This proves item (1).

Part (2): The first-order calculus is the inductive limit of local calculi. We now show the equality

$$\Omega_{D^{approx}}^1(A^{approx}) = \lim_F \rightarrow \Omega_{D_F}^1(A_F) \text{ inside } \mathcal{B}(H^{approx}).$$

(2.a) Inclusion of local forms into the global calculus. Fix  $F \in \mathcal{F}(X)$ , and let  $a_F, b_F \in \mathcal{A}_F$ . Consider the local one-form  $\omega_F := a_F [D_F, b_F] \in \Omega_{D_F}^1(A_F) \subseteq \mathcal{B}(H_F)$ . We view  $\omega_F$  as an operator on  $H^{approx}$  by  $\omega_F^{approx} := \iota_F \circ \omega_F \circ \iota_F^{-1}$  on the subspace  $\iota_F(H_F)$ , and extend it by zero on the orthogonal complement if we wish to think of it as an operator defined on all of  $H^{approx}$  (or, more precisely, identify  $\omega_F$  with its natural extension by continuity to the closure of  $\iota_F(H_F)$ ).

Now consider the corresponding elements  $a, b \in \mathcal{A}^{approx}$  given by the inductive limit embeddings:  $a := \jmath_F(a_F), b := \jmath_F(b_F)$ . By the definition of  $\delta^{approx}$  and part (1), we have  $\delta^{approx}(b) |_{H_F} = [D_F, b_F]$ . Therefore, the product  $a \delta^{approx}(b)$  satisfies

$$(a \delta^{approx}(b)) |_{H_F} = a_F [D_F, b_F] = \omega_F$$

In particular, the image of  $\omega_F$  in  $\mathcal{B}(H^{approx})$  is precisely the operator  $a \delta^{approx}(b)$  restricted to  $H_F$ , and hence lies in

$$\text{span}\{a' \delta^{approx}(b') : a', b' \in \mathcal{A}^{approx}\} = \Omega_{D^{approx}}^1(A^{approx})$$

By linearity, this shows that **every** element of the local calculi  $\Omega_{D_F}^1(A_F)$  embeds into  $\Omega_{D^{approx}}^1(A^{approx})$ . Passing to the inductive limit, we deduce

$$\lim_F \rightarrow \Omega_{D_F}^1(A_F) \subseteq \Omega_{D^{approx}}^1(A^{approx})$$

(2.b) Every global one-form is locally generated. Conversely, let  $\omega \in \Omega_{D^{approx}}^1(A^{approx})$ . By definition,

$$\omega = \sum_{k=1}^n a_k \delta^{approx}(b_k), a_k, b_k \in \mathcal{A}^{approx}$$

Write  $a_k = [(a_k)_F], b_k = [(b_k)_F]$  as compatible families, with  $(a_k)_F, (b_k)_F \in \mathcal{A}_F$ . Fix a separable component  $F_0 \in \mathcal{F}(X)$ . On the subspace  $H_{F_0}$ , by part (1),

$$\delta^{approx}(b_k) |_{H_{F_0}} = [D_{F_0}, (b_k)_{F_0}]$$

So,

$$\omega |_{H_{F_0}} = \sum_{k=1}^n (a_k)_{F_0} [D_{F_0}, (b_k)_{F_0}] \in \Omega_{D_{F_0}}^1(A_{F_0})$$

Thus, locally on each  $H_{F_0}$ , the restriction of  $\omega$  lies in the corresponding local calculus  $\Omega_{D_{F_0}}^1(A_{F_0})$ . Since  $\mathcal{F}(X)$  is directed and the union  $\bigcup_F H_F$  is dense in  $H^{approx}$ , we can approximate  $\omega$  in operator norm by operators supported on finitely many components. More precisely: for each  $\varepsilon > 0$ , we can approximate  $\omega$  uniformly on a finite set of vectors spanning a finite number of components  $H_{F_1}, \dots, H_{F_m}$  by elements of the form

$$\sum_{j=1}^m \omega_{F_j}, \omega_{F_j} \in \Omega_{D_{F_j}}^1(A_{F_j})$$

under the canonical embeddings into  $\mathcal{B}(H^{approx})$ . This shows that  $\omega$  lies in the norm-closure of the linear span of the images of the local calculi. But, by construction,  $\Omega_{D^{approx}}^1(A^{approx})$  is already defined as the norm-closed span of  $a \delta^{approx}(b)$ , and we have just seen that such operators are locally generated by  $\Omega_{D_F}^1(A_F)$ . Therefore,

$$\Omega_{D^{approx}}^1(A^{approx}) \subseteq \overline{\text{span} \bigcup_F \Omega_{D_F}^1(A_F)} = \lim_F \rightarrow \Omega_{D_F}^1(A_F)$$

where the last equality is precisely the description of the inductive limit in this context. Combining with the inclusion in (a), we conclude

$$\Omega_{D^{approx}}^1(A^{approx}) = \lim_F \rightarrow \Omega_{D_F}^1(A_F)$$

as claimed. This completes the proof of theorem 3.6.3.  $\square$

### 3.7. Conceptual Consequences

Approximate spectral triples provide:

- i) A geometric interpretation of  $A^{approx}$  as a weak noncommutative manifold;
- ii) Local differential and metric information inherited from separable windows;
- iii) The foundation for approximate K-homology and approximate index theory;
- iv) Compatibility with Connes' philosophy of encoding geometry through analytic data.

The framework is genuinely new: it replaces global compactness and separability with local compactness and directed separable geometry.

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## 4. Approximate Spectral Metric Geometry

In this section we develop the metric side of the theory associated with an approximate spectral triple  $(A^{approx}, H^{approx}, D^{approx})$ , constructed from a compatible family of local spectral triples  $(A_F, H_F, D_F)$  on separable components. The guiding idea is that the Connes distance on the state space of a  $C^*$ -algebra, defined in the separable framework, can be extended to the non-separable setting via an approximate Lipschitz seminorm built from the local commutators  $[D_F, a_F]$ .

We place our construction in the context of noncommutative metric geometry, as initiated by Connes with the spectral distance and further developed by Rieffel in terms of quantum compact metric spaces and Lipschitz seminorms, and by Latrémolière via the quantum Gromov–Hausdorff propinquity and its variants [16-21].

### 4.1. Classical Spectral Distance and Lipschitz Seminorms

For a classical spectral triple  $(A, H, D)$  with  $A$  a dense  $*$ -subalgebra of a unital  $C^*$ -algebra  $\mathfrak{A}$ , Connes defines a (possibly extended) metric on the state space  $\mathcal{S}(\mathfrak{A})$  by

$$d_D(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in A, \|[D, a]\| \leq 1\}$$

The associated Lipschitz seminorm is  $L_D(a) := \|[D, a]\|$ ,  $a \in A$ , so that the distance can be written in the Kantorovich–Rubinstein form

$$d_D(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : L_D(a) \leq 1\}$$

Rieffel abstracted this picture in the notion of a compact quantum metric space  $(\mathfrak{A}, L)$ , where  $L$  is a (densely defined) seminorm satisfying appropriate conditions such that the Monge–Kantorovich metric induced by  $L$  metrises the *weak\** topology on  $\mathcal{S}(\mathfrak{A})$ . This framework underlies subsequent developments such as the quantum Gromov–Hausdorff distance and the quantum propinquity [20,21].

In our setting, each local triple  $(A_F, H_F, D_F)$  yields a Lipschitz seminorm

$$L_F(a_F) := \|[D_F, a_F]\|, a_F \in \mathcal{A}_F$$

and the associated Connes metric  $d_{D_F}$  on the state space of the closure of  $A_F$ . Our goal is to construct an approximate Lipschitz structure on  $A^{approx}$  whose restrictions recover the  $L_F$ 's [1].

### 4.2. Approximate Lipschitz Seminorm

Let  $\mathcal{A}^{approx} \subset A^{approx}$  be the dense  $*$ -subalgebra of elements represented by compatible families  $(a_F)_{F \in \mathcal{F}(X)}$  with each  $a_F \in \mathcal{A}_F$  and such that the family of local commutators is uniformly bounded:

$$M(a) := \sup_{F \in \mathcal{F}(X)} \|[D_F, a_F]\| < \infty$$

By proposition 3.4.1, for each such  $a$  the approximate commutator

$$[D^{approx}, a] \in \mathcal{B}(H^{approx})$$

is well defined and satisfies

$$\| [D^{\text{approx}}, a] \| = \sup_F \| [D_F, a_F] \|$$

We use this to define the global seminorm.

*Definition 4.2.1 (Approximate Lipschitz Seminorm).* For  $a \in \mathcal{A}^{\text{approx}}$ , set

$$L^{\text{approx}}(a) := \| [D^{\text{approx}}, a] \| = \sup_{F \in \mathcal{F}(X)} \| [D_F, a_F] \|$$

This is a seminorm on  $\mathcal{A}^{\text{approx}}$ , possibly taking the value  $+\infty$  outside its natural domain. By construction, for each  $F$ ,

$$L^{\text{approx}}(a) \upharpoonright_{A_F} = \| [D_F, a_F] \| = L_F(a_F)$$

Thus,  $L^{\text{approx}}$  extends all the local Lipschitz seminorms simultaneously, and dominates them in norm.

A basic property is the compatibility with the inductive system [2,4]:

*Proposition 4.2.2 (Local–global compatibility).* For all  $F \subseteq G$  in  $\mathcal{F}(X)$  and all  $a \in \mathcal{A}^{\text{approx}}$ , one has

$$L_F(a_F) \leq L_G(a_G) \leq L^{\text{approx}}(a)$$

*Proof.* Let  $(A_F, H_F, D_F)_{F \in \mathcal{F}(X)}$  be a directed system of spectral triples in the sense of Connes [1], and let

$$A^{\text{approx}} = \lim_F \rightarrow A_F, H^{\text{approx}} = \lim_F \rightarrow H_F, D^{\text{approx}}$$

be the corresponding inductive-limit algebra, Hilbert space, and approximate Dirac operator.

The Lipschitz seminorms are defined by

$$L_F(a_F) = \| [D_F, a_F] \|, L^{\text{approx}}(a) = \sup_{E \in \mathcal{F}(X)} \| [D_E, a_E] \|.$$

We prove the two inequalities separately.

1. The inequality  $L_F(a_F) \leq L_G(a_G)$ . Let  $F \subseteq G$ , and let  $V_{FG}: H_F \rightarrow H_G$  denote the canonical isometric inclusion provided by the inductive-limit structure [2,4]. Compatibility of the system means that, for every  $\xi \in \text{Dom}(D_F)$ ,

$$D_G V_{FG} \xi = V_{FG} D_F \xi, a_G V_{FG} \xi = V_{FG} a_F \xi$$

We compute the commutator on  $H_F$ , transported into  $H_G$ :  $[D_G, a_G] V_{FG} \xi = D_G (a_G V_{FG} \xi) - a_G (D_G V_{FG} \xi) = D_G (V_{FG} a_F \xi) - a_G (V_{FG} D_F \xi) = V_{FG} D_F a_F \xi - V_{FG} a_F D_F \xi = V_{FG} ([D_F, a_F] \xi)$ . Since  $V_{FG}$  is an isometry, this identity implies

$$\| [D_F, a_F] \xi \| = \| V_{FG} [D_F, a_F] \xi \| = \| [D_G, a_G] V_{FG} \xi \|$$

Furthermore,  $\| V_{FG} \xi \| = \| \xi \|$ . Hence

$$\frac{\| [D_F, a_F] \xi \|}{\| \xi \|} = \frac{\| [D_G, a_G] V_{FG} \xi \|}{\| V_{FG} \xi \|} \leq \| [D_G, a_G] \|$$

Taking the supremum over all non-zero  $\xi \in H_F$ , we obtain  $\| [D_F, a_F] \| \leq \| [D_G, a_G] \|$ . Thus,  $L_F(a_F) \leq L_G(a_G)$ . This argument is a

straightforward application of the general fact that  $\|V^*TV\| \leq \|T\|$  for every bounded operator  $T$  and isometry  $V$ , as found in standard operator theory (cf. Reed & Simon, 1980; Murphy, 1990) [6,12].

2. The inequality  $L_G(a_G) \leq L^{approx}(a)$ . This follows immediately from the definition of the approximate Lipschitz seminorm. Indeed,  $L^{approx}(a) = \sup_{E \in \mathcal{F}(X)} \| [D_E, a_E] \|$ , and in particular the index  $E = G$  is included in the supremum. Therefore,

$$\| [D_G, a_G] \| \leq \sup_{E \in \mathcal{F}(X)} \| [D_E, a_E] \| = L^{approx}(a)$$

which yields the desired inequality.

Conclusion. Combining both parts, we conclude that for every  $F \subseteq G$  in the directed family  $\mathcal{F}(X)$  and every  $a \in A^{approx}$ ,

$$L_F(a_F) \leq L_G(a_G) \leq L^{approx}(a)$$

This establishes the local–global compatibility of the Lipschitz seminorm.  $\square$

Remarks and references. The proof relies fundamentally on:

- a) Connes' notion of Lipschitz seminorms in spectral triples (Connes, 1989; 1994).
- b) Properties of inductive limits of Hilbert spaces and  $C^*$ -algebras (Pedersen, 1979; Blackadar, 2006; Murphy, 1990) [2,4,6].
- c) Operator norm estimates via isometric embeddings [12].

These references will appear fully formatted in the “References” section once requested.

### 4.3. Approximate Connes Distance

Let  $\mathfrak{A}^{approx}$  be the  $C^*$ -completion of  $A^{approx}$  in the operator norm. Denote by  $\mathcal{S}(\mathfrak{A}^{approx})$  its state space.

*Definition 4.3.1 (Approximate Connes Distance).* For  $\varphi, \psi \in \mathcal{S}(\mathfrak{A}^{approx})$ , define

$$d^{approx}(\varphi, \psi) := \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}^{approx}, L^{approx}(a) \leq 1\}$$

This is an extended pseudo-metric (possibly taking the value  $+\infty$ ) on  $\mathcal{S}(\mathfrak{A}^{approx})$ , entirely analogous to the Connes distance in the separable case. For each separable component  $F$ , let  $\mathfrak{A}_F$  be the closure of  $A_F$ , and consider the canonical map

$$\iota_F^*: \mathcal{S}(\mathfrak{A}^{approx}) \rightarrow \mathcal{S}(\mathfrak{A}_F), \iota_F^*(\varphi) := \varphi \circ J_F$$

where  $J_F: \mathfrak{A}_F \rightarrow \mathfrak{A}^{approx}$  is the inclusion.

*Proposition 4.3.2 (Restriction to Local State Spaces).* Let  $(A^{approx}, H^{approx}, D^{approx})$  be the approximate spectral triple constructed from a compatible family  $(A_F, H_F, D_F)_{F \in \mathcal{F}(X)}$ . Let  $\mathfrak{A}^{approx}$  be the  $C^*$ -completion of  $A^{approx}$ , and let  $\mathfrak{A}_F$  be the  $C^*$ -completion of  $A_F$ . Denote by  $J_F: \mathfrak{A}_F \rightarrow \mathfrak{A}^{approx}$  the canonical  $*$ -monomorphism of the inductive system. For each state  $\varphi \in \mathcal{S}(\mathfrak{A}^{approx})$ , define its restriction to  $\mathfrak{A}_F$  by  $\iota_F^*(\varphi) := \varphi \circ J_F \in \mathcal{S}(\mathfrak{A}_F)$ . Let  $d_{D_F}$  be the Connes distance induced by the local spectral triple  $(A_F, H_F, D_F)$ , and let  $d^{approx}$  be the approximate Connes distance defined by the approximate Lipschitz seminorm  $L^{approx}$ . Then for all  $F \in \mathcal{F}(X)$  and all states  $\varphi, \psi \in \mathcal{S}(\mathfrak{A}^{approx})$ ,

$$d_{D_F}(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi)$$

*Proof.* For the local spectral triple  $(A_F, H_F, D_F)$ , the Lipschitz seminorm is defined by  $L_F(a_F) := \| [D_F, a_F] \|$ ,  $a_F \in A_F$ . For the approximate triple, elements  $a \in \mathcal{A}^{approx}$  are represented by compatible families  $a = (a_E)_{E \in \mathcal{F}(X)}$ ,  $a_E \in A_E$ , and the approximate Lipschitz seminorm is given by

$$L^{approx}(a) := \| [D^{approx}, a] \| = \sup_{E \in \mathcal{F}(X)} \| [D_E, a_E] \|$$

By Proposition 4.2.2, for each  $F$  and each such  $a$ , we have

$$L_F(a_F) \leq L^{approx}(a)$$

The local Connes distance on  $\mathcal{S}(\mathfrak{A}_F)$  is

$$d_{D_F}(\phi, \psi) = \sup\{ | \phi(a_F) - \psi(a_F) | : a_F \in A_F, L_F(a_F) \leq 1 \}$$

The approximate Connes distance on  $\mathcal{S}(\mathfrak{A}^{approx})$  is

$$d^{approx}(\phi, \psi) = \sup\{ | \phi(a) - \psi(a) | : a \in \mathcal{A}^{approx}, L^{approx}(a) \leq 1 \}$$

By definition of the restriction map,  $\iota_F^*(\phi) = \phi \circ ]_F$ ,  $\iota_F^*(\psi) = \psi \circ ]_F$ .

Let  $a_F \in A_F$  satisfy  $L_F(a_F) = \| [D_F, a_F] \| \leq 1$ . We consider its image  $a \in \mathcal{A}^{approx}$  obtained via the inductive-limit embedding:

- a) On the component  $F$ ,  $a_F$  is the given element.
- b) On each larger component  $E \supseteq F$ , the corresponding  $a_E$  is determined by the compatibility maps of the system.
- c) On unrelated components,  $a_E$  is determined in a coherent way by the inductive system [2,4].

By definition of  $L^{approx}$  and Proposition 4.2.2,

$$L^{approx}(a) = \sup_E \| [D_E, a_E] \| \geq \| [D_F, a_F] \| = L_F(a_F) \leq 1$$

Thus,  $L_F(a_F) \leq 1 \implies L^{approx}(a) \leq 1$ . In other words, every element of the local Lipschitz unit ball  $\{a_F \in A_F : L_F(a_F) \leq 1\}$  embeds into the global Lipschitz unit ball

$$\{a \in \mathcal{A}^{approx} : L^{approx}(a) \leq 1\}$$

Fix states  $\phi, \psi \in \mathcal{S}(\mathfrak{A}^{approx})$ . Their restrictions to  $\mathfrak{A}_F$  are

$$\phi := \iota_F^*(\phi) = \phi \circ ]_F, \psi' := \iota_F^*(\psi) = \psi \circ ]_F$$

For any  $a_F \in A_F$  with  $L_F(a_F) \leq 1$ , let  $a \in \mathcal{A}^{approx}$  be its image. Then:

- i) The evaluation of  $\phi$  on  $a_F$  satisfies  $\phi(a_F) = \phi(]_F(a_F)) = \phi(a)$ , because  $]_F(a_F)$  coincides with the  $F$ -component of  $a$  as an element of the inductive limit.
- ii) Similarly,  $\psi'(a_F) = \psi(]_F(a_F)) = \psi(a)$ . Hence,  $| \phi(a_F) - \psi'(a_F) | = | \phi(a) - \psi(a) |$ . Since  $L_F(a_F) \leq 1$  implies  $L^{approx}(a) \leq 1$ , the element  $a$  belongs to the admissible set defining  $d^{approx}(\phi, \psi)$ . Therefore

$$|\phi(a_F) - \psi'(a_F)| \leq d^{approx}(\phi, \psi)$$

Taking the supremum over all  $a_F \in A_F$  such that  $L_F(a_F) \leq 1$ , we obtain

$$d_{D_F}(\phi, \psi') = \sup\{|\phi(a_F) - \psi'(a_F)| : a_F \in A_F, L_F(a_F) \leq 1 \leq d^{approx}(\phi, \psi)\}$$

That is,  $d_{D_F}(i_F^*(\phi), i_F^*(\psi)) \leq d^{approx}(\phi, \psi)$ , which completes the proof.  $\square$

Remarks:

1. The structure of the argument follows the standard pattern used by Connes in the separable case: a spectral metric is defined via a Lipschitz seminorm, and restriction to subalgebras narrows the admissible set in the variational formula [1].
2. The use of local Lipschitz balls embedded in a global Lipschitz ball mirrors Rieffel's treatment of quantum metric spaces, where inclusions of  $C^*$ -subalgebras induce inequalities between the corresponding Monge–Kantorovich metrics [16,17].
3. The inductive-limit interpretation of the embeddings  $J_F$  is standard in the theory of  $C^*$ -algebras [2,4].

#### 4.4. Relation to Quantum Compact Metric Spaces

Rieffel's theory associates to a pair  $(\mathfrak{A}, L)$  a Monge–Kantorovich metric  $d_L$  on  $\mathcal{S}(\mathfrak{A})$ , and calls  $(\mathfrak{A}, L)$  a compact quantum metric space if  $d_L$  metrises the *weak\** topology. This has been extended in many directions, including quantum locally compact metric spaces and various Gromov–Hausdorff-type propinquities. In our approximate setting, we have the following structural result.

*Theorem 4.4.1 (Local compactness and approximate quantum metric structure).*

Let  $\mathcal{F}(X)$  be a directed family of indices, and for each  $F \in \mathcal{F}(X)$  let  $(\mathfrak{A}_F, L_F)$  be a compact quantum metric space in the sense of Rieffel [16,17]. That is:

- A.  $\mathfrak{A}_F$  is a unital  $C^*$ -algebra.
- B.  $L_F$  is a (possibly extended) seminorm on a *dense\**-subalgebra  $\mathcal{A}_F \subset \mathfrak{A}_F$ .
- C. The Monge–Kantorovich metric  $d_F(\phi, \psi) = \sup\{|\phi(a_F) - \psi(a_F)| : a_F \in \mathcal{A}_F, L_F(a_F) \leq 1\}$  is finite for all states  $\phi, \psi \in \mathcal{S}(\mathfrak{A}_F)$ , and induces the *weak\**-topology on  $\mathcal{S}(\mathfrak{A}_F)$ .

Assume moreover that:

- (1) The family  $(\mathfrak{A}_F, L_F)_F$  is inductively compatible:  $\mathfrak{A}^{approx} = \lim_{\rightarrow F} \mathfrak{A}_F$  is a  $C^*$ -algebra with canonical embeddings  $J_F: \mathfrak{A}_F \rightarrow \mathfrak{A}^{approx}$ .
- (2) There exists a *dense\**-subalgebra  $\mathcal{A}^{approx} \subset \mathfrak{A}^{approx}$  such that each  $a \in \mathcal{A}^{approx}$  is represented by a compatible family  $(a_F)_F$  with  $a_F \in \mathcal{A}_F$ , and the approximate Lipschitz seminorm

$$L^{approx}(a) := \sup_{F \in \mathcal{F}(X)} L_F(a_F)$$

is finite on  $\mathcal{A}^{approx}$ .

- (3) There exists a constant  $C > 0$  such that, for every  $F$ , the Lipschitz unit ball

$$\mathcal{B}_F := \{a_F \in \mathcal{A}_F : L_F(a_F) \leq 1, \|a_F\| \leq C\}$$

is norm–totally bounded in  $\mathfrak{A}_F$  (this is precisely Rieffel's compactness criterion, assumed with a uniform bound  $C$  across all  $F$ ).

Define the approximate Monge–Kantorovich metric on the state space  $\mathcal{S}(\mathfrak{A}^{approx})$  by

$$d^{approx}(\varphi, \psi) := \sup\{ |\varphi(a) - \psi(a)| : a \in \mathcal{A}^{approx}, L^{approx}(a) \leq 1 \}$$

Then:

- i) For all  $\varphi, \psi \in \mathcal{S}(\mathfrak{A}^{approx})$ , one has  $d^{approx}(\varphi, \psi) = \sup_{F \in \mathcal{F}(X)} d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$ , where  $\iota_F^*(\varphi) := \varphi \circ j_F$  is the restriction of  $\varphi$  to  $\mathfrak{A}_F$ .
- ii) The metric  $d^{approx}$  separates states and is finite on  $\mathcal{S}(\mathfrak{A}^{approx})$ .
- iii) The topology induced by  $d^{approx}$  is the projective limit of the local *weak\**-topologies: a net  $(\varphi_\alpha)$  converges to  $\varphi$  in  $d^{approx}$  if and only if, for every  $F$ , the restricted net  $\iota_F^*(\varphi_\alpha)$  converges to  $\iota_F^*(\varphi)$  in the metric  $d_F$  (equivalently, in the *weak\**-topology on  $\mathcal{S}(\mathfrak{A}_F)$ ). We refer to  $(\mathfrak{A}^{approx}, L^{approx})$  as an approximate quantum metric space.

**Proof.** Fix  $F \in \mathcal{F}(X)$ . For  $\varphi, \psi \in \mathcal{S}(\mathfrak{A}^{approx})$ , recall that we define

$$\iota_F^*(\varphi) := \varphi \circ j_F, \iota_F^*(\psi) := \psi \circ j_F$$

as states on  $\mathfrak{A}_F$ . By definition,

$$d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) = \sup\{ |\varphi(j_F(a_F)) - \psi(j_F(a_F))| : a_F \in \mathcal{A}_F, L_F(a_F) \leq 1 \}$$

Given  $a_F \in \mathcal{A}_F$  with  $L_F(a_F) \leq 1$ , consider its image  $a \in \mathcal{A}^{approx}$  in the inductive limit. By construction of  $L^{approx}$  and proposition 4.2.2, one has

$$L^{approx}(a) = \sup_E L_E(a_E) \geq L_F(a_F) \geq 0$$

In particular, if  $L_F(a_F) \leq 1$  then  $L^{approx}(a) \leq 1$ . Moreover,

$$\varphi(a) = \varphi(j_F(a_F)), \psi(a) = \psi(j_F(a_F))$$

Therefore,

$$|\iota_F^*(\varphi)(a_F) - \iota_F^*(\psi)(a_F)| = |\varphi(a) - \psi(a)|$$

Since  $a$  belongs to the global Lipschitz unit ball  $\{a \in \mathcal{A}^{approx} : L^{approx}(a) \leq 1\}$ , we have  $|\varphi(a) - \psi(a)| \leq d^{approx}(\varphi, \psi)$ .

Taking the supremum over all  $a_F \in \mathcal{A}_F$  with  $L_F(a_F) \leq 1$ , we obtain:

$$d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi)$$

Since  $F$  was arbitrary,  $\sup_{F \in \mathcal{F}(X)} d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi)$ . This proves one inequality:  $\sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi)$ .

This is precisely the content of proposition 4.3.2, specialized to the present setting.

We now show the converse inequality:

$$d^{approx}(\varphi, \psi) \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Let  $a \in \mathcal{A}^{approx}$  with  $L^{approx}(a) \leq 1$ . By definition,  $L^{approx}(a) = \sup_F L_F(a_F) \leq 1$ , so in particular  $L_F(a_F) \leq 1, \forall F$ .

Fix  $\varepsilon > 0$ . By the uniform compactness assumption on the Lipschitz unit balls (hypothesis (3)), for each  $F$  the set  $\mathcal{B}_F = \{a'_F \in \mathcal{A}_F: L_F(a'_F) \leq 1, \|a'_F\| \leq C\}$  is norm–totally bounded, and the constant  $C$  is independent of  $F$ .

For the given  $a$ , each component  $a_F$  belongs to the (not necessarily norm–bounded) unit ball  $\{L_F \leq 1\}$ . We may normalise or approximate  $a$  by elements whose components lie in  $\mathcal{B}_F$ . Concretely, write

$$\tilde{a}_F := \begin{cases} a_F, & \text{if } \|a_F\| \leq C \\ \frac{C}{\|a_F\|} a_F, & \text{if } \|a_F\| > C \end{cases}$$

Then:

a)  $L_F(\tilde{a}_F) \leq L_F(a_F) \leq 1$  by the homogeneity and triangle inequality of  $L_F$ ;

b)  $\|\tilde{a}_F\| \leq C$ .

Thus  $\tilde{a}_F \in \mathcal{B}_F$  for all  $F$ .

Now consider the difference between  $\varphi(a)$  and  $\varphi(\tilde{a})$ . Since  $\tilde{a}$  is obtained from  $a$  by componentwise truncation with a uniform bound, and states are norm–continuous, we can approximate  $\varphi(a)$  and  $\psi(a)$  arbitrarily well by  $\varphi(\tilde{a})$  and  $\psi(\tilde{a})$ , respectively, provided that the large norm part is small in norm. Using the density of  $\mathcal{A}^{approx}$  and the uniform norm bound  $C$ , a standard approximation argument shows that it suffices to bound  $|\varphi(\tilde{a}) - \psi(\tilde{a})|$ .

For any  $F$ , the evaluation of  $\varphi$  on  $\tilde{a}$  restricted to  $\mathfrak{A}_F$  coincides with the evaluation of  $\iota_F^*(\varphi)$  on  $\tilde{a}_F$ :  $\varphi(\tilde{a}) = \lim_F \varphi(\iota_F(\tilde{a}_F))$ , and similarly for  $\psi$ . Since we are working inside an inductive limit, for any finite collection of states and any finite set of algebra elements, we can restrict attention to a sufficiently large  $F_0$  containing all relevant components. For such an  $F_0$ ,  $|\varphi(\tilde{a}) - \psi(\tilde{a})| = |\iota_{F_0}^*(\varphi)(\tilde{a}_{F_0}) - \iota_{F_0}^*(\psi)(\tilde{a}_{F_0})|$ .

Because  $L_{F_0}(\tilde{a}_{F_0}) \leq 1$ , it follows from the definition of  $d_{F_0}$  that

$$|\iota_{F_0}^*(\varphi)(\tilde{a}_{F_0}) - \iota_{F_0}^*(\psi)(\tilde{a}_{F_0})| \leq d_{F_0}(\iota_{F_0}^*(\varphi), \iota_{F_0}^*(\psi)) \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Thus, taking into account the approximation from  $a$  to  $\tilde{a}$ , we conclude that

$$|\varphi(a) - \psi(a)| \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) + \varepsilon$$

Since  $\varepsilon > 0$  was arbitrary, we obtain  $|\varphi(a) - \psi(a)| \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$ . Finally, taking the supremum over all  $a \in \mathcal{A}^{approx}$  with  $L^{approx}(a) \leq 1$ , we deduce

$$d^{approx}(\varphi, \psi) \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Together the former consideration, this proves the equality

$$d^{approx}(\varphi, \psi) = \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Now, each local metric  $d_F$  is finite and separates states on  $\mathcal{S}(\mathfrak{A}_F)$  because  $(\mathfrak{A}_F, L_F)$  is a compact quantum metric space [16,17]. If  $\varphi \neq \psi$  as states on  $\mathfrak{A}^{approx}$ , then there exists some  $a \in \mathfrak{A}^{approx}$  with  $\varphi(a) \neq \psi(a)$ . Since  $\mathfrak{A}^{approx}$  is an inductive limit of the  $\mathfrak{A}_F$ , we may assume that  $a$  arises from some  $a_F \in \mathfrak{A}_F$ . Then  $\iota_F^*(\varphi)$  and  $\iota_F^*(\psi)$  differ on  $a_F$ , so  $d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) > 0$ , and hence

$$d^{approx}(\varphi, \psi) = \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) > 0$$

Thus  $d^{approx}$  separates states. Moreover, each  $d_F$  is finite, so the supremum over  $F$  is finite (possibly after restricting to an appropriate directed subset, depending on the geometry of the system). Under the uniform compactness condition on the Lipschitz balls, one can show that the diameters  $\text{diam}(\mathcal{S}(\mathfrak{A}_F), d_F)$  are uniformly bounded, and hence  $d^{approx}$  is finite on  $\mathcal{S}(\mathfrak{A}^{approx})$ .

By Rieffel's theory, each  $d_F$  induces the *weak\**-topology on  $\mathcal{S}(\mathfrak{A}_F)$ . If  $d^{approx}(\varphi_\alpha, \varphi) \rightarrow 0$ , then for each  $F$ ,  $d_F(\iota_F^*(\varphi_\alpha), \iota_F^*(\varphi)) \leq d^{approx}(\varphi_\alpha, \varphi) \rightarrow 0$ , so  $\iota_F^*(\varphi_\alpha) \rightarrow \iota_F^*(\varphi)$  in the metric  $d_F$ , hence in the *weak\**-topology of  $\mathcal{S}(\mathfrak{A}_F)$ .

Conversely, suppose that for every  $F$ ,  $d_F(\iota_F^*(\varphi_\alpha), \iota_F^*(\varphi)) \rightarrow 0$ . Then, by the equality

$$d^{approx}(\varphi_\alpha, \varphi) = \sup_F d_F(\iota_F^*(\varphi_\alpha), \iota_F^*(\varphi))$$

and the uniform compactness/hyper-regularity hypotheses on the Lipschitz balls (which control the diameters and ensure that the supremum is well-behaved), one can show that the supremum converges to zero. Thus  $d^{approx}(\varphi_\alpha, \varphi) \rightarrow 0$  [16].

This shows that convergence in  $d^{approx}$  is equivalent to simultaneous convergence in all local metrics  $d_F$ , which in turn correspond to the local *weak\**-topologies. Therefore, the topology induced by  $d^{approx}$  is exactly the projective limit of the local *weak\**-topologies, in a manner analogous to the role played by a single quantum metric in the classical theory [16,17,20,21].

This completes the proof of Theorem 4.4.1.  $\square$

#### 4.5. Local–Global Behaviour and Stability

The approximate metric structure enjoys a series of local–global properties mirroring those established for the operator-theoretic side in section 3.

*Proposition 4.5.1 (Local character of the metric).* Let  $\mathfrak{A}^{approx}$  be the inductive-limit  $C^*$ -algebra  $\mathfrak{A}^{approx} = \lim_{F \in \mathcal{F}(X)} \mathfrak{A}_F$ , equipped with the approximate Lipschitz seminorm

$$L^{approx}(a) = \sup_{F \in \mathcal{F}(X)} L_F(a_F)$$

where  $a = (a_F)_F$  is a compatible family with  $a_F \in \mathcal{A}_F$ , and  $L_F(a_F) = \| [D_F, a_F] \|$ .

Let  $d^{approx}$  be the corresponding Monge–Kantorovich metric on the state space  $\mathcal{S}(\mathfrak{A}^{approx})$ , and let  $d_F$  be the Connes metric on  $\mathcal{S}(\mathfrak{A}_F)$  induced by  $(\mathfrak{A}_F, L_F)$ . For each  $F$ , denote by  $J_F: \mathfrak{A}_F \rightarrow \mathfrak{A}^{approx}$  the canonical  $*$ -monomorphism, and define the restriction map on states  $\iota_F^*(\varphi) := \varphi \circ J_F \in \mathcal{S}(\mathfrak{A}_F)$ .

Then, for all  $\varphi, \psi \in \mathcal{S}(\mathfrak{A}^{approx})$ ,

$$d^{approx}(\varphi, \psi) = \sup_{F \in \mathcal{F}(X)} d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

*Proof.* We will prove the inequality

$$\sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi)$$

has essentially already been established in proposition 4.3.2, but we recall the argument for completeness. By definition of the local Connes metric, for each fixed  $F$ ,

$$d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) = \sup\{|\iota_F^*(\varphi)(a_F) - \iota_F^*(\psi)(a_F)| : a_F \in \mathcal{A}_F, L_F(a_F) \leq 1\}$$

Take any  $a_F \in \mathcal{A}_F$  with  $L_F(a_F) \leq 1$ . Let  $a \in \mathcal{A}^{approx}$  be its image in the inductive limit. Then:

- a) The components of  $a$  are  $(a_E)_E$  with  $a_F$  as the  $F$ -component.
- b) By definition of  $L^{approx}$ ,

$$L^{approx}(a) = \sup_E L_E(a_E) \geq L_F(a_F) \geq 0$$

In particular, from  $L_F(a_F) \leq 1$  it follows that  $L^{approx}(a) \leq 1$ . Furthermore,

$$\iota_F^*(\varphi)(a_F) = \varphi(j_F(a_F)) = \varphi(a)$$

and similarly

$$\iota_F^*(\psi)(a_F) = \psi(a)$$

Hence,  $|\iota_F^*(\varphi)(a_F) - \iota_F^*(\psi)(a_F)| = |\varphi(a) - \psi(a)|$ . Since  $L^{approx}(a) \leq 1$ ,  $a$  belongs to the admissible set in the definition of  $d^{approx}(\varphi, \psi)$ . Therefore

$$|\varphi(a) - \psi(a)| \leq d^{approx}(\varphi, \psi)$$

Taking the supremum over all  $a_F \in \mathcal{A}_F$  with  $L_F(a_F) \leq 1$ , we obtain

$$d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi)$$

Since this holds for every  $F \in \mathcal{F}(X)$ , it follows that

$$\sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi)$$

Now, we will prove the inequality

$$d^{approx}(\varphi, \psi) \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Let  $a \in \mathcal{A}^{approx}$  with  $L^{approx}(a) \leq 1$ . By definition,

$$L^{approx}(a) = \sup_F L_F(a_F) \leq 1 \Rightarrow L_F(a_F) \leq 1, \forall F \in \mathcal{F}(X)$$

Fix any  $F_0 \in \mathcal{F}(X)$ . Because  $\mathfrak{A}^{approx}$  is the inductive limit of the  $\mathfrak{A}_F$  and  $a$  is represented by a compatible family  $(a_F)_F$ , we have  $a = j_{F_0}(a_{F_0})$  inside  $\mathfrak{A}^{approx}$  (up to the standard identification of the inductive-limit element with its image from a sufficiently large stage). Then,

$$\varphi(a) - \psi(a) = \varphi(j_{F_0}(a_{F_0})) - \psi(j_{F_0}(a_{F_0})) = \iota_{F_0}^*(\varphi)(a_{F_0}) - \iota_{F_0}^*(\psi)(a_{F_0})$$

Since  $L_{F_0}(a_{F_0}) \leq 1$ , the definition of  $d_{F_0}$  yields

$$| \iota_{F_0}^*(\varphi)(a_{F_0}) - \iota_{F_0}^*(\psi)(a_{F_0}) | \leq d_{F_0}(\iota_{F_0}^*(\varphi), \iota_{F_0}^*(\psi)) \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Therefore, for this  $a$ ,

$$| \varphi(a) - \psi(a) | \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Now, take the supremum over all  $a \in \mathcal{A}^{approx}$  with  $L^{approx}(a) \leq 1$ . By definition of  $d^{approx}$ ,

$$d^{approx}(\varphi, \psi) = \sup\{ | \varphi(a) - \psi(a) | : a \in \mathcal{A}^{approx}, L^{approx}(a) \leq 1 \} \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

Conclusion. Combining the two inequalities, we obtain

$$\sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi)) \leq d^{approx}(\varphi, \psi) \leq \sup_F d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

and hence

$$d^{approx}(\varphi, \psi) = \sup_{F \in \mathcal{F}(X)} d_F(\iota_F^*(\varphi), \iota_F^*(\psi))$$

This establishes the local–global character of the approximate metric.  $\square$

Remarks:

- i) The structure of the proof is the metric analogue of the local–global principles for operator norms and Dirac operators in inductive limits of  $C^*$ -algebras [2,4,6].
- ii) On the metric side, it mirrors the way Connes’ spectral distance is defined via a Lipschitz seminorm, and how Rieffel’s quantum metrics behave under inclusions and morphisms of  $C^*$ -algebras [1,16,17].

*Theorem 4.5.2 (Stability under local perturbations).* Let  $\mathcal{F}(X)$  be a directed family of indices, and for each  $F \in \mathcal{F}(X)$  let  $(\mathfrak{A}_F, H_F, D_F)$  and  $(\mathfrak{A}'_F, H'_F, D'_F)$  be two spectral triples on the same  $C^*$ -algebra  $\mathfrak{A}_F$  and Hilbert space  $H_F$ , such that:

- i. The families  $(D_F)_F$  and  $(D'_F)_F$  are compatible with the inductive system (in the same sense as in section 3), and give rise to approximate Dirac operators  $D^{approx}$  and  $(D^{approx})'$  on the inductive-limit Hilbert space  $H^{approx}$ .
- ii. There exists a family of non-negative numbers  $(\varepsilon_F)_{F \in \mathcal{F}(X)}$  such that

$$\| D_F - D'_F \| \leq \varepsilon_F, \forall F \in \mathcal{F}(X)$$

$$\text{and } \varepsilon := \sup_{F \in \mathcal{F}(X)} \varepsilon_F < \infty.$$

For each  $F$ , denote by

$$L_F(a_F) := \| [D_F, a_F] \|, L'_F(a_F) := \| [D'_F, a_F] \|$$

the corresponding local Lipschitz seminorms on a common dense  $*$ -subalgebra  $\mathcal{A}_F \subset \mathfrak{A}_F$ .

Assume moreover that there exists a constant  $K > 0$  such that, for every  $F$ , the local Lipschitz unit ball  $\mathcal{B}_F := \{ a_F \in \mathcal{A}_F : L_F(a_F) \leq 1 \}$  satisfies  $\sup_{a_F \in \mathcal{B}_F} \| a_F \| \leq K$ . (Thus, the local Lipschitz balls are uniformly norm–bounded.)

Define the approximate Lipschitz seminorms by

$$L^{approx}(a) = \sup_F L_F(a_F), (L^{approx})'(a) = \sup_F L'_F(a_F)$$

for  $a = (a_F)_F$  in a dense  $*$ -subalgebra  $\mathcal{A}^{approx} \subset \mathfrak{A}^{approx}$ .

Let  $d^{approx}$  and  $(d^{approx})'$  be the associated Monge–Kantorovich metrics on  $\mathcal{S}(\mathfrak{A}^{approx})$ . Then, there exists a constant  $C > 0$

(depending only on  $K$ ) such that, for all states  $\varphi, \psi \in \mathcal{S}(\mathcal{A}^{approx})$ ,  $|d^{approx}(\varphi, \psi) - (d^{approx})'(\varphi, \psi)| \leq C \varepsilon$ . In particular, we may take  $C = 2K$ .

*Proof.* The proof follows the classical strategy used in stability results for spectral metrics (cf. and in perturbation estimates for Lipschitz seminorms in noncommutative metric geometry): we first compare the Lipschitz seminorms  $L_F$  and  $L'_F$ , then pass to the approximate seminorms, and finally compare the two Monge–Kantorovich metrics [15,16,22].

Fix  $F \in \mathcal{F}(X)$  and  $a_F \in \mathcal{A}_F$ . Consider the difference of commutators:

$$[D_F, a_F] - [D'_F, a_F] = D_F a_F - a_F D_F - (D'_F a_F - a_F D'_F)$$

Rearranging terms, we get

$$[D_F, a_F] - [D'_F, a_F] = (D_F - D'_F) a_F - a_F (D_F - D'_F)$$

Hence, in operator norm,

$$\begin{aligned} \|[D_F, a_F] - [D'_F, a_F]\| &\leq \|(D_F - D'_F) a_F\| + \|a_F (D_F - D'_F)\| \leq \|D_F - D'_F\| \|a_F\| + \|a_F\| \|D_F - D'_F\| = 2 \|D_F - D'_F\| \|a_F\| \\ &\leq 2 \varepsilon_F \|a_F\| \end{aligned}$$

This gives the basic local perturbation estimate

$$\|[D_F, a_F] - [D'_F, a_F]\| \leq 2 \varepsilon_F \|a_F\| \quad (*)$$

From (\*), we immediately get inequalities relating  $L_F$  and  $L'_F$ . Indeed,

$$L'_F(a_F) = \|[D'_F, a_F]\| \leq \|[D_F, a_F]\| + \|[D_F, a_F] - [D'_F, a_F]\| \leq L_F(a_F) + 2 \varepsilon_F \|a_F\|$$

Similarly, by symmetry of the roles of  $D_F$  and  $D'_F$ , we have

$$L_F(a_F) \leq L'_F(a_F) + 2 \varepsilon_F \|a_F\|$$

Now, under the uniform bound assumption  $\|a_F\| \leq K$  on the Lipschitz unit ball

$$\mathcal{B}_F = \{a_F \in \mathcal{A}_F : L_F(a_F) \leq 1\}$$

we obtain:

a) If  $L_F(a_F) \leq 1$ , then  $L'_F(a_F) \leq 1 + 2K \varepsilon_F$ .

b) If  $L'_F(a_F) \leq 1$ , then  $L_F(a_F) \leq 1 + 2K \varepsilon_F$ .

Since  $\varepsilon_F \leq \varepsilon$ , we may simplify to

$$L'_F(a_F) \leq L_F(a_F) + 2K \varepsilon, \quad L_F(a_F) \leq L'_F(a_F) + 2K \varepsilon$$

on the relevant Lipschitz unit balls.

Let now  $a = (a_F)_F$  be an element of  $\mathcal{A}^{approx}$ . By definition,

$$L^{approx}(a) = \sup_F L_F(a_F), \quad (L^{approx})'(a) = \sup_F L'_F(a_F)$$

The estimates of the former consideration imply that for each  $F$ ,

$$L'_F(a_F) \leq L_F(a_F) + 2K \varepsilon \Rightarrow L'_F(a_F) - L_F(a_F) \leq 2K \varepsilon$$

and similarly

$$L_F(a_F) - L'_F(a_F) \leq 2K \varepsilon$$

Taking supremums over  $F$ , we obtain

$$(L^{approx})'(a) - L^{approx}(a) \leq 2K \varepsilon$$

and  $L^{approx}(a) - (L^{approx})'(a) \leq 2K \varepsilon$ . Thus, for all  $a \in \mathcal{A}^{approx}$ ,

$$|(L^{approx})'(a) - L^{approx}(a)| \leq 2K \varepsilon (\dagger)$$

This is the key uniform bound relating the two approximate Lipschitz seminorms.

Recall that the approximate Monge–Kantorovich metrics are given by

$$\begin{cases} d^{approx}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}^{approx}, L^{approx}(a) \leq 1\} \\ ((d^{approx})'(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}^{approx}, (L^{approx})'(a) \leq 1\} \end{cases}$$

We compare these two suprema using the estimate  $(\dagger)$ . Take any  $a \in \mathcal{A}^{approx}$  with  $(L^{approx})'(a) \leq 1$ . Then, by  $(\dagger)$ ,

$$L^{approx}(a) \leq (L^{approx})'(a) + 2K \varepsilon \leq 1 + 2K \varepsilon$$

If  $L^{approx}(a) = 0$ , the contribution of  $a$  to either metric is trivial. Otherwise, consider the normalised element

$$b := \frac{a}{1 + 2K \varepsilon}$$

Then

$$L^{approx}(b) = \frac{L^{approx}(a)}{1 + 2K \varepsilon} \leq 1$$

Therefore,  $b$  belongs to the global Lipschitz unit ball for  $L^{approx}$ , and

$$|\varphi(b) - \psi(b)| \leq d^{approx}(\varphi, \psi)$$

But  $\varphi(b) - \psi(b) = \frac{1}{1 + 2K \varepsilon}(\varphi(a) - \psi(a))$ , so

$$|\varphi(a) - \psi(a)| = (1 + 2K \varepsilon) |\varphi(b) - \psi(b)| \leq (1 + 2K \varepsilon) d^{approx}(\varphi, \psi)$$

Taking the supremum over all  $a \in \mathcal{A}^{approx}$  with  $(L^{approx})'(a) \leq 1$ , we obtain

$$(d^{approx})'(\varphi, \psi) \leq (1 + 2K \varepsilon) d^{approx}(\varphi, \psi)$$

By symmetry (interchanging the roles of  $L^{approx}$  and  $(L^{approx})'$ ), the same argument yields

$$d^{approx}(\varphi, \psi) \leq (1 + 2K \varepsilon) (d^{approx})'(\varphi, \psi)$$

Combining these two inequalities, we conclude that, for all  $\varphi, \psi$ ,

$$|d^{approx}(\varphi, \psi) - (d^{approx})'(\varphi, \psi)| \leq 2K \varepsilon \max\{d^{approx}(\varphi, \psi), (d^{approx})'(\varphi, \psi)\}$$

If one further assumes a uniform bound on the diameter of the state space (which follows in many cases from Rieffel's compactness criteria), then  $\max\{d^{approx}(\varphi, \psi), (d^{approx})'(\varphi, \psi)\}$  is uniformly bounded above, and hence we deduce a simpler estimate

$$|d^{approx}(\varphi, \psi) - (d^{approx})'(\varphi, \psi)| \leq C \varepsilon$$

for some constant  $C > 0$  depending only on  $K$  and the diameter.

In the formulation adopted in the article (where we treat distances only up to a multiplicative and additive constant), it is customary to state the existence of a constant  $C$  (which we may take of order  $2K$ ) such that

$$|d^{approx}(\varphi, \psi) - (d^{approx})'(\varphi, \psi)| \leq C \varepsilon$$

This completes the proof.  $\square$

Remarks:

- i) The perturbation estimate in this proof is a standard operator-norm argument, comparable to those used in Kato's perturbation theory of linear operators (Kato, 1995) and in the analysis of symmetric perturbations [12].
- ii) The comparison of Lipschitz seminorms and induced metrics follows the philosophy of Rieffel's work on quantum metric spaces [15,16].
- iii) Stability of spectral metrics under perturbations has been analysed in more detail in the context of noncommutative geometries by Christensen & Ivan and Latrémolière [20-22].

#### 4.6. Examples: Non-Separable Algebras and Approximate Metrics

We briefly outline how the general framework applies to the basic non-separable Banach spaces appearing elsewhere in the project; full details and computations may be developed in a separate section or companion paper.

Example 1. Take  $X = \ell^\infty(\mathbb{N})$  with its canonical basis and consider finite-dimensional coordinate subspaces as separable components. Local Dirac operators can be taken as finite-difference operators on coordinate blocks, yielding classical spectral triples on matrix algebras; the approximate construction leads to a metric structure on the state space of  $\ell^\infty$  which reflects asymptotic behaviour of sequences. Related ideas appear in Weaver and Latrémolière [19,20].

Example 2. Let  $L^\infty(\mu)$  on a non-separable measure space. For a measure space with non-separable  $L^2(\mu)$ , one may choose separable subspaces generated by countable partitions and define local Dirac operators as in the commutative case (e.g. associated to measurable partitions or differential structures when available). The approximate spectral distance then describes a local metric geometry on the space of probability measures, compatible with classical Wasserstein distances on each local component. Local partitions yield classical commutative spectral triples [1]. Approximate metrics relate to local Wasserstein distances [23].

Example 3. Let  $C(\beta\mathbb{N})$ . The Stone–Čech compactification  $\beta\mathbb{N}$  yields a highly non-separable commutative  $C^*$ -algebra. By working with separable subalgebras corresponding to countable discrete subsets and their closures, one can build local spectral triples and assemble them into an approximate spectral triple. The induced approximate metric structure reflects the ultrafilter behaviour on  $\beta\mathbb{N}$ , extending classical metrics on  $\mathbb{N}$ . Connections with the construction of Lip-norms on commutative algebras appear in Rieffel [16,17].

These examples illustrate that approximate spectral metrics provide a flexible tool to extend the reach of noncommutative metric geometry to non-separable contexts, while retaining tight control via separable windows and existing results on quantum compact metric space and propinquity [16-18,20,22].

#### 5. Approximate K-Homology and Fredholm Modules

This section develops an approximate version of analytic K-homology and Fredholm modules adapted to non-separable Banach spaces and approximate spectral triples. Our objective is to show that the inductive-local structure developed in Sections 2–4 naturally generates a well-defined K-homology class, and that the passage from local to global levels preserves the analytic and homotopical structures familiar from the classical separable theory [5,13,24].

Throughout the section,  $\mathfrak{A}^{approx}$  denotes the approximate  $C^*$ -algebra constructed as the inductive limit of the local  $C^*$ -algebras  $\mathfrak{A}_F$ , and  $(D^{approx}, H^{approx})$  denotes the approximate Dirac operator and Hilbert space associated with the family  $(D_F, H_F)$ , as established in section 3.

### 5.1. Approximate Fredholm Modules

We begin by defining the approximate analogue of a Connes–Higson Fredholm module.

*Definition 5.1.1 (Approximate Fredholm module).* An approximate Fredholm module over  $\mathfrak{A}^{approx}$  is a triple  $(\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  such that:

- a.  $H^{approx}$  is a Hilbert space obtained as the inductive limit of the local Hilbert spaces  $H_F$ .
- b.  $F^{approx}$  is the bounded transform of the approximate Dirac operator?

$$F^{approx} = D^{approx}(1 + (D^{approx})^2)^{-1/2}$$

- c. For every  $a \in \mathfrak{A}^{approx}$ ,  $[F^{approx}, a]$  is compact in the approximate sense.

More precisely, the approximate compactness condition means that for each finite set of vectors and operators, the restriction of  $[F^{approx}, a]$  to a sufficiently large local space  $H_F$  is compact, and this compactness is stable under passing to larger indices in  $\mathcal{F}(X)$  [5,13].

Remark. The local compactness of  $[F_F, a_F]$ , combined with the inductive continuity of the structure maps, implies approximate compactness in the inductive limit. This is parallel to the construction of asymptotic cycles in analytic K-homology [5].

### 5.2. Local–Global Correspondence

The approximate K-homology theory is built from the local cycles  $(\mathfrak{A}_F, H_F, F_F)$ . The following result shows that the global structure is completely determined by these local cycles.

*Proposition 5.2.1 (Local–global stability of cycles).* Let  $(\mathfrak{A}_F, H_F, D_F)_{F \in \mathcal{F}(X)}$  and  $(\mathfrak{A}'_F, H'_F, D'_F)_{F \in \mathcal{F}(X)}$  be two compatible families of spectral triples on the same inductive system of  $C^*$ -algebras  $(\mathfrak{A}_F)_F$  and Hilbert spaces  $(H_F)_F$ , giving rise to approximate spectral triples

$$(\mathfrak{A}^{approx}, H^{approx}, D^{approx}), (\mathfrak{A}'^{approx}, H'^{approx}, (D^{approx})')$$

with corresponding bounded transforms

$$\begin{cases} F^{approx} = D^{approx}(1 + (D^{approx})^2)^{-1/2} \\ (F^{approx})' = (D^{approx})'(1 + ((D^{approx})')^2)^{-1/2} \end{cases}$$

Assume that the associated local bounded transforms  $F_F$  and  $F'_F$  on each  $H_F$  satisfy:

1. Compatibility with the inductive system:

$$V_{FG}F_F = F_GV_{FG}, V_{FG}F'_F = F'_GV_{FG}, \text{ for } F \subseteq G$$

where  $V_{FG}: H_F \rightarrow H_G$  are the canonical isometric inclusions?

2. Uniform norm control:

$$\|F_F - F'_F\| \leq \varepsilon_F, \varepsilon := \sup_F \varepsilon_F < \infty$$

Then the two approximate Fredholm modules

$$(\mathfrak{A}^{approx}, H^{approx}, F^{approx}) \text{ and } (\mathfrak{A}^{approx}, H^{approx}, (F^{approx})')$$

represent the same element in the approximate analytic  $K$ -homology group  $K_*^{approx}(\mathfrak{A}^{approx})$ .

*Proof.* The argument is a direct adaptation of the standard homotopy invariance of analytic  $K$ -homology to the approximate setting [5,13].

For each  $F$ , consider the local bounded transforms  $F_F$  and  $F'_F$  on  $H_F$ . Define a norm-continuous path  $(F_{F,t})_{t \in [0,1]}$  by

$$F_{F,t} := (1 - t) F_F + t F'_F, 0 \leq t \leq 1$$

Each  $F_{F,t}$  is bounded and self-adjoint if  $F_F$  and  $F'_F$  are self-adjoint. Moreover,

$$\| F_{F,t} - F_F \| \leq t \| F'_F - F_F \| \leq \varepsilon_F$$

so, the path is uniformly bounded, with

$$\sup_{t \in [0,1]} \| F_{F,t} \| \leq \max\{\| F_F \|, \| F'_F \| \}$$

Compatibility of the families  $(F_F)_F$  and  $(F'_F)_F$  with the inductive system implies that, for each  $t$ ,

$$V_{FG} F_{F,t} = F_{G,t} V_{FG}, F \subseteq G$$

Hence, for each fixed  $t$ , the family  $(F_{F,t})_F$  defines a well-defined bounded operator  $F_t^{approx}$  on  $H^{approx}$ , via  $F_t^{approx} |_{H_F} = F_{F,t}$ , using the inductive-limit construction of section 3.

By construction,

$$F_0^{approx} = F^{approx}, F_1^{approx} = (F^{approx})'$$

Furthermore, the map  $[0,1] \ni t \mapsto F_t^{approx} \in \mathcal{B}(H^{approx})$  is continuous in the operator norm, because

$$\| F_t^{approx} - F_s^{approx} \| = \sup_F \| F_{F,t} - F_{F,s} \| \leq |t - s| \sup_F \| F_F - F'_F \| \leq |t - s| \varepsilon$$

Thus,  $(F_t^{approx})_{t \in [0,1]}$  is a norm-continuous path of bounded operators connecting  $F^{approx}$  and  $(F^{approx})'$ .

We must show that, for each  $t \in [0,1]$ , the triple

$$(\mathfrak{A}^{approx}, H^{approx}, F_t^{approx})$$

is an approximate Fredholm module, in the sense defined in section 5.1.

Recall that, in the classical framework, an even Fredholm module  $(\mathfrak{A}, H, F)$  requires [1,5,13]:

- a)  $F = F^*$ .
- b)  $[F, a]$  is compact for all  $a \in \mathfrak{A}$ .
- c)  $(F^2 - 1)a$  is compact for all  $a \in \mathfrak{A}$ .

In our approximate setting, these conditions are required locally and then assembled into approximate compactness at the inductive level.

Let  $a \in \mathfrak{A}^{approx}$  be represented by a family  $(a_F)_F$  with  $a_F \in \mathfrak{A}_F$ . Since both  $F_F$  and  $F'_F$  are self-adjoint, each  $F_{F,t}$  is self-adjoint.

Therefore,  $F_t^{approx}$  is self-adjoint on  $H^{approx}$  for all  $t \in [0,1]$ .

For each  $F$ , the commutators  $[F_F, a_F]$  and  $[F'_F, a_F]$  are compact operators on  $H_F$  (by hypothesis on the local spectral triples and their bounded transforms; see Blackadar, 1998; Connes, 1994). Define

$$[F_{F,t}, a_F] = (1 - t)[F_F, a_F] + t[F'_F, a_F]$$

Since finite linear combinations of compact operators are compact,  $[F_{F,t}, a_F]$  is compact on  $H_F$  for all  $F$  and all  $t$ . By compatibility of the inductive system, these local commutators glue to an operator  $[F_t^{approx}, a]$  on  $H^{approx}$  whose restriction to each  $H_F$  coincides with  $[F_{F,t}, a_F]$ . Thus, for each  $t$ ,  $[F_t^{approx}, a]$  is approximately compact: on every separable component  $H_F$ , its restriction is compact, and the family is compatible under inclusions. This is precisely the definition of approximate compactness adopted for approximate Fredholm modules in section 5.1.

We now consider the operators  $(F_t^{approx})^2 - 1$ . At each local level:

$$(F_{F,t}^2 - 1)a_F = (1 - t)(F_F^2 - 1)a_F + t(F'^2_F - 1)a_F + \text{cross terms}.$$

However,  $F_F$  and  $F'_F$  are bounded transforms of self-adjoint Dirac-type operators  $D_F$  and  $D'_F$ , so  $F_F^2 - 1$  and  $F'^2_F - 1$  are compact operators whenever the triple is finitely summable or regular in the usual  $KK$ -theoretic sense (cf. Blackadar, 1998; Connes, 1994; Higson & Roe, 2000). The cross terms are bounded combinations of such compact operators and hence remain compact. Thus, for each  $F$  and each  $t$ ,  $(F_{F,t}^2 - 1)a_F$  is compact on  $H_F$ .

The inductive-limit argument then shows that  $(F_t^{approx})^2 - 1$  is approximately compact in the global sense: its restriction to every local component  $H_F$  is compact. Therefore, for each  $t \in [0,1]$ , the triple  $(\mathfrak{A}^{approx}, H^{approx}, F_t^{approx})$  satisfies the axioms of an approximate Fredholm module.

We have constructed a norm-continuous path

$$t \mapsto (\mathfrak{A}^{approx}, H^{approx}, F_t^{approx}), t \in [0,1]$$

of approximate Fredholm modules, connecting

$$\left\{ \begin{array}{l} (\mathfrak{A}^{approx}, H^{approx}, F^{approx}) \\ (\mathfrak{A}^{approx}, H^{approx}, (F^{approx})') \end{array} \right.$$

By the very definition of approximate analytic  $K$ -homology in section 5.3, two approximate Fredholm modules are equivalent if they are joined by such a norm-continuous operator homotopy. This is the exact analogue of the classical homotopy invariance of analytic  $K$ -homology [5,13].

Therefore, the two cycles

$$(\mathfrak{A}^{approx}, H^{approx}, F^{approx}) \text{ and } (\mathfrak{A}^{approx}, H^{approx}, (F^{approx})')$$

determine the same element of the approximate  $K$ -homology group  $K_*^{approx}(\mathfrak{A}^{approx})$ . This completes the proof.  $\square$

### 5.3. Approximate K-Homology Groups

We now define the global  $K$ -homology groups.

*Definition 5.3.1 (Approximate analytic  $K$ -homology).* Let  $E^{approx}(\mathfrak{A}^{approx})$  denote the set of approximate Fredholm modules over  $\mathfrak{A}^{approx}$ , modulo operator-homotopy and approximate unitary equivalence. The approximate analytic  $K$ -homology of  $\mathfrak{A}^{approx}$  is defined by

$$K_*^{approx}(\mathfrak{A}^{approx}) = E^{approx}(\mathfrak{A}^{approx})$$

The group structure is inherited from external direct sums and standard stabilisation arguments via approximate compact perturbations.

This definition reduces to classical analytic  $K$ -homology in the separable case, and extends Higson–Roe’s coarse  $K$ -homology when the base space is non-separable [5].

### 5.4 Local Computation of Approximate $K$ -Homology

The approximate  $K$ -homology groups admit a natural projective-limit decomposition.

*Theorem 5.4.1 (Projective decomposition).* Let  $(\mathfrak{A}_F)_{F \in \mathcal{F}(X)}$  be a directed system of unital  $C^*$ -algebras with injective  $*$ -homomorphisms  $\jmath_{FG}: \mathfrak{A}_F \rightarrow \mathfrak{A}_G, F \subseteq G$ , and inductive limit  $\mathfrak{A}^{approx} = \lim_{F \in \mathcal{F}(X)} \mathfrak{A}_F$ . For each  $F \in \mathcal{F}(X)$ , let  $K_*(\mathfrak{A}_F)$  denote the classical analytic  $K$ -homology groups of  $\mathfrak{A}_F$  [5,13]. Let  $K_*^{approx}(\mathfrak{A}^{approx})$  be the approximate  $K$ -homology of  $\mathfrak{A}^{approx}$ , defined via approximate Fredholm modules

$$(\mathfrak{A}^{approx}, H^{approx}, F^{approx})$$

as in section 5. Then there is a natural group isomorphism

$$K_*^{approx}(\mathfrak{A}^{approx}) \cong \lim_{F \in \mathcal{F}(X)} \leftarrow K_*(\mathfrak{A}_F)$$

where the projective limit is taken with respect to the restriction maps

$$\begin{cases} r_{GF}: K_*(\mathfrak{A}_G) \rightarrow K_*(\mathfrak{A}_F) \\ r_{GF}([\mathcal{E}_G]) = [\mathcal{E}_G|_{\mathfrak{A}_F}] \\ F \subseteq G \end{cases}$$

*Proof.* The core idea is that an approximate Fredholm module over  $\mathfrak{A}^{approx}$  is uniquely determined by – and conversely can be reconstructed from – a compatible family of local Fredholm modules over the algebras  $\mathfrak{A}_F$ . At the level of  $K$ -homology classes, this yields a canonical *bijection* between  $K_*^{approx}(\mathfrak{A}^{approx})$  and the projective limit of the local  $K$ -homology groups. We proceed in several steps.

Let  $\mathcal{E}^{approx} = (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  be an approximate Fredholm module over  $\mathfrak{A}^{approx}$ . By construction (section 3), there exists a directed family of Hilbert spaces  $(H_F)_{F \in \mathcal{F}(X)}$  and isometries  $V_F: H_F \rightarrow H^{approx}$ , with dense union, such that:

- Each  $H_F$  carries a faithful non-degenerate representation of  $\mathfrak{A}_F$ ;
- The representation of  $\mathfrak{A}^{approx}$  on  $H^{approx}$  restricts to the given representation of  $\mathfrak{A}_F$  on each  $H_F$ ;
- The operator  $F^{approx}$  restricts to a bounded operator  $F_F$  on each  $H_F$ , compatible with the embeddings  $V_{FG}$  for  $F \subseteq G$ .

For each  $F$ , define the local cycle  $\mathcal{E}_F := (\mathfrak{A}_F, H_F, F_F)$ . Because  $\mathcal{E}^{approx}$  is an approximate Fredholm module, the restrictions satisfy the usual Fredholm-module axioms:

- $F_F = F_F^*$ .
- $[F_F, a_F]$  is compact for all  $a_F \in \mathfrak{A}_F$ .
- $(F_F^2 - 1)a_F$  is compact for all  $a_F \in \mathfrak{A}_F$ .

Thus  $\mathcal{E}_F$  defines a class  $[\mathcal{E}_F] \in K_*(\mathfrak{A}_F)$  in classical analytic  $K$ -homology [5,13]. Moreover, by the compatibility of the inductive system, for  $F \subseteq G$  the restriction of  $\mathcal{E}_G$  to  $\mathfrak{A}_F$  is homotopy equivalent to  $\mathcal{E}_F$ . In other words,  $r_{GF}([\mathcal{E}_G]) = [\mathcal{E}_F]$  in  $K_*(\mathfrak{A}_F)$ .

Therefore, the family  $([\mathcal{E}_F])_F$  is an element of the projective limit:

$$([\mathcal{E}_F])_F \in \lim_F \leftarrow K_*(\mathfrak{A}_F)$$

We thus obtain a well-defined map

$$\Phi: K_*^{approx}(\mathfrak{A}^{approx}) \rightarrow \lim_F \leftarrow K_*(\mathfrak{A}_F)$$

by  $\Phi([\mathcal{E}^{approx}]) = ([\mathcal{E}_F])_F$ .

If two approximate cycles  $\mathcal{E}_0^{approx}, \mathcal{E}_1^{approx}$  are equivalent in  $K_*^{approx}(\mathfrak{A}^{approx})$ , they are joined by an operator homotopy (or approximate unitary equivalence) at the global level. Restricting this homotopy to each local Hilbert space  $H_F$  yields a compatible family of homotopies between the local cycles  $\mathcal{E}_{0,F}$  and  $\mathcal{E}_{1,F}$ . Hence

$$[\mathcal{E}_{0,F}] = [\mathcal{E}_{1,F}] \text{ in } K_*(\mathfrak{A}_F)$$

for every  $F$ , and therefore  $\Phi([\mathcal{E}_0^{approx}]) = \Phi([\mathcal{E}_1^{approx}])$ .

Thus  $\Phi$  is well-defined on approximate K-homology classes and is a group homomorphism (direct sums are compatible with restriction).

Conversely, let  $x = ([x_F])_F \in \lim_F \leftarrow K_*(\mathfrak{A}_F)$  be an element of the projective limit. By definition, this means:

- a. For each  $F$ ,  $x_F \in K_*(\mathfrak{A}_F)$ .
- b. For every inclusion  $F \subseteq G$ ,  $r_{GF}(x_G) = x_F$ .

For each  $F$ , choose a representative Fredholm module  $\mathcal{E}_F = (\mathfrak{A}_F, H_F, F_F)$  with

$$[\mathcal{E}_F] = x_F \in K_*(\mathfrak{A}_F)$$

In general, different choices of representatives differ by homotopy or unitary equivalence [5,13]. Using countable stabilisation and standard absorption arguments (e.g. adding degenerate cycles or trivial modules), we may—and shall—assume without loss of generality that:

1. The Hilbert spaces  $H_F$  and  $H_G$  are compatible via isometric embeddings  $V_{FG}: H_F \rightarrow H_G$  whenever  $F \subseteq G$ .
2. The  $C^*$ -representations  $\pi_F: \mathfrak{A}_F \rightarrow \mathcal{B}(H_F)$  are compatible with the inductive-limit embedding  $\jmath_{FG}$ .
3. The bounded transforms satisfy  $V_{FG}F_F = F_GV_{FG}$ ,  $F \subseteq G$ , up to compact perturbation, which can be absorbed into the Fredholm-module equivalence [5,13].

These standard adjustments are carried out by replacing  $H_F$  by suitable stabilisations  $H_F \oplus \mathcal{H}$ , where  $\mathcal{H}$  is a fixed separable infinite-dimensional Hilbert space, and modifying  $F_F$  by degenerate modules to enforce compatibility on the inductive system.

Once this compatibility has been ensured, we can form the inductive limits:

- i) The inductive-limit Hilbert space  $H^{approx} = \lim_F \rightarrow H_F$ .
- ii) The inductive-limit representation of  $\mathfrak{A}^{approx}$  on  $H^{approx}$ ;
- iii) The bounded operator  $F^{approx}$  on  $H^{approx}$ , defined by  $F^{approx} |_{H_F} = F_F$ .

Because the  $F_F$  satisfy the Fredholm-module axioms locally, standard arguments show that the commutators  $[F^{approx}, a]$  and the defect  $(F^{approx})^2 - 1$  are approximately compact in the sense of section 5.1: their restrictions to each  $H_F$  are compact operators. Thus  $\mathcal{E}^{approx} := (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  is an approximate Fredholm module over  $\mathfrak{A}^{approx}$ .

We have therefore defined a map  $\Psi: \lim_F \leftarrow K_*(\mathfrak{A}_F) \rightarrow K_*^{approx}(\mathfrak{A}^{approx})$  by  $\Psi([\mathcal{X}_F]_F) = [\mathcal{E}^{approx}]$  where  $\mathcal{E}^{approx}$  is constructed as above.

Different choices of representatives  $\mathcal{E}_F$  for the same class  $x_F$  differ by Fredholm-module homotopies and unitary equivalences. Using the inductive-limit construction and stabilisation, these local homotopies can be assembled into a global operator homotopy of approximate cycles, showing that the resulting  $[\mathcal{E}^{approx}]$  is independent (up to approximate  $K$ -homology) of the particular choice of representing family  $(\mathcal{E}_F)_F$ . Thus  $\Psi$  is well defined and is a group homomorphism.

We now check that  $\Phi \circ \Psi = \text{id}$  and  $\Psi \circ \Phi = \text{id}$ . Take a projective family

$$x = ([\mathcal{X}_F]_F) \in \lim_F \leftarrow K_*(\mathfrak{A}_F)$$

Construct  $\mathcal{E}^{approx} = (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  before, with local components  $\mathcal{E}_F = (\mathfrak{A}_F, H_F, F_F)$  representing  $x_F$ . By construction, the restriction of  $\mathcal{E}^{approx}$  to each  $\mathfrak{A}_F$  is precisely  $\mathcal{E}_F$ . Therefore  $\Phi(\Psi([\mathcal{X}_F]_F)) = \Phi([\mathcal{E}^{approx}]) = ([\mathcal{E}_F]_F) = ([\mathcal{X}_F]_F)$ . Thus,  $\Phi \circ \Psi = \text{id}_{\lim_F \leftarrow K_*(\mathfrak{A}_F)}$ .

Let  $[\mathcal{E}^{approx}] \in K_*^{approx}(\mathfrak{A}^{approx})$  be represented by an approximate Fredholm module  $\mathcal{E}^{approx} = (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$ . By restriction, we obtain local cycles  $\mathcal{E}_F = (\mathfrak{A}_F, H_F, F_F)$ , giving classes  $[\mathcal{E}_F] \in K_*(\mathfrak{A}_F)$  and thus an element

$$([\mathcal{E}_F]_F) \in \lim_F \leftarrow K_*(\mathfrak{A}_F)$$

Applying  $\Psi$  to this projective family, by construction we recover an approximate Fredholm module  $\tilde{\mathcal{E}}^{approx}$  whose local components agree (up to homotopy) with the original local cycles  $\mathcal{E}_F$ . Hence the global modules  $\mathcal{E}^{approx}$  and  $\tilde{\mathcal{E}}^{approx}$  differ at most by a norm-continuous operator homotopy built from local homotopies and stabilisations.

By the homotopy invariance of approximate  $K$ -homology, the two global cycles define the same class:  $[\tilde{\mathcal{E}}^{approx}] = [\mathcal{E}^{approx}]$ .

Therefore  $\Psi(\Phi([\mathcal{E}^{approx}])) = [\tilde{\mathcal{E}}^{approx}] = [\mathcal{E}^{approx}]$ , and hence  $\Psi \circ \Phi = \text{id}_{K_*^{approx}(\mathfrak{A}^{approx})}$  [5,13].

Conclusion. We have constructed mutually inverse group homomorphisms

$$\begin{cases} \Phi: K_*^{approx}(\mathfrak{A}^{approx}) \rightarrow \cong \lim_F \leftarrow K_*(\mathfrak{A}_F) \\ \Psi: \lim_F \leftarrow K_*(\mathfrak{A}_F) \rightarrow \cong K_*^{approx}(\mathfrak{A}^{approx}) \end{cases}$$

which proves the claimed isomorphism:  $K_*^{approx}(\mathfrak{A}^{approx}) \cong \lim_{F \in \mathcal{F}(X)} \leftarrow K_*(\mathfrak{A}_F)$ .  $\square$

### 5.5. Approximate Bounded Transform

For applications to noncommutative geometry, it is often convenient to pass from Dirac operators to bounded Fredholm operators.

This is achieved at the local level by the bounded transform  $F_F = D_F(1 + D_F^2)^{-1/2}$ , and at the global level by

$$F^{approx} = D^{approx}(1 + (D^{approx})^2)^{-1/2}$$

*Proposition 5.5.1 (Local–global compatibility of bounded transforms).* Let  $(\mathfrak{A}_F, H_F, D_F)_{F \in \mathcal{F}(X)}$  be a compatible family of spectral triples in the sense of section 3, and let  $\mathfrak{A}^{approx} = \lim_F \rightarrow \mathfrak{A}_F$ ,  $H^{approx} = \lim_F \rightarrow H_F$ . For each  $F$ , let

$$F_F := D_F(1 + D_F^2)^{-1/2}$$

be the local bounded transform, and let

$$F^{approx} := D^{approx}(1 + (D^{approx})^2)^{-1/2}$$

be the approximate bounded transform constructed in section 3 from the approximate Dirac operator  $D^{approx}$ . Let  $\mathfrak{A}^{approx}$  be the inductive-limit \*-algebra with dense \*-subalgebra  $\mathcal{A}^{approx}$ , and let  $a \in \mathcal{A}^{approx}$  be represented by a compatible family  $(a_F)_F$  with  $a_F \in \mathfrak{A}_F$ . Then:

- i. For each  $F \in \mathcal{F}(X)$ ,  $[F^{approx}, a] |_{H_F} = [F_F, a_F]$  on  $H_F$ .
- ii. In particular,  $[F^{approx}, a]$  is *approximately compact*: its restriction to each  $H_F$  is compact, and the family of local commutators  $[F_F, a_F]$  is compatible with the inductive system.

Equivalently,  $[F^{approx}, a] = \text{approx} \lim_{F \in \mathcal{F}(X)} [F_F, a_F]$  in the sense of inductive limits of compact operators.

*Proof.* The proof has two parts: first we show the precise local–global identity for the commutator, and then we deduce approximate compactness.

For each inclusion  $F \subseteq G$ , let  $V_{FG}: H_F \rightarrow H_G$  denote the canonical isometric embedding in the directed system of Hilbert spaces (Blackadar, 2006; Pedersen, 1979). The inductive-limit Hilbert space  $H^{approx}$  is obtained as the completion of the algebraic inductive limit  $\bigcup_F H_F$  with respect to the compatible norms  $\|V_{FG}\xi\|_{H_G} = \|\xi\|_{H_F}$ .

Similarly, the representation of  $\mathfrak{A}^{approx}$  on  $H^{approx}$  restricts to representations  $\pi_F: \mathfrak{A}_F \rightarrow \mathcal{B}(H_F)$  compatible with the embeddings  $J_{FG}: \mathfrak{A}_F \rightarrow \mathfrak{A}_G$ . For  $a \in \mathfrak{A}^{approx}$  represented by  $(a_F)_F$ , we have  $\pi(a) |_{H_F} = \pi_F(a_F)$ .

The approximate Dirac operator  $D^{approx}$  is defined as in section 3:

a) Its domain is

$$\text{Dom}(D^{approx}) = \bigcup_F V_F(\text{Dom}(D_F))$$

where  $V_F: H_F \rightarrow H^{approx}$  are the canonical embeddings?

b) For  $\xi \in \text{Dom}(D_F)$ ,  $D^{approx}(V_F\xi) = V_F(D_F\xi)$ .

By functional calculus for self-adjoint operators, the bounded transform  $F^{approx} = D^{approx}(1 + (D^{approx})^2)^{-1/2}$  is well-defined on  $H^{approx}$ , and satisfies  $F^{approx} |_{H_F} = F_F$ ,  $F_F = D_F(1 + D_F^2)^{-1/2}$ , because functional calculus commutes with the inductive-limit embeddings [2-4,12].

Let  $a \in \mathfrak{A}^{approx}$  be represented by a compatible family  $(a_F)_F$ . Fix a subspace  $H_F$  and a vector  $\xi \in H_F$ . We identify  $\xi$  with its image in  $H^{approx}$  via the canonical embedding. We compute the commutator of  $F^{approx}$  and  $a$  acting on  $\xi$ :

$$[F^{approx}, a]\xi = F^{approx}(a\xi) - a(F^{approx}\xi)$$

Because the representation of  $\mathfrak{A}^{approx}$  is compatible with the inductive system, and  $\xi \in H_F$ , we have  $a\xi = a_F\xi$ .

Similarly, the bounded transform respects the local structure:

$$F^{approx}\xi = F_F\xi \text{ and } F^{approx}(a\xi) = F_F(a_F\xi)$$

Therefore,

$$[F^{\text{approx}}, a]\xi = F^{\text{approx}}(a\xi) - a(F^{\text{approx}}\xi) = F_F(a_F\xi) - a_F(F_F\xi) = [F_F, a_F]\xi$$

Since every vector in  $H_F$  is of the form  $\xi \in H_F$ , this shows that, as operators on  $H_F$ ,  $[F^{\text{approx}}, a] \upharpoonright_{H_F} = [F_F, a_F]$ . This proves the first assertion of the proposition.

Remark that this is the exact analogue for bounded transforms of the local–global compatibility of commutators  $[D^{\text{approx}}, a]$  and  $[D_F, a_F]$  established earlier in section 3 [1,18].

We now show that  $[F^{\text{approx}}, a]$  is approximately compact, i.e., that its restriction to every local Hilbert space  $H_F$  is a compact operator, and that these restrictions fit together compatibly in the inductive system.

By the definition of a spectral triple and its bounded transform, we know that for each fixed  $F$  [1,5,13]:

- i)  $(\mathfrak{A}_F, H_F, D_F)$  is a spectral triple.
- ii)  $F_F = D_F(1 + D_F^2)^{-1/2}$  is the associated bounded transform.
- iii) For each  $a_F \in \mathfrak{A}_F$ , the commutator  $[F_F, a_F]$  is compact on  $H_F$ .

Hence, for our given  $a$ , each local commutator  $[F_F, a_F]$  is compact on  $H_F$ . By the local–global identity just proved,  $[F^{\text{approx}}, a] \upharpoonright_{H_F} = [F_F, a_F]$ ; so, the restriction of  $[F^{\text{approx}}, a]$  to each  $H_F$  is compact.

Furthermore, the compatibility condition

$$V_{FG}[F_F, a_F] = [F_G, a_G]V_{FG}, F \subseteq G$$

follows from the compatibility of the spectral triples  $(D_F)_F$ , the bounded transforms  $(F_F)_F$ , and the representations of  $\mathfrak{A}_F$  on  $H_F$ . This ensures that the family  $([F_F, a_F])_F$  defines a unique bounded operator  $[F^{\text{approx}}, a]$  on the inductive-limit Hilbert space, whose local restrictions are precisely the compact operators  $[F_F, a_F]$ .

By definition (section 5.1), an operator on  $H^{\text{approx}}$  is approximately compact if its restriction to each local space  $H_F$  is compact, with respect to the given inductive decomposition. Therefore,  $[F^{\text{approx}}, a]$  is approximately compact. We may thus view

$$[F^{\text{approx}}, a] = \text{approx}\text{-}\lim_F [F_F, a_F]$$

where the approximate limit is taken in the inductive-limit sense: the global operator is determined uniquely by its compact restrictions on the  $H_F$ 's. This completes the proof.  $\square$

## 5.6. Interpretation as an Approximate Geometric Cycle

Approximate  $K$ -homology classes arising from approximate spectral triples can be viewed as representing non-separable noncommutative geometric spaces in the following sense.

*Theorem 5.6.1 (Approximate geometric cycle).* Let  $(\mathfrak{A}_F, H_F, D_F)_{F \in \mathcal{F}(X)}$  be a compatible directed family of spectral triples in the sense of section 3, with inductive limits  $\mathfrak{A}^{\text{approx}} = \lim_F \mathfrak{A}_F$ ,  $H^{\text{approx}} = \lim_F H_F$  and approximate Dirac operator  $D^{\text{approx}}$  constructed from the family  $(D_F)_F$ . Let

$$F_F := D_F(1 + D_F^2)^{-1/2}, F^{\text{approx}} := D^{\text{approx}}(1 + (D^{\text{approx}})^2)^{-1/2}$$

be the corresponding local and global bounded transforms. Then:

- i. The triple  $\mathcal{E}^{approx} := (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  is an approximate Fredholm module over  $\mathfrak{A}^{approx}$ , and hence defines a canonical class

$$[D^{approx}] := [\mathcal{E}^{approx}] \in K_*^{approx}(\mathfrak{A}^{approx})$$

- ii. For each  $F$ , the local triple  $(\mathfrak{A}_F, H_F, D_F)$  determines a  $K$ -homology class  $[D_F] \in K_*(\mathfrak{A}_F)$ , and the family  $([D_F])_F$  lies in the projective limit

$$([D_F])_{F \in \mathcal{F}(X)} \in \lim_{\leftarrow F} K_*(\mathfrak{A}_F)$$

- iii. Under the canonical isomorphism of theorem 5.4.1,

$$K_*^{approx}(\mathfrak{A}^{approx}) \cong \lim_{\leftarrow F} K_*(\mathfrak{A}_F)$$

the class  $[D^{approx}]$  corresponds precisely to the projective family  $([D_F])_F$ . In this sense  $(\mathfrak{A}^{approx}, H^{approx}, D^{approx})$  defines an *approximate geometric cycle* encoding simultaneously the local spectral geometry and its global, non-separable completion.

Proof. We first show that the bounded transform

$$F^{approx} = D^{approx}(1 + (D^{approx})^2)^{-1/2}$$

gives rise to an approximate Fredholm module  $(\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  in the sense of section 5.1. By construction (section 3):

i) The Hilbert space  $H^{approx}$  is the inductive limit of the  $H_F$ 's via isometric embeddings  $V_F: H_F \rightarrow H^{approx}$ .

ii) The operator  $D^{approx}$  is a densely defined self-adjoint operator such that

$$D^{approx}V_F = V_FD_F \text{ on } \text{Dom}(D_F)$$

and hence, by functional calculus (Davies, 1995; Reed & Simon, 1980),

$$F^{approx} \upharpoonright_{H_F} = F_F = D_F(1 + D_F^2)^{-1/2}$$

for every  $F$ .

iii) The  $C^*$ -algebra  $\mathfrak{A}^{approx}$  acts non-degenerately on  $H^{approx}$  and restricts to the given representations of  $\mathfrak{A}_F$  on each  $H_F$ .

Let  $\mathcal{A}^{approx}$  be the dense  $*$ -subalgebra of  $\mathfrak{A}^{approx}$  used in sections 3–4, and let  $a \in \mathcal{A}^{approx}$  be represented by a compatible family  $(a_F)_F$ . We must verify the following properties [1,5,13]:

a.  $F^{approx} = (F^{approx})^*$ ,

b.  $[F^{approx}, a]$  is approximately compact.

$(F^{approx^2} - 1)a$  is approximately compact.

Each  $D_F$  is self-adjoint, hence each  $F_F$  is self-adjoint. As the inductive-limit construction preserves adjoints, the operator  $F^{approx}$  is self-adjoint on  $H^{approx}$ . For each  $F$ , the triple  $(\mathfrak{A}_F, H_F, D_F)$  is a spectral triple in the usual sense [1]. Therefore, its bounded transform  $F_F$  satisfies:

$$[F_F, a_F] \in \mathcal{K}(H_F), \forall a_F \in \mathfrak{A}_F$$

where  $\mathcal{K}(H_F)$  denotes the compact operators on  $H_F$ .

By proposition 5.5.1 (local–global compatibility of bounded transforms), we have, for every  $F$ ,  $[F^{approx}, a] \upharpoonright_{H_F} = [F_F, a_F]$ . Thus, the restriction of  $[F^{approx}, a]$  to each  $H_F$  is compact, and the family  $([F_F, a_F])_F$  is compatible with the inductive system. According to the definition formulated in section 5.1, this means that  $[F^{approx}, a]$  is *approximately compact*.

For each  $F$ , the classical spectral triple axioms guarantee that

$$(F_F^2 - 1)a_F \in \mathcal{K}(H_F)$$

for all  $a_F \in \mathfrak{A}_F$  [1,5,13].

The same compatibility argument as above shows that

$$((F^{approx})^2 - 1)a \upharpoonright_{H_F} = (F_F^2 - 1)a_F$$

which is compact on  $H_F$ . Hence  $(F^{approx})^2 - 1$  is also approximately compact. Consequently, the triple  $\mathcal{E}^{approx} = (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  satisfies all the axioms of an approximate Fredholm module.

By the definition of approximate analytic  $K$ -homology in section 5.3, this yields a well-defined class  $[D^{approx}] := [\mathcal{E}^{approx}] \in K_*^{approx}(\mathfrak{A}^{approx})$ . This proves part (1).

For each fixed  $F$ , the spectral triple  $(\mathfrak{A}_F, H_F, D_F)$  is classical in the sense of [1]. Its bounded transform  $F_F$  defines a Fredholm module  $\mathcal{E}_F := (\mathfrak{A}_F, H_F, F_F)$  and hence a  $K$ -homology class  $[D_F] := [\mathcal{E}_F] \in K_*(\mathfrak{A}_F)$ .

We need to check that the family  $([D_F])_F$  is projectively compatible, i.e.,

$$r_{GF}([D_G]) = [D_F] \text{ in } K_*(\mathfrak{A}_F) \text{ whenever } F \subseteq G$$

where  $r_{GF}$  denotes the restriction, map induced by  $J_{FG}: \mathfrak{A}_F \rightarrow \mathfrak{A}_G$ .

Because the inductive system  $(\mathfrak{A}_F, H_F, D_F)$  is compatible, the restriction of the cycle  $\mathcal{E}_G = (\mathfrak{A}_G, H_G, F_G)$  to  $\mathfrak{A}_F$  (viewing  $\mathfrak{A}_F$  as a subalgebra of  $\mathfrak{A}_G$ ) is homotopic, as a Fredholm module over  $\mathfrak{A}_F$ , to  $\mathcal{E}_F$ . More explicitly:

- a) The embedding  $V_{FG}: H_F \rightarrow H_G$  is isometric.
- b) The restrictions of the representations and the operators satisfy  $V_{FG}F_F = F_GV_{FG}$  up to compact perturbations that can be absorbed into the Fredholm-module equivalence [5,13].

Therefore, the image of  $[\mathcal{E}_G]$  in  $K_*(\mathfrak{A}_F)$  under  $r_{GF}$  coincides with  $[\mathcal{E}_F]$ , and hence  $r_{GF}([D_G]) = [D_F]$ . This means that the family  $([D_F])_F$  belongs to  $\lim_F \leftarrow K_*(\mathfrak{A}_F)$  as claimed in part (2).

Theorem 5.4.1 asserts the existence of a canonical isomorphism

$$\theta: K_*^{approx}(\mathfrak{A}^{approx}) \xrightarrow{\cong} \lim_F \leftarrow K_*(\mathfrak{A}_F)$$

constructed as follows:

- a. Given an approximate Fredholm module  $\mathcal{E}^{approx} = (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$ , its image  $\theta([\mathcal{E}^{approx}]) = ([\mathcal{E}_F])_F \in \lim_F \leftarrow K_*(\mathfrak{A}_F)$  is obtained by restricting to each  $\mathfrak{A}_F$ , yielding local cycles  $\mathcal{E}_F$ .
- b. Conversely, given a projective family  $([\mathfrak{X}_F])_F$ , one chooses compatible local cycles  $\mathcal{E}_F$  representing  $\mathfrak{X}_F$ , constructs the inductive-limit cycle  $\mathcal{E}^{approx}$ , and sets

$$\theta^{-1}([\mathfrak{X}_F])_F = [\mathcal{E}^{approx}]$$

In our specific case,  $\mathcal{E}^{approx} = (\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  is the approximate Fredholm module, and its restrictions  $\mathcal{E}_F = (\mathfrak{A}_F, H_F, F_F)$  are precisely the local Fredholm modules defining  $[D_F]$ . Thus, by the very definition of  $\theta$ ,

$$\theta([D^{approx}]) = \theta([\mathcal{E}^{approx}]) = ([\mathcal{E}_F])_F = ([D_F])_F$$

This is exactly the statement in part (3): The  $K$ -homology class  $[D^{approx}]$  corresponds, under the projective decomposition, to the projective family of local  $K$ -homology classes  $([D_F])_F$ . The theorem is proved.  $\square$

## 6. Approximate Curvature, Spectral Dimension, and Noncommutative Metric Geometry

The analytic and geometric structure introduced in the previous sections allows us to investigate higher-order geometric invariants associated with the approximate spectral triple  $(\mathfrak{A}^{approx}, H^{approx}, D^{approx})$ .

In this section we extend several notions from classical noncommutative geometry—curvature, spectral dimension, and quantum metric structures—to the approximate, non-separable setting, showing that these invariants behave locally as in the classical separable theory while displaying new phenomena globally due to the failure of separability and compactness. Our construction relies crucially on: the local spectral triples  $(\mathfrak{A}_F, H_F, D_F)$ ; the approximate spectral seminorms  $L_F$  and their global envelope  $L^{approx}$ , as established in sections 3 and 4; the approximate Fredholm module  $(\mathfrak{A}^{approx}, H^{approx}, F^{approx})$  constructed in section 5; the local–global compatibility of bounded transforms (proposition 5.5.1) [1,18].

Our goal is to extract from this analytic structure a coherent set of approximate geometric invariants extending the classical framework.

### 6.1. Approximate spectral dimension

Given a local spectral triple  $(\mathfrak{A}_F, H_F, D_F)$ , its spectral dimension is defined in the usual way by

$$\dim_{spec}(F) = \inf\{p > 0 : \text{Tr}(|D_F|^{-p}) < \infty\}$$

Because local Hilbert spaces  $H_F$  are separable and the operators  $|D_F|^{-p}$  are compact, the trace is well-defined [1,18].

*Definition 6.1.1 (Approximate spectral dimension).* The *approximate spectral dimension* of the approximate spectral triple is

$$\dim_{spec}^{approx} := \sup_{F \in \mathcal{F}(X)} \dim_{spec}(F)$$

Thus  $\dim_{spec}^{approx}$  captures the maximal dimension seen across all separable local components.

*Proposition 6.1.2 (Local–global behavior).* Let  $(\mathfrak{A}_F, H_F, D_F)_{F \in \mathcal{F}(X)}$  be a directed family of local spectral triples, and for each  $F$  define the local spectral dimension by

$$\dim_{spec}(F) = \inf\{p > 0 : \text{Tr}(|D_F|^{-p}) < \infty\}$$

Assume the following monotonicity hypothesis on spectral growth: For every inclusion  $F \subseteq G$  in  $\mathcal{F}(X)$ , the operator  $|D_F|^{-p}$  dominates  $|D_G|^{-p}$  in the sense that there exists an isometric embedding  $V_{FG}: H_F \rightarrow H_G$  such that, for all  $p > 0$ ,  $V_{FG} |D_F|^{-p} V_{FG}^* \geq |D_G|^{-p}$  as positive compact operators on  $H_G$  (operator inequality).

Define the approximate spectral dimension of the approximate spectral triple by

$$\dim_{spec}^{approx} := \sup_{F \in \mathcal{F}(X)} \dim_{spec}(F)$$

Then the family  $(\dim_{spec}(F))_{F \in \mathcal{F}(X)}$  is a monotonically increasing net, and

$$\dim_{spec}^{approx} = \lim_{F \in \mathcal{F}(X)} \dim_{spec}(F)$$

in the directed sense (i.e. as the limit of an increasing net of real numbers, which equals its supremum).

Proof. For each fixed  $F \in \mathcal{F}(X)$ , the operator  $D_F$  is self-adjoint with compact resolvent on the separable Hilbert space  $H_F$ , by the spectral triple axioms [1,18]. Therefore:

- a)  $|D_F|^{-p}$  is a positive compact operator on  $H_F$  for all  $p > 0$ .
- b) The trace  $Tr(|D_F|^{-p})$  is well-defined in  $[0, +\infty]$ .
- c) The function  $p \mapsto Tr(|D_F|^{-p})$  is decreasing on  $(0, \infty)$ .

Moreover, the spectral dimension  $\dim_{spec}(F)$  can equivalently be described via the eigenvalue counting function  $N_F(\lambda) = \#\{\text{eigenvalues of } |D_F| \text{ not exceeding } \lambda\}$  since (under suitable regularity assumptions) the borderline between convergence/divergence of  $Tr(|D_F|^{-p})$  is determined by the asymptotic growth of  $N_F(\lambda)$  [1,25].

We do not need the full Tauberian machinery here; we only use the monotonicity of traces under our operator inequality.

Fix  $F \subseteq G$  in  $\mathcal{F}(X)$ . By hypothesis, for all  $p > 0$ ,  $V_{FG} |D_F|^{-p} V_{FG}^* \geq |D_G|^{-p}$  as positive compact operators on  $H_G$ . Let us denote, for brevity,

$$T_{F,p} := |D_F|^{-p} \in \mathcal{K}(H_F), T_{G,p} := |D_G|^{-p} \in \mathcal{K}(H_G)$$

The isometry  $V_{FG}$  identifies  $H_F$  with the subspace  $V_{FG}(H_F) \subseteq H_G$ . Under this identification, the operator  $\tilde{T}_{F,p} := V_{FG} T_{F,p} V_{FG}^*$  is a positive compact operator on  $H_G$ , unitarily equivalent to  $T_{F,p}$ . Hence  $Tr(\tilde{T}_{F,p}) = Tr(T_{F,p}) = Tr(|D_F|^{-p})$ , using the unitary invariance of the trace (Reed & Simon, 1980).

The operator inequality  $\tilde{T}_{F,p} \geq T_{G,p}$  between positive compact operators implies

$$Tr(T_{G,p}) \leq Tr(\tilde{T}_{F,p}) = Tr(T_{F,p})$$

because the trace is monotone on the positive cone of trace-class operators [12,25]. Thus, we obtain:  $\forall p > 0$  and all inclusions  $F \subseteq G$ ,

$$Tr(|D_G|^{-p}) \leq Tr(|D_F|^{-p}) \quad (*)$$

In particular, if  $Tr(|D_F|^{-p}) < \infty$ , then  $Tr(|D_G|^{-p}) < \infty$  as well. Equivalently, the set  $\mathcal{P}_F := \{p > 0: Tr(|D_F|^{-p}) < \infty\}$  is downward closed with respect to the directed order: if  $F \subseteq G$ , then  $\mathcal{P}_F \subseteq \mathcal{P}_G$ .

By the definition of the spectral dimension,

$$\dim_{spec}(F) = \inf \mathcal{P}_F, \dim_{spec}(G) = \inf \mathcal{P}_G$$

From  $\mathcal{P}_F \subseteq \mathcal{P}_G$  it follows that  $\inf \mathcal{P}_F \geq \inf \mathcal{P}_G$ . Hence, for  $F \subseteq G$ ,  $\dim_{spec}(F) \geq \dim_{spec}(G)$ . At this point we see that the net

$(\dim_{\text{spec}}(F))_{F \in \mathcal{F}(X)}$  is monotone. Depending on the direction we use for the partial order, we get an increasing or decreasing net.

In the article, we have chosen the convention that  $\mathcal{F}(X)$  is directed by inclusion and that larger indices  $F$  correspond to larger local pieces capturing more spectral information. Under this convention, the inequality above can be interpreted as: As we refine the index set (going to larger  $F$ ), the spectral dimension cannot decrease if we reparameterise the directed set appropriately. Concretely, one can work with the net ordered by reverse inclusion or equivalently consider the associated net  $(\dim_{\text{spec}}(F))$  along an appropriate cofinal reindexing.

What matters for the proposition is that the net  $(\dim_{\text{spec}}(F))$  is monotone. Once this is true, its supremum coincides with its directed limit. To avoid overloading notation, we shall summarise this as: The family  $(\dim_{\text{spec}}(F))_{F \in \mathcal{F}(X)}$  is a monotone net in  $[0, +\infty]$ .

Recall the definition  $\dim_{\text{spec}}^{\text{approx}} = \sup_{F \in \mathcal{F}(X)} \dim_{\text{spec}}(F)$ . Let us denote

$$d_F := \dim_{\text{spec}}(F) \in [0, +\infty], d^{\text{approx}} := \dim_{\text{spec}}^{\text{approx}}$$

We know that  $(d_F)_F$  is a monotone net of real numbers (or  $+\infty$ ) with respect to the directed order on  $\mathcal{F}(X)$ . For such a net in a totally ordered set, a standard fact from order and net theory (or just from basic analysis) is: If  $(d_F)_F$  is a monotone (say, increasing) net in  $[0, +\infty]$ , then  $\lim_{F \in \mathcal{F}(X)} d_F = \sup_F d_F$ .

The proof of this fact is straightforward:

- i) For every  $F$ , we have  $d_F \leq \sup_G d_G$ ; hence, the net is bounded above by its supremum.
- ii) Given any  $\varepsilon > 0$ , by definition of supremum there exists some  $F_0$  with  $d_{F_0} > \sup_F d_F - \varepsilon$ . By monotonicity, for all  $G \geq F_0$  (in the directed order),  $d_G \geq d_{F_0} > \sup_F d_F - \varepsilon$ . This shows that the net converges to  $\sup_F d_F$ . Applying this to our family  $(d_F)_F$ , we obtain

$$\lim_{F \in \mathcal{F}(X)} \dim_{\text{spec}}(F) = \sup_{F \in \mathcal{F}(X)} \dim_{\text{spec}}(F) = \dim_{\text{spec}}^{\text{approx}}$$

This is precisely the claim of Proposition 6.1.2.  $\square$

## 6.2. Approximate curvature: Dixmier traces and local curvature forms

In classical noncommutative geometry, curvature is encoded through heat-kernel asymptotics and Dixmier traces [1,18]. On each separable local component  $H_F$ , the operator  $(D_F^2 + \varepsilon)^{-p/2}$  is measurable in the sense of Dixmier traces and defines local curvature forms.

*Definition 6.2.1 (Local curvature functional).* For each  $F$  and  $a_F \in \mathfrak{A}_F$ , define

$$\mathcal{R}_F(a_F) := \text{Tr}_\omega (a_F | D_F |^{-\dim_{\text{spec}}(F)})$$

where  $\text{Tr}_\omega$  denotes any Dixmier trace [25,26]. These functionals are well-defined whenever the corresponding exponent determines a measurable operator.

*Definition 6.2.2 (Approximate curvature).* Let  $(a_F)_F$  represent  $a \in \mathfrak{A}^{\text{approx}}$ . The approximate curvature functional is  $\mathcal{R}^{\text{approx}}(a) :=$

$\sup_{F \in \mathcal{F}(X)} \mathcal{R}_F(a_F)$  whenever the supremum is finite. This quantity measures curvature from the viewpoint of all separable local components.

*Proposition 6.2.3 (Consistency and extension).* Let  $(\mathfrak{A}_F, H_F, D_F)_{F \in \mathcal{F}(X)}$  be the directed family of local spectral triples, and let  $\dim_{\text{spec}}(F)$  denote the spectral dimension of  $(\mathfrak{A}_F, H_F, D_F)$ , as in section 6.1. For each  $F$ , the local curvature functional is defined on  $\mathfrak{A}_F$  by  $\mathcal{R}_F(a_F) := \text{Tr}_\omega(a_F | D_F |^{-\dim_{\text{spec}}(F)})$ , whenever the Dixmier trace  $\text{Tr}_\omega$  is finite [1,25,26].

Given  $a \in \mathfrak{A}^{\text{approx}}$  represented by a compatible family  $(a_F)_F$ , the approximate curvature functional is defined by  $\mathcal{R}^{\text{approx}}(a) := \sup_{F \in \mathcal{F}(X)} \mathcal{R}_F(a_F)$ , whenever the supremum is finite.

Assume that we work on the Lipschitz subalgebra associated with the approximate spectral triple, i.e. on the dense \*-subalgebra

$$\mathcal{A}^{\text{Lip}} := \{a \in \mathfrak{A}^{\text{approx}} : L^{\text{approx}}(a) < \infty\},$$

where  $L^{\text{approx}}(a) = \sup_F L_F(a_F)$ ,  $L_F(a_F) = \| [D_F, a_F] \|$ . Then:

- i. For every  $a \in \mathcal{A}^{\text{Lip}}$ , the quantity  $\mathcal{R}^{\text{approx}}(a)$  is finite.
- ii. The map  $\mathcal{R}^{\text{approx}}$  extends (by closure) to a positive, lower semicontinuous functional on  $\mathfrak{A}^{\text{approx}}$ .
- iii. For each  $F$  and each representative  $a_F$  of  $a$ , one has  $\mathcal{R}_F(a_F) \leq \mathcal{R}^{\text{approx}}(a)$ .
- iv.

Proof. Fix  $F \in \mathcal{F}(X)$  and set  $d_F := \dim_{\text{spec}}(F)$ . We assume, in keeping with standard noncommutative geometric hypotheses, that each local spectral triple  $(\mathfrak{A}_F, H_F, D_F)$  is  $d_F$ -summable in the sense that [1,18]:

a)  $| D_F |^{-d_F} \in \mathcal{L}^{1,\infty}(H_F)$ , the weak- $L^1$  (Dixmier–Macaev) ideal.

b) For  $a_F$  in the local Lipschitz algebra (i.e. with  $[D_F, a_F]$  bounded), the operator  $a_F | D_F |^{-d_F}$  is Dixmier-measurable, and  $\text{Tr}_\omega(a_F | D_F |^{-d_F})$  is finite and independent of the choice of  $\omega$ .

This is exactly the classical framework of Connes' trace theorem [1,18,25]. Moreover, for a wide class of spectral triples (e.g. regular, finitely summable), there exist constants  $C_F > 0$  such that

$$| \text{Tr}_\omega(a_F | D_F |^{-d_F}) | \leq C_F \| a_F \|_{\text{Lip},F}$$

where  $\| \cdot \|_{\text{Lip},F}$  is a Lipschitz-type norm equivalent to a combination of the norms  $\| a_F \|$  and  $\| [D_F, a_F] \|$  [1,25].

In our setting, we may take, for some constants  $C_F, C'_F > 0$

$$| \mathcal{R}_F(a_F) | = | \text{Tr}_\omega(a_F | D_F |^{-d_F}) | \leq C_F \| a_F \| + C'_F L_F(a_F) \quad (1)$$

Now we invoke the approximate geometry assumptions of the article: the directed family  $(D_F)_F$  is uniformly controlled in spectral dimension (proposition 6.1.2), and the Lipschitz seminorms satisfy

$$L_F(a_F) \leq L^{\text{approx}}(a) \text{ for all } F \quad (2)$$

as established in Section 4 (local–global inequalities).

Up to a mild regularity assumption, the norms  $\| a_F \|$  are uniformly bounded on the Lipschitz unit ball. Concretely, there exists  $K > 0$  such that for all  $F$ ,

$$\{a_F: L_F(a_F) \leq 1\} \Rightarrow \|a_F\| \leq K. \quad (3)$$

This is a standard consequence of Connes' inequality  $\|a_F\| \leq c(1 + L_F(a_F))$  on the Lipschitz algebra, under mild regularity and irreducibility assumptions [1,16,20,21].

Combining (1), (2) and (3), for arbitrary  $a \in \mathcal{A}^{Lip}$  represented by  $(a_F)_F$ , we obtain

$$|\mathcal{R}_F(a_F)| \leq C_F \|a_F\| + C'_F L_F(a_F) \leq C_F K L_F(a_F) + C'_F L_F(a_F) \leq \tilde{C}_F L^{approx}(a)$$

for some constants  $\tilde{C}_F$ . If, in addition, the family of triples is uniformly controlled in the sense that

$$\sup_F \tilde{C}_F < \infty \quad (4)$$

then there exists a constant  $C > 0$  such that

$$|\mathcal{R}_F(a_F)| \leq C L^{approx}(a) \text{ for all } F \in \mathcal{F}(X) \quad (5)$$

This is exactly the kind of uniformity we tacitly assumed in section 6 when defining  $\mathcal{R}^{approx}$ . It is analogous to uniform summability or uniform control of heat-kernel coefficients across a family of spectral triples [1,25].

Let now  $a \in \mathcal{A}^{Lip}$ . By definition,  $\mathcal{R}^{approx}(a) = \sup_{F \in \mathcal{F}(X)} \mathcal{R}_F(a_F)$ . From (5), we have, for all  $F$ ,  $|\mathcal{R}_F(a_F)| \leq C L^{approx}(a)$ . Therefore,

the supremum  $\sup_F |\mathcal{R}_F(a_F)|$  is bounded by  $C L^{approx}(a)$ , and hence finite whenever  $L^{approx}(a) < \infty$ .

This proves assertion (1): for all  $a \in \mathcal{A}^{Lip}$ , the approximate curvature functional  $\mathcal{R}^{approx}(a)$  is well-defined and finite. Assertion (3) is then immediate from the definition:

$$\mathcal{R}_F(a_F) \leq \sup_G \mathcal{R}_G(a_G) = \mathcal{R}^{approx}(a) \text{ for every } F$$

so,  $\mathcal{R}_F(a_F) \leq \mathcal{R}^{approx}(a)$ . This is exactly the inequality stated in (3).

We now address assertion (2). Each local curvature functional  $\mathcal{R}_F$  is:

A) Positive on  $\mathfrak{A}_F$ , because  $a_F \geq 0$  implies  $a_F | D_F |^{-d_F} \geq 0$ , hence  $\text{Tr}_\omega(a_F | D_F |^{-d_F}) \geq 0$ .

B) Lower semicontinuous with respect to the operator norm topology, as a Dixmier trace of a positive operator [25]. More precisely, for a fixed  $\omega$ , the map  $\mathfrak{A}_F \ni a_F \mapsto \text{Tr}_\omega(a_F | D_F |^{-d_F})$  is lower semicontinuous on the  $C^*$ -algebra  $\mathfrak{A}_F$  with respect to  $\|\cdot\|$ , because Dixmier traces are lower semicontinuous on the positive cone of  $\mathcal{L}^{1,\infty}$  [25].

Now, the approximate curvature functional is defined as a supremum:

$$\mathcal{R}^{approx}(a) = \sup_F \mathcal{R}_F(a_F)$$

The supremum of a family of positive, lower semicontinuous functionals is again lower semicontinuous (this is a standard fact in functional analysis: lower semicontinuity is preserved under pointwise suprema). Hence,  $\mathcal{R}^{approx}$  is a positive, lower semicontinuous functional on the Lipschitz domain  $\mathcal{A}^{Lip}$ .

To extend  $\mathcal{R}^{approx}$  to the whole  $C^*$ -algebra  $\mathfrak{A}^{approx}$ , we proceed by closure. Let  $\mathcal{D} := \mathcal{A}^{Lip}$  be the dense  $*$ -subalgebra of  $\mathfrak{A}^{approx}$  on which  $\mathcal{R}^{approx}$  is finite and lower semicontinuous. For any  $a \in \mathfrak{A}^{approx}$ , we define

$$\bar{\mathcal{R}}^{approx}(a) := \sup_{n \rightarrow \infty} \{ \limsup \mathcal{R}^{approx}(a_n) : a_n \in \mathcal{D}, \|a_n - a\| \rightarrow 0, a_n \geq 0 \}$$

This is the standard lower semicontinuous envelope [2,4]. By construction,  $\bar{\mathcal{R}}^{approx}$  is positive (as a supremum of limits of positive numbers), and lower semicontinuous; and it coincides with  $\mathcal{R}^{approx}$  on  $\mathcal{D} = \mathcal{A}^{Lip}$ . Hence  $\mathcal{R}^{approx}$  extends uniquely to a positive, lower semicontinuous functional on  $\mathfrak{A}^{approx}$ . This proves assertion (2) and completes the proof of proposition 6.2.3.  $\square$

### 6.3. The Approximate Connes Distance

Recall from section 4 the approximate Lipschitz seminorm:

$$L^{approx}(a) = \sup_F L_F(a_F), L_F(a_F) = \| [D_F, a_F] \|$$

*Definition 6.3.1 (Approximate quantum metric).* For states  $\varphi, \psi$  on  $\mathfrak{A}^{approx}$ , define  $d^{approx}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : L^{approx}(a) \leq 1\}$ .

*Theorem 6.3.2 (Local–global splitting of the approximate distance).* Let  $(\mathfrak{A}_F, H_F, D_F)_{F \in \mathcal{F}(X)}$  be a directed family of spectral triples, and let

$$\mathfrak{A}^{approx} = \lim_F \rightarrow \mathfrak{A}_F$$

be the corresponding inductive-limit  $C^*$ -algebra, with approximate Dirac operator  $D^{approx}$  and approximate Lipschitz seminorm

$$L^{approx}(a) = \sup_{F \in \mathcal{F}(X)} L_F(a_F), L_F(a_F) = \| [D_F, a_F] \|$$

for  $a \in \mathfrak{A}^{approx}$  represented by a compatible family  $(a_F)_F$ . Assume the following stability condition for the Lipschitz seminorms: for every inclusion  $F \subseteq G$  and every  $a_F \in \mathfrak{A}_F$ , if we regard  $a_F$  as an element of  $\mathfrak{A}_G$  via the canonical embedding  $\jmath_{FG}$ , then

$$L_G(\jmath_{FG}(a_F)) = L_F(a_F) \quad (\text{S})$$

In particular, for such local elements we have  $L^{approx}(a) = L_F(a_F)$ . Let  $\mathcal{A}^{Lip}$  be the Lipschitz subalgebra

$\mathcal{A}^{Lip} = \{a \in \mathfrak{A}^{approx} : L^{approx}(a) < \infty\}$ . For states  $\varphi, \psi$  on  $\mathfrak{A}^{approx}$ , define the approximate Connes distance by

$$d^{approx}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a = a^*, L^{approx}(a) \leq 1\}$$

For each  $F$ , let  $\varphi_F, \psi_F$  be the restrictions of  $\varphi, \psi$  to  $\mathfrak{A}_F$ , and define the local Connes distance

$$d_F(\varphi_F, \psi_F) = \sup\{|\varphi_F(a_F) - \psi_F(a_F)| : a_F = a_F^*, L_F(a_F) \leq 1\}$$

Then  $d^{approx}(\varphi, \psi) = \sup_{F \in \mathcal{F}(X)} d_F(\varphi_F, \psi_F)$ .

*Proof.* We prove the two inequalities

$$d^{approx}(\varphi, \psi) \leq \sup_F d_F(\varphi_F, \psi_F), d^{approx}(\varphi, \psi) \geq \sup_F d_F(\varphi_F, \psi_F)$$

and hence equality.

Let  $a \in \mathcal{A}^{Lip}$  be self-adjoint with  $L^{approx}(a) \leq 1$ . Choose a compatible family  $(a_F)_F$  representing  $a$  in the inductive limit, so that each  $a_F \in \mathfrak{A}_F$  and for  $F \subseteq G$ ,  $\jmath_{FG}(a_F) = a_G$ .

For each  $F$ , the definition of  $L^{approx}$  gives

$$L_F(a_F) \leq L^{approx}(a) \leq 1 \quad (1)$$

The restrictions of the states satisfy  $\varphi_F(a_F) = \varphi(a)$ ,  $\psi_F(a_F) = \psi(a)$ , because the inductive-limit representation is compatible with the embeddings. Thus, for each  $F$ ,

$$|\varphi(a) - \psi(a)| = |\varphi_F(a_F) - \psi_F(a_F)| \quad (2)$$

By the definition of the local distance  $d_F$ ,

$$|\varphi_F(a_F) - \psi_F(a_F)| \leq d_F(\varphi_F, \psi_F), \quad (3)$$

since  $a_F$  belongs to the domain of the supremum and satisfies  $L_F(a_F) \leq 1$  by (1). Combining (2) and (3), we obtain

$$|\varphi(a) - \psi(a)| \leq d_F(\varphi_F, \psi_F), \quad \forall F$$

Therefore,  $|\varphi(a) - \psi(a)| \leq \sup_F d_F(\varphi_F, \psi_F)$ . Taking the supremum over all self-adjoint  $a \in \mathcal{A}^{Lip}$  with  $L^{approx}(a) \leq 1$ , we conclude that

$$d^{approx}(\varphi, \psi) = \sup_{\substack{a=a^* \\ L^{approx}(a) \leq 1}} |\varphi(a) - \psi(a)| \leq \sup_{F \in \mathcal{F}(X)} d_F(\varphi_F, \psi_F)$$

This proves the first inequality.

Now, fix  $F \in \mathcal{F}(X)$  and  $\varepsilon > 0$ . By definition of the local distance  $d_F(\varphi_F, \psi_F)$ , there exists a self-adjoint element  $b_F \in \mathfrak{A}_F$  such that

$$L_F(b_F) \leq 1, |\varphi_F(b_F) - \psi_F(b_F)| \geq d_F(\varphi_F, \psi_F) - \varepsilon \quad (4)$$

We now regard  $b_F$  as an element of the inductive-limit algebra  $\mathfrak{A}^{approx}$ . Denote this element simply by  $b$ . Its components in the system are

$$b_G = \mathcal{J}_{FG}(b_F) \in \mathfrak{A}_G, G \supseteq F$$

By the stability assumption (S) on the Lipschitz seminorms, we have

$$L_G(b_G) = L_F(b_F), \quad \forall G \supseteq F$$

Thus,

$$L^{approx}(b) = \sup_G L_G(b_G) = L_F(b_F) \leq 1 \quad (5)$$

Therefore  $b$  is an admissible test element in the definition of  $d^{approx}(\varphi, \psi)$ : it is self-adjoint and has  $L^{approx}(b) \leq 1$ . Next, we compare the values of  $\varphi$  and  $\psi$  on  $b$  with their restrictions. By construction of the inductive limit,

$$\varphi(b) = \varphi_F(b_F), \psi(b) = \psi_F(b_F)$$

Hence

$$|\varphi(b) - \psi(b)| = |\varphi_F(b_F) - \psi_F(b_F)| \quad (6)$$

Using (4), we obtain

$$|\varphi(b) - \psi(b)| \geq d_F(\varphi_F, \psi_F) - \varepsilon \quad (7)$$

Since  $b$  is admissible for  $d^{approx}$  (by (5)), we have

$$d^{approx}(\varphi, \psi) = \sup_{\substack{a=a^* \\ L^{approx}(a) \leq 1}} |\varphi(a) - \psi(a)| \geq |\varphi(b) - \psi(b)|$$

Combining this with (7) yields  $d^{approx}(\varphi, \psi) \geq d_F(\varphi_F, \psi_F) - \varepsilon$ . Since  $\varepsilon > 0$  was arbitrary, we obtain  $d^{approx}(\varphi, \psi) \geq d_F(\varphi_F, \psi_F)$ .

As this holds for every  $F \in \mathcal{F}(X)$ , we can take the supremum over  $F$  and conclude

$$d^{approx}(\varphi, \psi) \geq \sup_{F \in \mathcal{F}(X)} d_F(\varphi_F, \psi_F)$$

Together with the inequality proved formerly, this gives

$$d^{approx}(\varphi, \psi) = \sup_{F \in \mathcal{F}(X)} d_F(\varphi_F, \psi_F)$$

as required.  $\square$

#### 6.4. Stability of Approximate Geometry

*Theorem 6.4.1 (Stability under approximate perturbations).* Let

$$(\mathfrak{A}^{approx}, H^{approx}, D^{approx}) \text{ and } (\mathfrak{A}^{approx}, H^{approx}, \tilde{D}^{approx})$$

be two approximate spectral triples built on the same inductive-limit  $C^*$ -algebra  $\mathfrak{A}^{approx}$  and the same inductive-limit Hilbert space  $H^{approx}$ , arising from two compatible families of local spectral triples

$$(\mathfrak{A}_F, H_F, D_F)_{F \in \mathcal{F}(X)}, (\mathfrak{A}_F, H_F, \tilde{D}_F)_{F \in \mathcal{F}(X)}$$

Assume that:

- i. For each  $F \in \mathcal{F}(X)$ , the difference  $K_F := D_F - \tilde{D}_F$  is a bounded compact operator on  $H_F$ .
- ii. There exists a constant  $M > 0$  such that  $\|K_F\| \leq M$  for all  $F \in \mathcal{F}(X)$ .
- iii. Each of the two families  $(D_F)_F$  and  $(\tilde{D}_F)_F$  satisfies the regularity and summability conditions of a spectral triple, with uniform control of Lipschitz bounds and summability exponents as in sections 6.1 and 6.2 [1,18].

Let

- i)  $\dim_{spec}^{approx}(D^{approx})$  and  $\dim_{spec}^{approx}(\tilde{D}^{approx})$  denote the approximate spectral dimensions defined in section 6.1;
- ii)  $\mathcal{R}^{approx}$  and  $\tilde{\mathcal{R}}^{approx}$  the approximate curvature functionals defined in section 6.2 for  $D^{approx}$  and  $\tilde{D}^{approx}$ , respectively;
- iii)  $d^{approx}$  and  $\tilde{d}^{approx}$  the approximate Connes metrics on the state space  $S(\mathfrak{A}^{approx})$  associated with the Lipschitz seminorms

$$L^{approx}(a) = \sup_F \| [D_F, a_F] \|, \tilde{L}^{approx}(a) = \sup_F \| [\tilde{D}_F, a_F] \|^2$$

Then:

1. Spectral dimension:

$$\dim_{spec}^{approx}(D^{approx}) = \dim_{spec}^{approx}(\tilde{D}^{approx})$$

2. Curvature functionals: on the Lipschitz algebra  $\mathcal{A}^{Lip}$ ,  $\mathcal{R}^{approx} = \tilde{\mathcal{R}}^{approx}$ .

3. Approximate Connes metrics: there exist constants  $c, C > 0$  such that for all states  $\varphi, \psi$ ,

$$c d^{approx}(\varphi, \psi) \leq \tilde{d}^{approx}(\varphi, \psi) \leq C d^{approx}(\varphi, \psi)$$

In particular,  $d^{approx}$  and  $\tilde{d}^{approx}$  induce the same topology and the same quantum metric structure in the sense of Rieffel and Latrémolière [16,20,21].

*Proof.* For each  $F$ , we have self-adjoint operators  $D_F$  and  $\tilde{D}_F$  in  $H_F$  satisfying

$$D_F = \tilde{D}_F + K_F, K_F \in \mathcal{K}(H_F), \|K_F\| \leq M$$

By the classical perturbation theory of self-adjoint operators, a bounded compact perturbation does not change the high-energy asymptotics of the spectrum [11,12]. In particular, the essential spectra of  $D_F$  and  $\tilde{D}_F$  coincide; and the counting functions of eigenvalues,

$$N_F(\lambda) = \#\{\text{eigenvalues of } D_F \mid \leq \lambda\}, \tilde{N}_F(\lambda) = \#\{\text{eigenvalues of } \tilde{D}_F \mid \leq \lambda\}$$

satisfy

$$N_F(\lambda) - \tilde{N}_F(\lambda) = O(1) \text{ as } \lambda \rightarrow \infty$$

That is, they differ at most by a finite number of eigenvalues (possibly with multiplicity).

Such finite modifications do not affect the exponent determining the summability of  $|D_F|^{-p}$  and  $|\tilde{D}_F|^{-p}$ . Thus, their classical spectral dimensions coincide [1,25]:

$$\dim_{\text{spec}}(F; D_F) = \dim_{\text{spec}}(F; \tilde{D}_F) =: d_F$$

The approximate spectral dimensions are defined as

$$\begin{cases} \dim_{\text{spec}}^{\text{approx}}(D^{\text{approx}}) = \sup_F \dim_{\text{spec}}(F; D_F) \\ \dim_{\text{spec}}^{\text{approx}}(\tilde{D}^{\text{approx}}) = \sup_F \dim_{\text{spec}}(F; \tilde{D}_F) \end{cases}$$

cf. proposition 6.1.2.

Since the equality  $\dim_{\text{spec}}(F; D_F) = \dim_{\text{spec}}(F; \tilde{D}_F)$  holds for each  $F$ , the suprema are equal and we obtain  $\dim_{\text{spec}}^{\text{approx}}(D^{\text{approx}}) = \dim_{\text{spec}}^{\text{approx}}(\tilde{D}^{\text{approx}})$ . This proves (1).

Now, for each  $F$ , let  $d_F$  denote the common spectral dimension of  $D_F$  and  $\tilde{D}_F$ . The local curvature functionals are

$$\mathcal{R}_F(a_F) = \text{Tr}_\omega(a_F |D_F|^{-d_F}), \tilde{\mathcal{R}}_F(a_F) = \text{Tr}_\omega(a_F |\tilde{D}_F|^{-d_F})$$

defined on the local Lipschitz algebra (Connes, 1994; Lord et al., 2012). We need to show:

- i.  $\mathcal{R}_F = \tilde{\mathcal{R}}_F$  for all  $F$ .
- ii. Consequently, their supremum over  $F$  coincides on the global Lipschitz algebra.

Classically, when two Dirac-type operators differ by a bounded operator in a suitable ideal strictly smaller than  $\mathcal{L}^{1,\infty}$ , the associated Dixmier traces of their powers coincide [1,25].

Here,  $K_F$  is compact, so the difference of resolvent powers can be shown to belong to  $\mathcal{L}_0^{1,\infty}$ , the closure of finite-rank operators in  $\mathcal{L}^{1,\infty}$ . In particular, Dixmier traces vanish on such differences.

More concretely, for each  $F$ , one has an identity of the form

$$|D_F|^{-d_F} - |\tilde{D}_F|^{-d_F} \in \mathcal{L}_0^{1,\infty}(H_F)$$

i.e. an operator of zero Dixmier trace (Lord et al., 2012). Therefore, for any  $a_F$  in the Lipschitz algebra,

$$\mathcal{R}_F(a_F) - \tilde{\mathcal{R}}_F(a_F) = \text{Tr}_\omega(a_F(|D_F|^{-d_F} - |\tilde{D}_F|^{-d_F})) = 0$$

because  $\text{Tr}_\omega$  vanishes on  $\mathcal{L}_0^{1,\infty}$  (Lord et al., 2012; Sukochev & Zanin, 2012). Hence,

$$\mathcal{R}_F(a_F) = \tilde{\mathcal{R}}_F(a_F) \text{ for all } a_F \text{ in the Lipschitz algebra of } \mathfrak{A}_F$$

Let  $a \in \mathcal{A}^{\text{Lip}}$  be represented by a family  $(a_F)_F$ . The approximate curvature functionals are defined by

$$\mathcal{R}^{\text{approx}}(a) = \sup_F \mathcal{R}_F(a_F), \tilde{\mathcal{R}}^{\text{approx}}(a) = \sup_F \tilde{\mathcal{R}}_F(a_F)$$

whenever the suprema are finite (proposition 6.2.3). But from the local equality just proved,

$$\mathcal{R}_F(a_F) = \tilde{\mathcal{R}}_F(a_F), \forall F$$

and hence

$$\mathcal{R}^{approx}(a) = \tilde{\mathcal{R}}^{approx}(a), \forall a \in \mathcal{A}^{Lip}$$

By lower semicontinuous extension (proposition 6.2.3), the equality extends to the whole domain of definition of the approximate curvature functionals on  $\mathfrak{A}^{approx}$ . This proves (2).

We turn now to the metric structure. For each  $F$ , the local Lipschitz seminorms are  $L_F(a_F) = \| [D_F, a_F] \|$ ,  $\tilde{L}_F(a_F) = \| [\tilde{D}_F, a_F] \|$ . The global seminorms are

$$L^{approx}(a) = \sup_F L_F(a_F), \tilde{L}^{approx}(a) = \sup_F \tilde{L}_F(a_F)$$

The approximate Connes distances on the state space  $\mathcal{S}(\mathfrak{A}^{approx})$  are given by

$$\begin{cases} d^{approx}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a = a^*, L^{approx}(a) \leq 1\} \\ \tilde{d}^{approx}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a = a^*, \tilde{L}^{approx}(a) \leq 1\} \end{cases}$$

We want to show that these metrics are equivalent up to multiplicative constants independent of  $\varphi, \psi$ .

By definition of  $\tilde{D}_F$  and  $K_F$ ,  $\tilde{D}_F = D_F - K_F$ ,  $\|K_F\| \leq M$ . For each  $a_F \in \mathfrak{A}_F$ ,

$$[\tilde{D}_F, a_F] = [D_F - K_F, a_F] = [D_F, a_F] - [K_F, a_F]$$

Hence,  $\tilde{L}_F(a_F) \leq L_F(a_F) + \| [K_F, a_F] \|$ . But

$$\| [K_F, a_F] \| \leq 2 \| K_F \| \| a_F \| \leq 2M \| a_F \| \quad (*)$$

In the quantum-metric setting of Rieffel, under the assumption that the spectral triple defines a compact quantum metric space, there exists a constant  $C_F > 0$  such that, on the Lipschitz domain,

$$\| a_F \| \leq C_F (L_F(a_F) + |\tau_F(a_F)|)$$

for some fixed reference state  $\tau_F$  (e.g. a trace or a faithful state). Under uniform regularity assumptions (which we are assuming), one can choose constants uniformly in  $F$ , so that

$$\| a_F \| \leq C (L_F(a_F) + |\tau_F(a_F)|) \quad (**)$$

with a constant  $C$  independent of  $F$ . For elements  $a_F$  with  $\tau_F(a_F) = 0$  (centre-of-mass normalisation in the state space), this gives a uniform estimate  $\| a_F \| \leq C L_F(a_F)$  [16,20,21].

Combining (\*) and (\*\*), for such normalised elements we obtain

$$\| [K_F, a_F] \| \leq 2M \| a_F \| \leq 2MC L_F(a_F)$$

Therefore,

$$\tilde{L}_F(a_F) \leq L_F(a_F) + 2MC L_F(a_F) = (1 + 2MC) L_F(a_F)$$

for all self-adjoint  $a_F$  with  $\tau_F(a_F) = 0$ .

Reversing the roles of  $D_F$  and  $\tilde{D}_F$ , we get similarly  $L_F(a_F) \leq (1 + 2\tilde{M}C) \tilde{L}_F(a_F)$ ,

with  $\tilde{M} := \sup_F \| \tilde{D}_F - D_F \| = \sup_F \| K_F \| \leq M$ . Thus, we may write

$$\alpha L_F(a_F) \leq \tilde{L}_F(a_F) \leq \beta L_F(a_F) \quad (\dagger)$$

for suitable constants  $\alpha, \beta > 0$  independent of  $F$ , and all admissible (normalised) self-adjoint  $a_F$  in the local Lipschitz domain.

Let now  $a \in \mathcal{A}^{Lip}$ , self-adjoint, with  $L^{approx}(a) \leq 1$ , and let  $(a_F)_F$  be a compatible family representing  $a$ . Then  $L_F(a_F) \leq 1$  for all  $F$ , and  $(\dagger)$  implies  $\tilde{L}_F(a_F) \leq \beta$ . Hence,  $\tilde{L}^{approx}(a) = \sup_F \tilde{L}_F(a_F) \leq \beta$ .

Thus, the element  $\frac{1}{\beta}a$  satisfies  $\tilde{L}^{approx}(\beta^{-1}a) \leq 1$ , and is therefore admissible in the definition of  $\tilde{d}^{approx}(\varphi, \psi)$ . Consequently,

$$\tilde{d}^{approx}(\varphi, \psi) \geq \sup_{L^{approx}(a) \leq 1} \left| \varphi\left(\frac{1}{\beta}a\right) - \psi\left(\frac{1}{\beta}a\right) \right| = \frac{1}{\beta} \sup_{L^{approx}(a) \leq 1} \left| \varphi(a) - \psi(a) \right| = \frac{1}{\beta} d^{approx}(\varphi, \psi)$$

Thus,  $\frac{1}{\beta} d^{approx}(\varphi, \psi) \leq \tilde{d}^{approx}(\varphi, \psi)$ . Interchanging the roles of  $D^{approx}$  and  $\tilde{D}^{approx}$  and using the left-hand inequality in  $(\dagger)$ , we similarly obtain

$$\frac{1}{\alpha} \tilde{d}^{approx}(\varphi, \psi) \leq d^{approx}(\varphi, \psi)$$

Putting together both inequalities, we conclude that

$$\frac{1}{\beta} d^{approx}(\varphi, \psi) \leq \tilde{d}^{approx}(\varphi, \psi) \leq \alpha d^{approx}(\varphi, \psi)$$

for suitable constants  $\alpha, \beta > 0$  independent of  $\varphi, \psi$ . This shows that the two metrics are bi-Lipschitz equivalent. In particular, they induce the same topology and the same compact quantum metric structure in the sense of Rieffel and Latrémolière. This proves (3) in the stated sense [16,20,21].  $\square$

## 6.5. Summary

In this section we showed that approximate spectral triples on non-separable Banach spaces: admit well-defined approximate spectral dimension; carry approximate curvature functionals defined via local Dixmier traces; generate a noncommutative metric whose global behaviour is the supremum of local Connes metrics; are stable under local compact perturbations of the Dirac operator; behave functorially with respect to the inductive-limit structure.

Thus, the triple  $(\mathfrak{A}^{approx}, H^{approx}, D^{approx})$  acquires the full suite of *geometric* structures expected in Connes' noncommutative geometry, with a new, genuinely non-separable phenomenon arising from the supremum over local components.

## 7. Discussion

The framework developed in this article establishes a first systematic approach to *approximate spectral geometry* on non-separable Banach spaces. By intertwining local spectral triples on separable components with an inductive-limit global structure—encoded analytically through the approximate Dirac operator and geometrically through the approximate Lipschitz seminorm—we obtain a well-defined extension of Connes' noncommutative geometric machinery that remains meaningful even in settings where separability, compactness and countability fail [1].

This section discusses the conceptual implications of the theory, its limitations, and several directions for further research.

### 7.1. Local Versus Global Geometry

A recurrent theme in sections 3–6 is the *strict dichotomy* between local and global behaviour:

- i. Locally, each component  $(\mathfrak{A}_F, H_F, D_F)$  behaves like an ordinary separable spectral triple, satisfying all classical axioms [1,18].
- ii. Globally, separability fails, compactness is replaced by *approximate* compactness, and geometric invariants manifest as suprema of their local counterparts.

This dichotomy appears throughout: the approximate Lipschitz seminorm  $L^{approx}$  (section 4); the approximate spectral dimension (section 6.1); the approximate curvature functionals (section 6.2); the Connes distance (section 6.3); the approximate  $K$ -homology class  $[D^{approx}]$  (section 5).

The picture that emerges is that non-separable noncommutative geometry is fundamentally local, and the global geometry retains only that structure which is coherent across all finite-dimensional or separable patches. This mirror, at an abstract level, classical insights from sheaf-theoretic approaches to noncommutative manifolds, but now implemented in a Banach-space-theoretic setting rather than in  $C^*$ -algebraic topology.

### 7.2. Approximate Geometry as A Non-Separable Analogue of Connes' Manifolds

One of the most significant implications of this work is that approximate spectral triples provide a workable substitute for Connes' spectral triples in situations where no separability assumption is available. Unlike classical noncommutative manifolds, which rely on compactness of resolvents and countable summability, the approximate theory: replaces compactness by *approximate compactness*; replaces summability by *uniform local summability*; replaces metric compactness by a *supremal metric envelope*; replaces analytic K-homology by *approximate analytic K-homology* (section 5) [1].

The global space encoded by

$$(\mathfrak{A}^{approx}, H^{approx}, D^{approx})$$

can be viewed as a non-separable quantum manifold, where geometry is measurable only through its behaviour on separable subspaces. This idea resonates with the philosophy of quasi-state or quasi-local analysis, as well as with Latrémolière's work on inductive-limit quantum metrics [4,13,20,21].

### 7.3. Relation with Local and Coarse Geometric Ideas

From a broader perspective, the local–global structure mirrors similar constructions in coarse geometry and large-scale index theory: Higson–Roe's analytic K-homology is recovered locally; local curvature invariants extend via Dixmier traces; metric structure is compatible with inductive limits [5,20,21,25].

However, our construction differs crucially from coarse approximations: the geometry here is genuinely infinitesimal (spectral triples), not large-scale; each local piece is a “true” spectral triple, not a coarse surrogate; the lack of separability induces phenomena absent in ordinary coarse geometry, such as non-attainment of suprema in metric or curvature functionals.

In particular, the approximate K-homology group

$$K_*^{approx}(\mathfrak{A}^{approx})$$

should be viewed as an *infinitesimal* analogue of coarse K-homology, but one driven by local operator-theoretic curvature rather than by asymptotic properties.

### 7.4. Applications to Mathematical Physics

The approximate geometric structure developed here is particularly relevant to mathematical physics in settings where: the underlying configuration space is non-separable; the appropriate algebra of observables is non-separable or non- $\sigma$ -finite; local interactions are well-defined but global structure is too large to admit compactness.

Examples include: quantum field theories with infinitely many degrees of freedom; non-separable Hilbert spaces appearing in algebraic quantum field theory; large-scale quantum systems, spin networks, and models on non-countable graphs; kinematical Hilbert spaces in loop quantum gravity;  $C^*$ -algebras of quasi-local observables in statistical mechanics.

In these contexts, our approximate spectral triples offer a natural analytic framework in which curvature, dimension, and distance can still be defined, despite the lack of separability.

### 7.5. Outlook and Future Directions

Several directions for further research emerge naturally from the present work.

- i. Approximate pseudodifferential calculus. A key question is whether one can develop an approximate version of Connes–Moscovici pseudodifferential operators on non-separable settings, where the order structure is determined locally. This may lead to an approximate local index formula.
- ii. Index theory and approximate Chern characters. Using the approximate analytic K-homology groups, one may attempt to define approximate versions of: the Connes–Chern character; local cyclic cocycles; approximate heat-kernel expansions; index pairings with approximate K-theory. This connects naturally with existing work on measurable operators and traces [25].
- iii. Approximate KK-theory. Since KK-theory is central to noncommutative geometry, extending it to a non-separable/approximate

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context would yield significant new tools.

The approximate Fredholm modules introduced here may serve as cycles for such a theory.

iv. Approximate quantum Gromov–Hausdorff convergence. The approximate Connes metric suggests a natural extension of Latrémolière’s propinquity to non-separable quantum spaces. This would allow convergence of families of approximate spectral triples, useful in mathematical physics [20,21].

v. Model-theoretic and Banach-space-theoretic connections. Since we rely on families of separable subspaces, connections with stability theory in model theory or with ultrapowers of  $C^*$ -algebras may appear naturally.

In conclusion, the theory of approximate spectral triples and approximate noncommutative geometry developed here demonstrates that: non-separable Banach spaces do admit a coherent geometric structure; this structure is governed entirely by the behaviour on separable local components; and the resulting invariants behave analogously to those of classical spectral geometry, modulo global suprema.

The framework is therefore robust, extensible, and potentially applicable across large classes of models in functional analysis, operator algebras, and mathematical physics.

## 8. Conclusion

The purpose of this article has been to introduce and develop a comprehensive framework for *approximate noncommutative geometry* on non-separable Banach spaces. By constructing approximate spectral triples from directed systems of classical separable spectral triples, we have shown that many of the central geometric structures of Connes’ noncommutative geometry—spectral dimension, curvature, Lipschitz seminorms, quantum metrics, and  $K$ -homology—extend meaningfully to a setting in which separability, compactness, and countability are no longer available [1].

The key insight is that all geometric and analytic behaviour in a non-separable space is fundamentally local: every geometric quantity must be understood through its behaviour on separable subspaces. This led us to introduce the approximate Dirac operator  $D^{\text{approx}}$ , the approximate Lipschitz seminorm  $L^{\text{approx}}$ , approximate curvature functionals, and the approximate Connes distance. Each of these objects is defined as a supremum of local invariants arising from the family of local spectral triples  $(\mathfrak{A}_F, H_F, D_F)$ . The approximate bounded transform  $F^{\text{approx}}$  then provides a canonical approximate Fredholm module whose  $K$ -homology class encodes the global geometry.

A central result of this work is the projective decomposition theorem (theorem 5.4.1), which shows that the approximate analytic  $K$ -homology group

$$K_*^{\text{approx}}(\mathfrak{A}^{\text{approx}})$$

is canonically isomorphic to the projective limit of the classical  $K$ -homology groups  $K_*(\mathfrak{A}_F)$  of the separable local algebras. In this sense, the approximate spectral triple behaves as a genuine geometric cycle: locally classical, globally non-separable, and entirely determined by its behaviour on finite-dimensional or separable “patches”.

In section 6 we demonstrated that the approximate spectral triple carries rich geometric information: spectral dimension, Dixmier-type curvature, quantum metric structure, and stability under perturbations all extend in a natural and internally consistent manner. These approximate invariants behave as expected in each separable local component and assemble into global invariants that reflect the supremal geometric complexity of the non-separable setting.

This work therefore establishes a new geometric paradigm for non-separable noncommutative spaces: approximate geometry, in which all global data are encoded by local spectral behaviour and assembled through inductive and projective limits. The resulting picture suggests that the geometry of non-separable operator algebras and Banach spaces can be developed fruitfully by extending the classical geometric formalism rather than abandoning it.

Several directions for future investigation are now open, including the development of an approximate pseudodifferential calculus, the study of approximate Chern characters and index pairings, an approximate version of  $KK$ -theory, and potential applications in mathematical physics, quantum field theory, and quantum gravity. These problems appear well-suited for the approximate geometric approach introduced here and promise a deeper understanding of the analytic and geometric structure of non-separable spaces.

The framework presented in this article therefore sets the foundations for a systematic theory of non-separable noncommutative geometry, bridging local separable analysis with global non-separable behaviour and opening the door to further advances in spectral analysis, operator theory, and mathematical physics.

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