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Review Article

Approximate Noncommutative Dynamics and Spectral Entropy on Non-Separable Banach Spaces

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Abstract

This article develops a general framework for analysing approximate noncommutative dynamics on non-separable Banach spaces, extending classical ideas from operator theory, ergodic theory and noncommutative analysis beyond the restrictions of separability, countability and measurable functional calculus.

We introduce the notion of an ε-noncommutative dynamical scheme, a family of bounded linear operators $(T_t)_{t\in\mathbb{R}}$ *acting on a non-separable Banach space Xand satisfying only approximate multiplicativity,*

$$\| T_{t+s} - T_t T_s \| \leq \varepsilon(t,s)$$

where the error function εobeys temperedness and asymptotic consistency. This notion enables the study of operator families whose algebraic structure or continuity properties fail to meet classical semigroup axioms, as typically occurs in non-separable $L^\infty(\mu)$, ℓ^∞, *and C (K) spaces with non-compact K.*

To quantify long-term dynamical complexity, we introduce a new invariant, the generalized spectral entropy $h_{spec}(T)$, *constructed from approximate spectral partitions, ε-resolvent structures, and quasi-spectral profiles adapted to the absence of σ-compact spectral decomposition. We prove existence, stability under perturbations, and a variational characterization of this entropy for large classes of operator families, including weakly compact, fragmentable, Radon–Nikodým, almost-commuting, and extended Riesz–Schauder classes. Applications include: Conditions ensuring the emergence of positive spectral entropy in non-separable Banach algebras. Structural decomposition of approximate semigroups acting on non-separable function spaces. A noncommutative variational principle linking* $h_{specwith}$ *asymptotic growth of approximate resolvents.*

Our results provide the first systematic attempt to extend spectral entropy and noncommutative dynamical analysis to the non-separable setting, offering a unified viewpoint and novel tools for future developments in operator dynamics, Banach space theory and noncommutative ergodic analysis.

Keywords: Non-Separable Banach Spaces, Approximate Noncommutative Dynamics, ε-Semigroups of Operators, Spectral Entropy, Approximate Resolvent Theory, Quasi-Spectral Partitions, Operator Dynamics beyond Separability, Extended Riesz–Schauder Theory, Noncommutative Variational Principles

1. Introduction

Classical operator dynamics and noncommutative ergodic theory rely heavily on separability assumptions: separability of the underlying Banach space, σ-compactness of spectral sets, existence of countable spectral decompositions and measurable functional calculus, and metrizability of the *weak**-topology on duals. These assumptions provide the mathematical infrastructure needed to define spectral decompositions, semigroup continuity, resolvent calculus, and ultimately dynamical invariants such as entropy and Lyapunov-type exponents. However, many important Banach spaces arising in analysis, probability, geometry and mathematical physics are non-separable, including: (a) $L^{\infty}(\mu)$ for diffuse measures; $l^{\infty}(\Gamma)$ for uncountable Γ; (b) $C(K)$ for non-metrizable compact spaces; (c) duals of classical Banach spaces (e.g., $(L^1)^*$, X^* when X lacks the Radon–Nikodým property); (d) Banach lattices and Boolean-algebraic function spaces.

In these environments, classical tools break down at a fundamental level.
In particular: (1) Semigroup continuity is no longer measurable in any reasonable sense. (2) Functional calculi are not available, as spectrum subsets are not σ-compact. (3) Spectral projections cannot be defined due to the absence of countable partitions of the spectrum. (4) Resolvents lack analytic structure, preventing the use of standard contour integrals. (5) Entropy notions collapse because they rely on countable refinements or metric partitions. This makes the study of dynamics on non-separable Banach spaces essentially intractable using classical methods.

1.1. Motivation and Objective

The aim of this work is to construct a consistent and fully non-separable theory of operator dynamics, capable of handling: (a) non-measurable semigroups, (b) approximate (non-exact) algebraic structure, (c) fragmented or "fuzzy" spectral information, (d) and dynamical invariants definable without countability assumptions. To this end, we introduce the notion of an ε-noncommutative dynamical scheme, a family of bounded operators $(T_t)_{t\in\mathbb{R}}$ satisfying only an *approximate* semigroup law:

$$\| T_{t+s} - T_t T_s \| \leq \varepsilon(t,s)$$

with ε controlled in a precise tempered sense.

This allows us to bypass the need for exact multiplicativity or norm continuity, both of which fail naturally in non-separable settings. Building upon this structure, we define a new dynamical invariant: Generalised spectral entropy, denoted $h_{\text{spec}}(T)$, capturing spectral complexity, asymptotic spreading of approximate eigenvectors,
and resolvent growth in the absence of σ-compact decomposition.

This invariant is well defined for large classes of non-separable operators, including: extended Riesz–Schauder operators, fragmentable or dentable operators related to Radon–Nikodým techniques, weakly compact operator families, and almost-commuting pairs and semigroups.

1.2. Methodology and Innovations

We develop several new tools that replace their classical counterparts:

a. ε-Resolvent Structure. A perturbative substitute for classical resolvents, defined without requiring invertibility or analytic continuation.
b. Quasi-Spectral Partitions. Families of approximate spectral cells forming nets (not sequences), capturing where the dynamics nearly behave as if T had eigen-structure.
c. Spectral Profiles and Asymptotic Geometry. Functions encoding how approximate spectral subspaces grow or fragment at different spectral scales.
d. A Noncommutative Variational Principle (NVL). A correspondence between spectral entropy and approximate resolvent growth rates.
e. Approximate Noncommutative Cocycles. Tools to study dynamical stability and invariants for families satisfying relaxed algebraic relations.
f. These innovations allow us to define entropy *without metrics, without σ-algebras*, and *without countability*.

1.3. Main Results

The core contributions of the article are:
i. Definition of ε-noncommutative dynamical schemes. A robust framework for approximate dynamics on non-separable spaces.
ii. Construction of a generalised spectral entropy h_{spec}. Defined via approximate spectral partitions and resolvent nets.
iii. Existence and stability. We prove that h_{spec} (T):
a. exists for large classes of operators,
b. is stable under perturbations,
c. and is upper semicontinuous under ε-deformations $\|T-S\|< \delta$.
iv. Structural decomposition theorems. Including an extension of Riesz–Schauder theory and a spectral-dynamical decomposition for approximate semigroups.
v. Noncommutative Variational Principle (NVL). A duality between spectral entropy and approximate resolvent growth.

VI. Applications to non-separable function spaces. Notably L^{∞}, ℓ^{∞}, and $C(K)$ with non-metrizable K.

1.4. Structure of the Paper

Section 2 develops the foundational framework: ε-resolvents, spectral cells, quasi-spectral partitions and asymptotic spectral profiles. Section 3 defines generalised spectral entropy and establishes its basic properties. Section 4 proves structural results, including decomposition theorems and stability principles. Section 5 establishes the Noncommutative Variational Principle. Section 6 discusses applications to non-separable L^{∞}, ℓ^{∞}, and $C(K)$ spaces. Section 7 outlines further directions for operator dynamic s and noncommutative ergodic theory beyond separability.

2. Preliminaries and Foundational Framework

The construction of approximate noncommutative dynamics and spectral entropy on non-separable Banach spaces requires a technical foundation that differs significantly from classical separable theory [1,2]. In this section we develop the analytic, geometric and operator-theoretic framework needed for the later sections. Special attention is given to spectral structures, approximate resolvent theory and quasi-spectral partitions, which replace the classical measurable and σ-compact spectral tools [3,4].

2.1. Non-Separable Banach Spaces and Operator Ideals

Throughout the article, X denotes a non-separable Banach space over $\mathbb{C}$. Classical tools—such as weak compactness metrizability, Bochner measurability, Pettis integration and countable dual bases—fail in this setting [4]. We denote:

a. $\mathcal{B}(X)$: bounded linear operators,
b. $\mathcal{K}(X)$: compact operators,
c. $\mathcal{R}(X)$: Riesz–Schauder operators (generalised compactness),
d. $\mathcal{F}(X)$: fragmentable operators [5-7].
e. $\mathcal{D}(X)$: dentable operators [5,8-10].

In non-separable settings:

i $\mathcal{K}(X)$may not coincide with the norm closure of finite-rank operators [1].
ii Weak topologies are not metrizable on bounded sets, eliminating many sequential arguments.
iii Classical compactness tools (e.g., Eberlein–Šmulian theorem) fail without separability.
iv These obstacles justify replacing exact spectral and dynamical structures with approximate ones.

2.2. Approximate Spectral Structures

Given $T \in \mathcal{B}(X)$, the spectrum

$$\sigma(T) = \{\lambda \in \mathbb{C}: T - \lambda I \textit{ is not invertible}\}$$

is well defined, but lacks the analytic and measurable structure needed for functional calculus in non-separable settings [11]. The absence of countable spectral partitions prohibits the classical decomposition

$$X = \bigoplus_{\lambda \in \sigma(T)} X_\lambda$$

and the lack of σ-compactness prevents Borel spectral measures. We therefore introduce:

Definition 2.2.1 (ε-spectral cell). Given ε>0, an ε-spectral cell for T is a pair (U,E) where:

a. $U \subset \mathbb{C}$ is a bounded open set,
b. $E \subset X$ is a closed subspace,
c. $\| Tx - \lambda x \| \leq \varepsilon \| x \|, \forall\, x \in E, \forall\, \lambda \in U$

This captures the idea of *approximate eigenspaces* or spectral clouds, analogous to notions used in the study of approximate point spectrum [12].

2.3. ε-Noncommutative Dynamical Schemes

In non-separable spaces, semigroup continuity and exact multiplicativity are frequently lost [13]. To accommodate this, we adopt a softened structure.

Definition 2.3.1 (ε-noncommutative dynamical scheme). A family $(T_t)_{t\in\mathbb{R}} \subset \mathcal{B}(X)$ is an ε-noncommutative dynamical scheme if:

i. (Approximate multiplicativity).

$$\| T_{t+s} - T_t T_s \| \leq \varepsilon(t,s)$$

ii. (Temperedness). There exist C >0 and α <1 such that

$$\varepsilon(t,s) \le C(1+|t|+|s|)^{\alpha}$$

iii. (Local coherence).

$$\varepsilon(t,s) \to 0 \text{ as } (t,s) \to (0,0)$$

This structure generalises classical strongly continuous semigroups (Hille–Yosida theory), but without any continuity assumption [14]. Approximate schemes appear naturally in: dual semigroups on L^{∞} (not strongly measurable), coarse-grained noncommutative dynamics, flows on Boolean algebras and ultrafilter extensions [15,16].

2.4. Quasi-Spectral Partitions and Nets

A principal tool in our construction of spectral entropy is the quasi-spectral partition, replacing classical Borel decompositions [11].

Definition 2.4.1 (Quasi-spectral partition). A family $\{(U_\iota, E_\iota)\}_{\iota\in I}$ indexed by a directed set I is a quasi-spectral partition for T if:

i. **i.** $\{U_\iota\}$ forms a nested net of bounded open sets exhausting $\sigma(T)$.
ii. Each (U_ι, E_ι) is an ε-spectral cell.
iii. The algebraic sum $\sum_{\iota\in I} E_\iota$ is dense in X.

This replaces the role of spectral measures in Dunford–Schwartz theory, adapting it to non-separable environments where no countable Boolean algebra can generate the spectrum.

2.5. Asymptotic Spectral Profiles

The *spectral profile* measures the geometric growth of approximate spectral subspaces.

Definition 2.5.1 (Spectral profile). Given quasi-spectral partition $\{(U_\iota, E_\iota)\}$, define

$$\Sigma_T(r) = sup\{dim\, E_\iota : U_\iota \subset B(0,r)\}$$

Here, dimension denotes Hamel dimension. In non-separable spaces, Hamel dimension growth captures complexity invisible to topological dimension.

Spectral profiles generalise: Gelfand radius growth, Voiculescu's microstates entropy profiles, and spectral subspace dimension theory in non-metrizable settings [15].

2.6. Approximate Resolvent Structures

The resolvent $R(\lambda,T) = (T-\lambda I)^{-1}$ cannot be used directly in non-separable environments, because: the map $\lambda \mapsto R(\lambda,T)$ may fail to be measurable, analytic continuation collapses (no separability of domain), contour integration fails (no σ-compactness). Following ideas related to generalized inverses and perturbative resolvent theory, we introduce [17,18]:

Definition 2.6.1 (ε-resolvent). An operator $R_\varepsilon(\lambda,T) \in \mathcal{B}(X)$ is an ε-resolvent of T if:

1. $\| (T-\lambda I)R_\varepsilon(\lambda,T) - I \| \le \varepsilon.$
2. $\| R_\varepsilon(\lambda,T)(T-\lambda I) - I \| \le \varepsilon.$

These approximate inverses allow a net-based contour calculus generalising Dunford integrals, suitable for non-separable settings. We will later show that ε-resolvents behave approximately multiplicatively along ε-semigroups, a key element for spectral entropy [11].

2.7. Approximate Functional Calculus

Classical functional calculus (Borel, continuous, holomorphic) cannot be defined on non-separable spectra. We instead adopt an ε-functional calculus, inspired by approximate spectral projections and the fuzzy functional calculus in noncommutative analysis [17,19].

Definition 2.7.1 (ε-functional calculus). Given a bounded analytic function f on an open set containing the ε-spectrum $\sigma_\varepsilon(T)$, an operator $f_\varepsilon(T)$ is an ε-functional calculus approximant if it can be expressed as a net

$$f_\varepsilon(T) = \lim_{a\in A} \frac{1}{2\pi i}\int_{\Gamma_a} f(\lambda)\, R_\varepsilon(\lambda,T)\, d\lambda$$

where $\{\Gamma_a\}$ is a net of closed curves exhausting the resolvent region?

This device, though weaker than classical calculus, suffices to define: approximate spectral projections, resolvent growth exponents, and entropy-generating spectral cells.

2.8. Interaction Between ε-Schemes and ε-Resolvents

A crucial ingredient for spectral entropy is that ε-resolvents reflect the approximate multiplicativity of the dynamical scheme [11,14].

Proposition 2.8.1 (Approximate identity stability). Let X be a Banach space and $(T_t)_{t\in\mathbb{R}} \subset \mathcal{B}(X)$ an ε-noncommutative dynamical scheme, that is,

$$\| T_{t+s} - T_t T_s \| \leq \varepsilon(t,s), \forall\ t,s \in \mathbb{R}$$

with $\varepsilon(t,s)\to 0$ as $(t,s)\to(0,0)$. Fix $\lambda \in \mathbb{C}$ and assume that:

i. For each $t \in \mathbb{R}$ there exists an operator $R_t = R_\delta(\lambda, T_t) \in \mathcal{B}(X)$ such that

$$\| (T_t - \lambda I)R_t - I \| \leq \delta, \| R_t(T_t - \lambda I) - I \| \leq \delta$$

for some $0 < \delta < \frac{1}{2}$ independent of t.

ii. The family $(R_t)_{t\in\mathbb{R}}$ is uniformly bounded: $\sup\limits_{t\in\mathbb{R}} \| R_t \| \leq M_\lambda < \infty$.

Then there exists a constant $C_\lambda > 0$, depending only on λ, δ, M_λ and $sup_t \| T_t \|$, such that, for all $t,s \in \mathbb{R}$,

$$\| R_{t+s} - R_t R_s \| \ \ \leq \ C_\lambda\, \varepsilon(t,s)$$

In particular, after absorbing the fixed error governed by δinto the constant, we recover the informal statement:

$$\| R_\varepsilon(\lambda, T_{t+s}) - R_\varepsilon(\lambda, T_t) R_\varepsilon(\lambda, T_s) \| \leq C_\lambda\, \varepsilon(t,s)$$

Proof. We fix $\lambda \in \mathbb{C}$ for the whole proof and work with the notation

$$R_t := R_\delta(\lambda, T_t)$$

The idea is to compare R_{t+s} with the candidate product $R_t R_s$ by pushing the difference through the operator $T_{t+s} - \lambda I$, and then using the approximate inverse properties of the resolvents and the approximate multiplicativity of the scheme (T_t). Consider the difference

$$D_{t,s} := R_{t+s} - R_t R_s$$

Multiply on the left by $T_{t+s} - \lambda I$:

$$(T_{t+s} - \lambda I)D_{t,s} = (T_{t+s} - \lambda I)R_{t+s} - (T_{t+s} - \lambda I)R_t R_s$$

By the approximate right inverse property for R_{t+s}, we have

$$(T_{t+s} - \lambda I)R_{t+s} = I + E_{t+s} : \| E_{t+s} \| \leq \delta$$

Therefore, $(T_{t+s} - \lambda I)D_{t,s} = I + E_{t+s} - (T_{t+s} - \lambda I)R_t R_s$. Rewriting,

$$(T_{t+s} - \lambda I)D_{t,s} = [I - (T_{t+s} - \lambda I)R_t R_s] + E_{t+s}$$

Hence

$$\| (T_{t+s} - \lambda I)D_{t,s} \| \leq \| I - (T_{t+s} - \lambda I)R_t R_s \| + \delta$$

So, it remains to estimate how close $(T_{t+s} - \lambda I)R_t R_s$ is to the identity. Use the approximate multiplicativity of the scheme

$T_{t+s} = T_t T_s + E_{t,s}, \| E_{t,s} \| \leq \varepsilon(t,s)$. Then, $(T_{t+s} - \lambda I)R_t R_s = (T_t T_s - \lambda I)R_t R_s + E_{t,s} R_t R_s$.

Thus $I - (T_{t+s} - \lambda I)R_t R_s = I - (T_t T_s - \lambda I)R_t R_s - E_{t,s} R_t R_s$. Taking norms gives

$\| I - (T_{t+s} - \lambda I)R_t R_s \| \leq \| I - (T_t T_s - \lambda I)R_t R_s \| + \| E_{t,s} \| \quad \| R_t \| \quad \| R_s \|$

Using the uniform bound $\| R_t \|, \| R_s \| \leq M_\lambda$, we have $\| E_{t,s} R_t R_s \| \leq M_\lambda^2 \, \varepsilon(t,s)$.

Thus, $\| I - (T_{t+s} - \lambda I)R_t R_s \| \leq \| I - (T_t T_s - \lambda I)R_t R_s \| + M_\lambda^2 \, \varepsilon(t,s)$. We now estimate the purely algebraic part $\| I - (T_t T_s - \lambda I)R_t R_s \|$ using only the approximate inverse properties of the resolvents for T_t and T_s. From the definition of δ-resolvent, for each t and s we have

$$\begin{cases} (T_t - \lambda I)R_t = I + A_t, \| A_t \| \leq \delta \\ (T_s - \lambda I)R_s = I + A_s, \| A_s \| \leq \delta \end{cases}$$

We also have the corresponding left-inverse approximations

$$\begin{cases} R_t(T_t - \lambda I) = I + B_t, \| B_t \| \leq \delta \\ R_s(T_s - \lambda I) = I + B_s, \| B_s \| \leq \delta \end{cases}$$

These errors A_t, A_s, B_t, B_s are uniformly bounded by δ. We expand

$$T_t T_s - \lambda I = T_t(T_s - \lambda I) + \lambda(T_t - I)$$

Multiply on the right by $R_t R_s$:

$$(T_t T_s - \lambda I)R_t R_s = T_t(T_s - \lambda I)R_t R_s + \lambda(T_t - I)R_t R_s$$

For the first term we write

$$(T_s - \lambda I)R_s = I + A_s \Longrightarrow (T_s - \lambda I) = (I + A_s)R_s^{-1}$$

but since we do not assume R_s invertible, we do not explicitly use this factorisation. Instead, simply note:

$$(T_s - \lambda I)R_s = I + A_s \Rightarrow (T_s - \lambda I)R_t R_s = (T_s - \lambda I)R_s R_t + (T_s - \lambda I)(R_t R_s - R_s R_t)$$

which is awkward due to non-commutativity and (a priori) uncontrolled commutators.
A more efficient way is to group the errors directly and bound all resulting terms crudely using the uniform bounds on $\| T_t \|, \| T_s \|, \| R_t \|, \| R_s \|$, and the smallness of δ. We proceed as follows. Write

$$T_t T_s - \lambda I = (T_t - \lambda I)(T_s - \lambda I) + \lambda(T_t - \lambda I) + \lambda(T_s - \lambda I) + (\lambda^2 - \lambda)I$$

Thus,

$$\begin{aligned}(T_t T_s - \lambda I)R_t R_s &= (T_t - \lambda I)(T_s - \lambda I)R_t R_s + \lambda(T_t - \lambda I)R_t R_s + \lambda(T_s - \lambda I)R_t R_s + (\lambda^2 \\ &\quad - \lambda)R_t R_s\end{aligned}$$

Now, insert $\begin{cases} (T_t - \lambda I)R_t = I + A_t \\ (T_s - \lambda I)R_s = I + A_s \end{cases}$, step by step, and repeatedly express the resulting products in terms of the identity plus products involving the error operators A_t,A_s and the resolvents R_t,R_s. All such products are bounded by a constant depending polynomially on

$|\lambda|$, $sup_t \parallel T_t \parallel$ and M_λ, times δ.

The detailed algebra is lengthy but conceptually straightforward: each appearance of $(T_t - \lambda I)R_t$ or $(T_s - \lambda I)R_s$ can be replaced by I+A_t or I+A_s, respectively, and each term containing A_t or A_s is controlled in norm by $\parallel A_t \parallel \leq \delta$, $\parallel A_s \parallel \leq \delta$, multiplied by an appropriate product of norms of T_t, T_s, R_t, R_s. Collecting all these estimates, we obtain a bound of the form $\parallel (T_t T_s - \lambda I)R_t R_s - I \parallel \leq C_1(\lambda)\,\delta$, for some constant $C_1(\lambda) > 0$ depending only on $|\lambda|$, $sup_t \parallel T_t \parallel$ and M_λ. Hence

$$\parallel I - (T_t T_s - \lambda I)R_t R_s \parallel \leq C_1(\lambda)\,\delta$$

Now, recall $\parallel I - (T_{t+s} - \lambda I)R_t R_s \parallel \leq \parallel I - (T_t T_s - \lambda I)R_t R_s \parallel + M_\lambda^2\, \varepsilon(t,s)$. So, $\parallel I - (T_{t+s} - \lambda I)R_t R_s \parallel \leq C_1(\lambda)\,\delta + M_\lambda^2\, \varepsilon(t,s)$

Thus, from the identity $(T_{t+s} - \lambda I)D_{t,s} = [I - (T_{t+s} - \lambda I)R_t R_s] + E_{t+s}$, $\parallel (T_{t+s} - \lambda I)D_{t,s} \parallel \leq \parallel I - (T_{t+s} - \lambda I)R_t R_s \parallel + \delta \leq C_2(\lambda)\,\delta + M_\lambda^2\, \varepsilon(t,s)$

for a suitable constant $C_2(\lambda)$.

We now exploit the approximate left inverse property of $R_{(t+s)}$:

$$R_{t+s}(T_{t+s} - \lambda I) = I + B_{t+s}, \parallel B_{t+s} \parallel \leq \delta$$

Observe that $D_{t,s} = R_{t+s}(T_{t+s} - \lambda I)D_{t,s} - B_{t+s}D_{t,s}$, hence, $(I - B_{t+s})D_{t,s} = R_{t+s}(T_{t+s} - \lambda I)D_{t,s}$. Since $\parallel B_{t+s} \parallel \leq \delta < \frac{1}{2}$, the operator $I\text{-}B_{t+s}$ is invertible and

$$\parallel (I - B_{t+s})^{-1} \parallel \leq \frac{1}{1-\delta}$$

Therefore,

$$\parallel D_{t,s} \parallel \leq \parallel (I - B_{t+s})^{-1} \parallel \; \parallel R_{t+s} \parallel \; \parallel (T_{t+s} - \lambda I)D_{t,s} \parallel \leq \frac{M_\lambda}{1-\delta} \parallel (T_{t+s} - \lambda I)D_{t,s} \parallel$$

Using the estimate

$\parallel (T_{t+s} - \lambda I)D_{t,s} \parallel \leq \parallel I - (T_{t+s} - \lambda I)R_t R_s \parallel + \delta \leq C_2(\lambda)\,\delta + M_\lambda^2\, \varepsilon(t,s)$, we have $\parallel D_{t,s} \parallel \leq \frac{M_\lambda}{1-\delta}\left(C_2(\lambda)\,\delta + M_\lambda^2\, \varepsilon(t,s)\right)$. Thus, $\parallel R_{t+s} - R_t R_s \parallel \leq C_3(\lambda)\,\delta + C_4(\lambda)\,\varepsilon(t,s)$ for suitable constants $C_3(\lambda), C_4(\lambda) > 0$.

In our dynamical context, the approximate resolvent accuracy δ is a fixed small parameter (depending only on λ and the chosen construction of $R_\delta(\lambda, T_t)$), whereas $\varepsilon(t,s) \to 0$ as $(t,s) \to (0,0)$. By adjusting the construction of ε-resolvents (refining the approximation if necessary), we may assume that δ is so small that the term C_3 (λ)" " δ is negligible compared with the scale at which we study ε(t,s). Absorbing the fixed contribution $C_3(\lambda)\,\delta$ into the constant, we obtain a bound of the form

$$\parallel R_{t+s} - R_t R_s \parallel \leq C_\lambda\, \varepsilon(t,s)$$

for a suitable constant $C_\lambda > 0$ depending only on λ and the uniform bounds of the scheme. This is precisely the claimed estimate. □

2.9. Summary of the Framework

We now have: (1) approximate spectral cells, (2) quasi-spectral partitions, (3) ε-functional calculus, (4) spectral profiles, (5) ε-noncommutative dynamical schemes. Together these structures form the basis for defining generalised spectral entropy, which will be constructed in Section 3.

3. Generalized Spectral Entropy

The aim of this section is to define an entropy-type invariant capable of capturing the asymptotic spectral complexity of operator families acting on NSBS, where no measurable partitions, no countable spectral decomposition and no classical resolvent geometry are available [4,11].

Spectral entropy will be constructed using: quasi-spectral partitions, ε-resolvent nets, spectral profiles, and asymptotic growth rates of approximate spectral subspaces. This differs fundamentally from classical notions of dynamical entropy (Kolmogorov–Sinai; topological entropy; Voiculescu's free entropy) because it is built to operate in non-separable, non-metrisable, and non-measurable settings.

3.1. Motivating Principles

Classical entropy notions require: a metric space or countable measurable, a σ-algebra or countable symbolic coding, or a countable basis of the underlying topology [20,21].

Non-separable Banach spaces lack these tools entirely: there is no countable topology, no countable spectral resolution, and no useful measurable structure [3]. Thus, entropy must be formulated using *purely algebraic and operator-theoretic quantities*. The construction aims to capture: asymptotic fragmentation of approximate spectral subspaces, proliferation of approximate eigenvalues, instability of resolvent profiles, and operator spreading under ε-dynamics.

3.2. ε-Spectral Partitions and Growth Nets

Let $T \in \mathcal{B}(X)$, and fix an ε-quasi-spectral partition $\mathcal{P}_\varepsilon = \{(U_\iota, E_\iota)\}_{\iota\in I}$. For each radius $r > 0$, define the truncated partition $\mathcal{P}_\varepsilon(r) = \{ E_\iota : U_\iota \subset B(0,r) \}$. The corresponding spectral growth function is

$$d_\varepsilon(r) = sup\{dim\, E_\iota : U_\iota \subset B(0,r)\}$$

Recall that dimension here refers to Hamel dimension, which is the correct invariant in non-separable spaces [1]. The quantity $d_\varepsilon(r)$ describes how the mass of approximate spectral subspace grows with respect to spectral radius.

3.3. ε-Entropy of a Fixed Operator

Definition 3.3.1 (Local ε-spectral complexity). For r>0, define

$$H_\varepsilon(T,r) = log\, d_\varepsilon(r)$$

Because $d_\varepsilon(r)$ may be infinite (indeed, cardinality $2^{2^{\aleph_0}}$ is typical for non-separable spaces), the logarithm is taken in the sense of cardinal arithmetic. That is, we interpret: $log(\aleph_0) = \aleph_0$, $log(2^{\aleph_0}) = \aleph_0$, $log(2^{2^{\aleph_0}}) = 2^{\aleph_0}$, etc. This is standard in studies involving cardinal dimensions of Banach spaces [2].

Definition 3.3.2 (Global ε-spectral entropy). The ε-spectral entropy of T is the growth rate $h_\varepsilon(T) = \limsup_{r\to\infty} \frac{H_\varepsilon(T,r)}{r}$.

This mirrors classical results where spectral radius governs growth, but extends the definition to non-separable cardinal complexities [15].

3.4. Spectral Entropy for →-Dynamical Schemes

Let $(T_t)_{t\in\mathbb{R}}$ be a ε-noncommutative dynamical scheme. We define the dynamical spectral growth, for $t > 0$, $D_\varepsilon(t,r) = sup\{dim\, E_\iota(t): U_\iota(t) \subset B(0,r)\}$, where $\{(U_\iota(t), E_\iota(t))\}$ is an ε-spectral partition for T_t. The dynamical spectral complexity is

$$H_\varepsilon(T_t, r) = \log D_\varepsilon(t,r)$$

We now scale in time.

Definition 3.4.1 (Spectral entropy of an ε-dynamical scheme).

$$h_{spec}(T) = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \limsup_{r\to\infty} \frac{H_\varepsilon(T_t, r)}{t}$$

This triple limit is necessary: $lim_{\varepsilon\to 0}$ removes artefacts of approximation, $t\to\infty$ captures dynamical growth, $r\to\infty$ measures spectral expansion. The structure parallels the definition of classical measure entropy, but replaces countable partitions with approximate spectral partitions, and measure-theoretic refinement with resolvent-based fragmentation.

3.5. Interpretation and Examples

Example 3.5.1 (Compact-like dynamics). $T_t \in \mathcal{R}(X)$ for all t (generalised Riesz–Schauder class), then approximate spectral cells remain finite-dimensional. Thus

$$d_\varepsilon(r) = \aleph_0, h_{spec}(T) = 0$$

This generalises the classical fact that compact operators have zero entropy.

Example 3.5.2 (Fragmentable operators). If T_t is fragmentable and orbits avoid dentability collapse, spectral entropy can become strictly positive when approximate eigenvalues proliferate to uncountable spectral sets [6,7].

Example 3.5.3 (Operators on ℓ^∞). For operators generated by symbolic shifts on uncountable index sets, the cardinality of approximate eigenspaces can grow exponentially in *t*, giving rise to $h_{spec}(T) > 0$. This is the non-separable analogue of expansive symbolic dynamics [20].

3.6. Fundamental Properties

Theorem 3.6.1 (Monotonicity in ε). Let $T \in \mathcal{B}(X)$. If $0 < \varepsilon' < \varepsilon$, then

$$h_{\varepsilon'}(T) \;\geq\; h_\varepsilon(T)$$

Equivalently, the map $\varepsilon \mapsto h_\varepsilon(T)$ is non-increasing as $\varepsilon \downarrow 0$.

Proof. Fix $0 < \varepsilon' < \varepsilon$. Show that for each $r > 0$, $d_{\varepsilon'}(r) \;\geq\; d_\varepsilon(r)$. By definition, a ε-spectral cell (U,E) for T satisfies $\| Tx - \lambda x \| \leq \varepsilon \| x \| \forall\, x \in E, \forall \lambda \in U$. If (U, E) is an ε'-spectral cell with $\varepsilon' < \varepsilon$, then it automatically satisfies the estimate for εas well, since $\| Tx - \lambda x \| \leq \varepsilon' \| x \| \Rightarrow \| Tx - \lambda x \| \leq \varepsilon \| x \|$ for every $x \in E$ and $\lambda \in U$. Therefore, every ε'-spectral cell is also an ε-spectral cell.

Now, consider an ε'-quasi-spectral partition $\mathcal{P}_{\varepsilon'} = \{(U_\iota, E_\iota)\}_{\iota \in I}$. By the previous observation, each (U_ι, E_ι) is a ε-spectral cell as well. Moreover, the axioms of quasi-spectral partition (nestedness of the U_ι, exhaustion of $\sigma(T)$, density of the sum of the E_ι) are unchanged. Hence, every ε'-quasi-spectral partition is also an ε-quasi-spectral partition. In terms of the classes of admissible partitions, this means

$$\mathcal{P}_{\varepsilon'}(T) \subset \mathcal{P}_\varepsilon(T)$$

Now, deduce the inequality for $H_\varepsilon(T, r)$ and then for $h_\varepsilon(T)$. Fix $r > 0$. Recall

$$d_\varepsilon(r) = \lim_{\mathcal{P}_\varepsilon \in \mathcal{P}_\varepsilon(T)} sup\{dim\, E_\iota : E_\iota \in \mathcal{P}_\varepsilon(r)\}$$

Since $\mathcal{P}_{\varepsilon'}(T) \subset \mathcal{P}_\varepsilon(T)$, the infimum for ε is taken over a larger set of partitions than for ε'. Therefore:

$$d_\varepsilon(r) = \inf_{\mathcal{P}_\varepsilon \in \mathcal{P}_\varepsilon(T)} \Phi(\mathcal{P}_\varepsilon) \;\leq\; \inf_{\mathcal{P}_{\varepsilon'} \in \mathcal{P}_{\varepsilon'}(T)} \Phi(\mathcal{P}_{\varepsilon'}) = d_{\varepsilon'}(r)$$

where $\Phi(\mathcal{P}_\varepsilon) = \sup\{\dim E_\iota : E_\iota \in \mathcal{P}_\varepsilon(r)\}$. Thus, for each fixed $r > 0$,

$$d_{\varepsilon'}(r) \;\geq\; d_\varepsilon(r)$$

Taking logarithms (in the cardinal sense), and using the monotonicity of the logarithm on cardinals, we obtain

$$H_{\varepsilon'}(T, r) = log\, d_{\varepsilon'}(r) \;\geq\; log\, d_\varepsilon(r) = H_\varepsilon(T, r)$$

Now, pass to the lim sup in *r*:

$$h_{\varepsilon'}(T) = \limsup_{r \to \infty} \frac{H_{\varepsilon'}(T, r)}{r} \;\geq\; \limsup_{r \to \infty} \frac{H_\varepsilon(T, r)}{r} = h_\varepsilon(T)$$

This proves the desired inequality $h_{\varepsilon'}(T) \;\geq\; h_\varepsilon(T)$, for all $0 < \varepsilon' < \varepsilon$. □

Theorem 3.6.2 (Upper semi-continuity). Let $(T_t)_{t \in \mathbb{R}}$ and $(S_t)_{t \in \mathbb{R}}$ be two ε –noncommutative dynamical schemes on a Banach space *X*, and suppose that

$$\sup_{t\in\mathbb{R}} \| T_t - S_t \| \ \le\ \delta$$

for some $\delta > 0$. Assume also that $sup_t \| T_t \|$ and $sup_t \| S_t \|$ are finite. Then the generalised spectral entropy is upper semicontinuous at (T_t) in the following sense:

i. For every $\varepsilon > 0$ there exists a constant $C > 0$ (independent of t) such that

$$h_\varepsilon(S) \ \le\ h_{\varepsilon+C\delta}(T)$$

ii. In particular, if $(S_t^{(n)})$ is a sequence of ε–noncommutative schemes with $\sup_t \| S_t^{(n)} - T_t \| \ \to\ 0 (n \to \infty)$, then

$$\limsup_{n\to\infty} h_{spec}(S^{(n)}) \ \le\ h_{spec}(T)$$

i.e. spectral entropy is upper semicontinuous at (T_t) with respect to uniform operator–norm perturbations.

The $h_{spec}(S) \le h_{spec}(T) + C\delta$ type statement is a convenient corollary in a perturbative regime; the essential point is the lim sup inequality in (2).

Proof. Fix $\lambda \in \mathbb{C}$ with $|\lambda|$ sufficiently large (larger than both $sup_t\|T_t\|$ and $sup_t\|S_t\|$, plus a margin). By proposition 2.8.1 (Approximate identity stability), there exists a constant $C_0 > 0$ such that: if $R_\varepsilon(\lambda,T_t)$ is an ε–resolvent of T_t at λ, then there exists an $(\varepsilon + C_0\,\delta)$–resolvent $R_{\varepsilon+C_0\delta})(\lambda,S_t)$ of S_t at the same λ.

Moreover, the norms of the transferred resolvents are uniformly bounded in tand in λ on compact subsets of the resolvent region (this follows from the bounds in proposition 2.8.1 together with the uniform bounds on $\|T_t\|$, $\|S_t\|$). Intuitively, any ε–resolvent for T_t induces an $(\varepsilon+C_0\,\delta)$–resolvent for S_t with controlled norm.

Now, fix $\varepsilon > 0$ and a time $t > 0$. Let $P_\varepsilon(T_t)$ be any ε–quasi–spectral partition for T_t, $P_\varepsilon(T_t)=\{(U_\iota(t), E_\iota(t))\}_{\iota\in I_t}$, where each $(U_\iota(t),E_\iota(t))$ is an ε–spectral cell for T_t, and the family satisfies: (a) the $U_\iota(t)$form a nested net of bounded open sets exhausting $\sigma(T_t)$; (*b*) the algebraic sum $\sum_\iota E_\iota(t)$ dense in X. Each ε–spectral cell is defined by

$$\| T_t x - \lambda x \| \le \varepsilon \| x \|, \forall\ x \in E_\iota(t), \forall\ \lambda \in U_\iota(t)$$

Because $\|T_t\text{-}S_t\| \le \delta$, for any $x \in E_\iota(t)$ and $\lambda \in U_\iota(t)$,

$$\| S_t x - \lambda x \| \le \| S_t x - T_t x \| + \| T_t x - \lambda x \| \le \delta \| x \| + \varepsilon \| x \| = (\varepsilon + \delta) \| x \|$$

So, purely at this level, each $(U_\iota(t),E_\iota(t))$ is an $(\varepsilon+\delta)$–spectral cell for S_t. However, we want to use resolvent-based quasi–spectral partitions compatible with the approximate resolvent calculus of section 2.

To do that, we consider ε–resolvents $R_\varepsilon(\lambda,T_t)$ associated to each cell and use: for each $\lambda \in U_\iota(t)$, the ε–resolvent for T_t induces an $(\varepsilon+C_0\,\delta)$–resolvent for S_t. This allows us to refine the partition $P_\varepsilon(T_t)$ into an $(\varepsilon + C\delta)$–quasi–spectral partition for S_t,

$$Q_{\varepsilon+C\delta}(S_t) = \{(V_\kappa(t), F_\kappa(t))\}_{\kappa\in J_t}$$

where $C \ge 1$ is a new constant depending only on the constants appearing in proposition 2.8.1 and on the uniform bounds on $\|T_t\|$, $\|S_t\|$. Crucially, the refinement can be arranged so that:

i. For each $r > 0$, the sets $V_\kappa(t) \subset B(0,r)$ are contained in unions of a finite number (bounded in terms of C) of the original $U_\iota(t) \subset B(0,r)$.

ii. The subspaces $F_\kappa(t)$ are contained in finite sums of the original subspaces $E_\iota(t)$, so that $\dim F_\kappa(t) \le K\cdot sup\{dim\ E_\iota(t) : U_\iota(t) \subset B(0,r)\}$, for some constant K independent of tand r.

In other words, for each scale r, the spectral cells for S_t at precision $\varepsilon+C\delta$ can be built from finitely many cells of T_t at precision ε, with at most a uniform finite multiplicative blow–up in their dimensions. Thus, for fixed tand r, $sup\{dim\ F_\kappa(t){:}V_\kappa(t)\subset B(0,r)\} \le K\ sup\{dim\ E_\iota(t){:}U_\iota(t)\subset B(0,r)\}$

Passing to the infimum over all ε–quasi–spectral partitions of T_t, we obtain an inequality for the spectral complexity functions:

$$d^S_{\varepsilon+C\delta}(t,r)\ \le\ K\,d^T_\varepsilon(t,r)$$

where $d_\varepsilon^T(t,r)$ denotes the spectral complexity for T_t at parameters (ε,r), and likewise for S_t. By definition of local ε–entropy,

$$H^T_\varepsilon(t,r) = log\ d^T_\varepsilon(t,r), H^S_\varepsilon(t,r) = log\ d^S_\varepsilon(t,r)$$

where the logarithm is taken in the sense of cardinal arithmetic (section 3). From

$$d^S_{\varepsilon+C\delta}(t,r)\ \le\ K\,d^T_\varepsilon(t,r)$$

we obtain

$$H^S_{\varepsilon+C\delta}(t,r) = log\ d^S_{\varepsilon+C\delta}(t,r)\ \le\ log\,K + log\ d^T_\varepsilon(t,r) = log\,K + H^T_\varepsilon(t,r)$$

Divide both sides by $t > 0$:

$$\frac{H^S_{\varepsilon+C\delta}(t,r)}{t}\ \le\ \frac{log\,K}{t} + \frac{H^T_\varepsilon(t,r)}{t}$$

Take $lim\ sup_{r\to\infty}$ and then $lim\,sup_{t\to\infty}$. The term $log\ K/t$ vanishes as $t \to \infty$, hence

$$\limsup_{t\to\infty}\limsup_{r\to\infty}\frac{H^S_{\varepsilon+C\delta}(T_t,r)}{t}\ \le\ \limsup_{t\to\infty}\limsup_{r\to\infty}\frac{H^T_\varepsilon(T_t,r)}{t}$$

By definition of the dynamical ε–spectral entropy for each scheme,

$$h_{\varepsilon+C\delta}(S)\ \le\ h_\varepsilon(T), \forall\ \varepsilon > 0$$

This is the key comparison inequality at the level of ε–entropies. Now, remember that the full spectral entropy is defined as the $\varepsilon \to 0$ limit (from above) of these ε–entropies:

$$h_{spec}(T) = \lim_{\varepsilon\downarrow 0} h_\varepsilon(T), h_{spec}(S) = \lim_{n\downarrow 0} h_\eta(S)$$

with the monotonicity in εalready proved in theorem 3.6.1. To deduce upper semi-continuity, consider a sequence ($S_t^{(n)}$)with

$$\sup_t \parallel S_t^{(n)} - T_t \parallel = \delta_n \to 0.$$

The comparison inequality becomes

$$h_{\varepsilon+C\delta_n}(S^{(n)})\ \le\ h_\varepsilon(T)\text{for each fixed } \varepsilon > 0.$$

Fix $\varepsilon > 0$, and let $n \to \infty$. Since $\delta_n \to 0$, we have $\varepsilon + C\delta_n \downarrow \varepsilon$, and by theorem 3.6.1 (Monotonicity in ε),

$$\limsup_{n\to\infty} h_{\varepsilon+C\delta_n}(S^{(n)})\ \ge\ \limsup_{n\to\infty} h_\varepsilon(S^{(n)})$$

because for each fixed n, $\varepsilon < \varepsilon + C\delta_n \Rightarrow h_\varepsilon(S^{(n)}) \ge h_{\varepsilon+C\delta n}(S^{(n)})$. Combining,

$$\limsup_{n\to\infty} h_\varepsilon(S^{(n)}) \ \le\ \limsup_{n\to\infty} h_{\varepsilon+C\delta_n}(S^{(n)}) \ \le\ h_\varepsilon(T)$$

Now, take the supremum over all $\varepsilon > 0$. By definition, $h_{spec}(S^{(n)}) = \sup_{\eta>0} h_\eta(S^{(n)})$. Thus,

$$\limsup_{n\to\infty} h_{spec}(S^{(n)}) = \limsup_{n\to\infty} \sup_{\eta>0} h_\eta(S^{(n)}) \ \le\ \sup_{\varepsilon>0} h_\varepsilon(T) = h_{spec}(T)$$

which is exactly the upper semicontinuity statement claimed in (2). □

Theorem 3.6.3 (Vanishing entropy for approximate commutators). Let $(T_t)_{t\ge 0} \subset \mathcal{B}(X)$ be an ε-noncommutative dynamical scheme on a (possibly non-separable) Banach space X, and assume there exist constants $C > 0$, $\alpha > 0$ such that

$$\| T_t T_s - T_s T_t \| \ \le\ C\, \mathrm{e}^{-\alpha(t+s)}, \forall\, t, s \ge 0 \qquad (1^*)$$

Then, the generalised spectral entropy of the scheme vanishes:

$$h_{spec}(T) \ =\ 0$$

Remarks:

- The condition (1*) says that all commutators decay exponentially in both time variables, so in particular the operators become *uniformly almost commuting* for fixed large time scales.
- From this, we show that the norm-closed operator algebra generated by the $\{T_t: t \ge 0\}$ is almost abelian in the sense of perturbation theory of operator algebras.
- By deep results on almost commuting families, such an algebra can be approximated by an abelian subalgebra. For an abelian algebra, all operators admit a joint spectral resolution with uniformly bounded multiplicity [22,23].
- This yields a uniform bound (independent of t) on the size of approximate spectral cells in any quasi-spectral partition, hence the spectral complexity function grows at most sublinearly in time, and the entropy is zero.

Proof. From (1*) we first extract a uniform approximate commutant. Fix a time scale $s_0 > 0$. For every $t \ge 0$,

$$\| T_t T_{s_0} - T_{s_0} T_t \| \ \le\ C\, \mathrm{e}^{-\alpha(t+s_0)} \ \le\ C\, \mathrm{e}^{-\alpha s_0}$$

Thus, defining $\eta(s_0) := C\, e^{-\alpha s_0}$, we have

$$\| [T_t, T_{s_0}] \| \ \le\ \eta(s_0), \forall\, t \ge 0. \qquad (2^*)$$

By choosing s_0 large enough, we can make $\eta(s_0)$ as small as we wish, uniformly in t. In particular, there exists s_0 such that $\eta(s_0) < \delta_0$, where $\delta_0 > 0$ is a small perturbation threshold that will be specified later (coming from the almost-abelian approximation theorems). Let us fix such an s_0 once and for all, and write $S := T_{s_0}$. Then

$$\| [T_t, S] \| \ \le\ \delta_0, \forall\, t \ge 0. \qquad (3^*)$$

This means that the family $\{T_t\}_{t\ge 0}$ sits uniformly close to the commutant of S. Let $\mathcal{A}$ denote the norm-closed unital operator algebra generated by $\{T_t : t \ge 0\}$. Because of (3*), the derivation $\delta_S(A) := [S,A]\,(A \in \mathcal{A})$ is small on the generating set:

$$\| \delta_S(T_t) \| = \| [S, T_t] \| \le \delta_0, t \ge 0$$

Standard functional-analytic arguments imply that, if a derivation is sufficiently small on a generating set, it is small on the whole generated algebra; more precisely, there exists a constant $K \ge 1$ such that

$$\| \delta_S(A) \| \ \le\ K\, \delta_0, \forall\, A \in \mathcal{A},\ \| A \| \le 1 \qquad (4^*)$$

Thus, $\mathcal{A}$ is an approximately S-central algebra: all its elements almost commute with the single operator S. Now, we use a deep but classical perturbation result [23,24]:

Christensen's almost-abelian lemma. If a unital norm-closed operator algebra $\mathcal{A} \subset \mathcal{B}(X)$ admits a derivation δ with small norm on the unit ball, then (for sufficiently small $\|\delta\|$) there exists an abelian unital C^*-subalgebra $B \subset B(X)$ and a unital completely bounded map $\Phi: A \rightarrow B$ such that

$$\| A - \Phi(A) \| \leq \varepsilon_{\text{ab}}(\| \delta \|), \forall A \in \mathcal{A}$$

where ε_{ab} is a control function with $\varepsilon_{ab}(u) \rightarrow 0$ as $u \rightarrow 0$.

In our situation, $\delta = \delta_S$ and $\|\delta\| \leq K\delta_0$ by (4*). Choosing $\delta_0 > 0$ small enough, we obtain an abelian C^*-subalgebra $B \subset B(X)$ and a unital completely bounded map Φ: $A \rightarrow B$ such that for every $t \geq 0$,

$$\| T_t - B_t \| \quad \leq \quad \rho, \text{where } B_t := \Phi(T_t) \in \mathcal{B}, \qquad (5^*)$$

and $0 < \rho \ll 1$is a fixed perturbation constant (depending on $\delta_0$but independent of t). Intuitively, the entire dynamical family $\{T_t\}$ is uniformly close, in operator norm, to a commuting family $\{B_t\} \subset B$.

Because $\mathcal{B}$ is abelian, the family $\{B_t\}_{t\geq 0}$ admits a joint spectral representation [11,19]: there exists a compact Hausdorff space K and a representation $\pi : B \rightarrow C(K)$, such that each B_t corresponds to a multiplication operator by a continuous function $f_t \in C(K)$:

$$B_t \;\cong\; M_{f_t} \text{on } C(K) \textit{ (or on an appropriate Banach-lattice model of X)}$$

In particular, for any finite collection of time instants $t_1,\dots,t_n$, the operators $B_{t_1}, \dots, B_{t_n}$ commute and possess a joint spectral resolution realised by Borel subsets of K. Spectral projections (or spectral bands) are given by multiplication by characteristic functions of measurable subsets of K. From this we may construct, for any $\varepsilon > 0$ and any radius $r > 0$, an ε-quasi-spectral partition for each B_t with uniformly finite multiplicity of spectral cells:

A) Partition K into finitely many Borel´s sets $K_1,\dots,K_m$ such that on each K_j, the functions f_t (for all relevant t) vary by at most a small amount $\delta(\varepsilon,r)$ within the ball $B(0,r) \subset \mathbb{C}$.

B) Let $E_j := \{x \in X$: "support" $(x) \subset K_j\}$ (or its analogue in the concrete realisation of B). Then each E_j is invariant under B, and on E_j the operators B_t act almost as multiplication by scalars whose values are constrained to a small disc in $\mathbb{C}$.

By construction:

- The family $\{(U_j, E_j)\}_{j=1}^{m}$, where each $U_j \subset \mathbb{C}$ is a small disc containing the range of f_t on $K_j \cap f_t^{-1}(B(0,r))$, forms an ε–quasi-spectral partition for each B_t in the spectral ball $B(0,r)$.
- The dimensions $dimE_j$ are determined entirely by the representation of B on X and do not depend on t. In particular, for each fixed r,

$$sup\{dim\, E_j : U_j \subset B(0,r)\} =: M(r) < \infty$$

where $M(r)$ is a cardinal independent of t.

Now, we transfer this quasi-spectral structure from the commuting family $\{B_t\}$ back to the almost commuting family $\{T_t\}$. Recall that $\|T_t\text{-}B_t\| \leq \rho$ for all t, with $\rho > 0$ fixed. Let (U_j,E_j) be an ε – spectral cell for B_t. For any $x \in E_j$ and $\lambda \in U_j$,

$$\| B_t x - \lambda x \| \leq \varepsilon \| x \|$$

Then,

$$\| T_t x - \lambda x \| \leq \| T_t x - B_t x \| + \| B_t x - \lambda x \| \leq \rho \| x \| + \varepsilon \| x \| = (\varepsilon + \rho) \| x \|$$

Thus, each (U_j,E_j) is an $(\varepsilon+\rho)$ –spectral cell for T_t. Since the collection $\{(U_j,E_j)\}$ still exhausts the relevant spectral region and the algebraic sum $\sum_j E_j$ is dense in *X*, we obtain an $(\varepsilon+\rho)$-quasi-spectral partition for T_t with the same family of subspaces E_j as for B_t.

Consequently, for each $t \geq 0$ and each $r > 0$, the spectral complexity function of T_t satisfies $d^T_{\varepsilon+\rho}(t, r) \;\leq\; M(r)$, where $M(r)$ depends only on the abelian representation of *B*, not on *t*.

Hence $H^T_{\varepsilon+\rho}(t, r) = log\, d^T_{\varepsilon+\rho}(t, r) \;\leq\; log\, M(r)$, and therefore

$$\frac{H^T_{\varepsilon+\rho}(t, r)}{t} \;\leq\; \frac{log\, M(r)}{t}$$

Now, take $lim\, sup_{t\to\infty}$ for fixed *r* :

$$\limsup_{t\to\infty} \frac{H^T_{\varepsilon+\rho}(t, r)}{t} \;\leq\; \limsup_{t\to\infty} \frac{log\, M(r)}{t} = 0$$

Then, take $lim\, sup_{r\to\infty}$. Since the right-hand side is identically zero in *t*, we still obtain zero:

$$\limsup_{r\to\infty} \limsup_{t\to\infty} \frac{H^T_{\varepsilon+\rho}(t, r)}{t} = 0$$

so that

$$h_{\varepsilon+\rho}(T) = 0, \forall\; \varepsilon > 0$$

Finally, by theorem 3.6.1 (monotonicity in ε),

$$h_{spec}(T) = \lim_{\eta\downarrow 0} h_\eta(T) \leq h_{\varepsilon+\rho}(T) = 0$$

and therefore $h_{spec}(T) = 0$. This proves theorem 3.6.3. □

3.7. Spectral Entropy and Approximate Resolvent Growth

A central result linking resolvent-based geometry and dynamical spectral complexity is established below. This theorem is the backbone of the Noncommutative Variational Principle to be developed in section 5.

Theorem 3.7.1 (Resolvent–Entropy Correspondence). Let $(T_t)_{t\geq 0} \subset \mathcal{B}(X)$ be an ε-noncommutative dynamical scheme on a (possibly non-separable) Banach space *X*, and let $h_{spec}(T)$ denote the generalised spectral entropy defined in section 3 via quasi-spectral partitions and spectral profiles. Fix $R > 0$ sufficiently large so that, for all $t \geq 0$, the approximate spectrum of T_t at scale ε is contained in $B(0,R)$ (for small ε), and define

$$\Phi_\varepsilon(T, t, R) = log(\sup_{|\lambda|<R} \parallel R_\varepsilon(\lambda, T_t) \parallel)$$

Then,

$$h_{spec}(T) = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, \Phi_\varepsilon(T, t, R)$$

and the right-hand side does not depend on the particular choice of *R*, provided *R* is large enough.
Proof. The proof naturally splits into two inequalities:

$$h_{spec}(T) \;\leq\; \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, \Phi_\varepsilon(T, t, R) \qquad \text{(U)}$$

$$h_{spec}(T) \ \geq \ \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, \Phi_\varepsilon(T,t,R) \qquad \text{(L)}$$

Together, (Upper bound) and (Lower bound) give the desired equality. Throughout we use the definitions from section 3: (a) ε–quasi-spectral partitions; (b) the spectral complexity function $d_\varepsilon(t,r)$; (c) the local spectral complexity $H_\varepsilon(T_t, r) = \log d_\varepsilon(t,r)$; (d) and the spectral entropy

$$h_{spec}(T) = \lim_{\varepsilon\to 0}(\limsup_{t\to\infty}\ \limsup_{r\to\infty} \frac{H_\varepsilon(T_t,r)}{t})$$

We also rely on the approximate resolvent theory developed in section 2 (in particular, ε–resolvents and their stability).

<u>Upper bound: controlling entropy by resolvent growth</u>. Fix $\varepsilon > 0$ and $R > 0$ large. For each $t \geq 0$, let

$$M_\varepsilon(t,R) = \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t) \|, \Phi_\varepsilon(T,t,R) = \log M_\varepsilon(t,R)$$

We show that the spectral growth of ε – quasi-spectral partitions at radius $r \leq R$ is bounded in terms of $M_\varepsilon(t,R)$, and hence in terms of $\Phi_\varepsilon(T,t,R)$. Let $\mathcal{P}_\varepsilon(T_t) = \{(U_\iota(t), E_\iota(t))\}_{\iota\in I_t}$ be an arbitrary ε–quasi-spectral partition for T_t. For each ι with $U_\iota(t) \subset B(0,R)$, we have by definition:

$$\| T_t x - \lambda x \| \leq \ \varepsilon \| x \|, \forall\, x \in E_\iota(t), \forall\, \lambda \in U_t(t) \qquad (1\#)$$

Fix such an index ι, pick any $\lambda_\iota \in U_\iota(t)$, and consider the ε – resolvent $R_\varepsilon(\lambda_\iota, T_t)$. Applying $R_\varepsilon(\lambda_\iota, T_t)$ to (1#), we obtain, for $x \in E_\iota$ (t), $R_\varepsilon(\lambda_\iota, T_t)(T_t - \lambda_\iota I)\, x = x + \theta_\iota(x)$

where $\theta_\iota(x)$ is the error coming from the approximate inverse identity

$$\| R_\varepsilon(\lambda_\iota, T_t)(T_t - \lambda_\iota I) - I \| \leq \varepsilon$$

Hence, $\|\theta_\iota(x)\| \leq \varepsilon\|x\|$. On the other hand,

$$\| R_\varepsilon(\lambda_\iota, T_t)(T_t - \lambda_\iota I)x \| \leq \| R_\varepsilon(\lambda_\iota, T_t) \| \ \ \| (T_t - \lambda_\iota I)x \| \leq M_\varepsilon(t,R)\, \varepsilon \| x \|$$

Thus,

$$\| x \| \leq \| \theta_\iota(x) \| + \| R_\varepsilon(\lambda_\iota, T_t)(T_t - \lambda_\iota I)x \| \leq (\varepsilon + M_\varepsilon(t,R)\, \varepsilon) \ \| x \|$$

For $\varepsilon > 0$ fixed, and $M_\varepsilon(t,R)$ possibly large, on its own this inequality is not yet informative. However, it shows that the inclusion

$$E_\iota(t) \ni x \longmapsto (T_t - \lambda_\iota I)x$$

has image contained in a subspace on which $R_\varepsilon(\lambda_\iota, T_t)$ acts with norm $M_\varepsilon(t,R)$. A more useful form comes from considering finite-dimensional subspaces. If $F \subset E_\iota(t)$ is finite dimensional, then the operator $(T_t - \lambda_\iota I)$: $F \to X$ has norm at most ε, while the restriction of $R_\varepsilon(\lambda_\iota, T_t)$ to its image has norm at most $M_\varepsilon(t,R)$. Therefore, the composition

$$F \overset{T_t - \lambda_\iota I}{\to} (T_t - \lambda_\iota I)(F) \overset{R_\varepsilon(\lambda_\iota, T_t)}{\to} X$$

differs from the identity on F by at most ε in operator norm. In particular, the map $T_t - \lambda_\iota I$ is almost injective with controlled distortion governed by $M_\varepsilon(t,R)$.

Standard linear algebra in Banach spaces then yields the following: the dimension of any subspace on which both $\| (T_t - \lambda_\iota I)x \| \leq \varepsilon \| x \|$, and $\| R_\varepsilon(\lambda_\iota, T_t) \| \leq M_\varepsilon(t, R)$ hold uniformly, is bounded by a function depending on $M_\varepsilon(t,R)$ and ε. In particular, there exists a function $\Psi_\varepsilon : [1,\infty) \to [1,\infty)$, increasing, such that

$$dim\, E_\iota(t) \leq \Psi_\varepsilon(M_\varepsilon(t,R)) whenever\ U_\iota(t) \subset B(0,R) \qquad (2\#)$$

Taking the supremum over all ι with $U_\iota(t) \subset B(0,R)$, we obtain

$$d_\varepsilon(t,R) \ \leq \ \Psi_\varepsilon(M_\varepsilon(t,R))$$

where $d_\varepsilon(t,R) = sup\{dim\, E_\iota(t) \colon U_\iota(t) \subset B(0,R)\}$. Therefore,

$$H_\varepsilon(T_t, R) = log\, d_\varepsilon(t,R) \leq log\, \Psi_\varepsilon(M_\varepsilon(t,R))$$

Since Ψ_ε is increasing and $M_\varepsilon(t,R) \geq 1$, there exists a constant $C_\varepsilon > 0$ such that

$$log\, \Psi_\varepsilon(x) \leq C_\varepsilon + log\, x, \ \forall x \geq 1$$

Thus,

$$H_\varepsilon(T_t, R) \leq C_\varepsilon + log\, M_\varepsilon(t,R) = C_\varepsilon + \Phi_\varepsilon(T,t,R). \qquad (3\#)$$

Dividing (3#) by $t > 0$ and taking $lim\,sup_{t\to\infty}$, the constant C_ε/t vanishes, yielding

$$lim\, sup \frac{H_\varepsilon(T_t, R)}{} \leq lim\, sup \frac{\Phi_\varepsilon(T,t,R)}{}$$

Now, by monotonicity in the radius r and the fact that we have chosen Rlarge enough to see the relevant spectral region, the $lim\,sup_{(r\to\infty)}$ in the definition of $h_\varepsilon(T)$ is controlled by the value at $r = R$ up to a harmless multiplicative constant (which disappears after dividing by tand taking $lim\,sup_{t\to\infty}$). Thus:

$$h_\varepsilon(T) = \limsup_{t\to\infty} \limsup_{r\to\infty} \frac{H_\varepsilon(T_t, r)}{t} \ \leq \limsup_{t\to\infty} \frac{\Phi_\varepsilon(T,t,R)}{t}.$$

Finally, letting $\varepsilon \to 0$ and using monotonicity of h_ε (T) in ε (theorem 3.6.1), we obtain

$$h_{spec}(T) = \lim_{\varepsilon\to 0} h_\varepsilon(T) \leq \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, \Phi_\varepsilon(T,t,R)$$

which proves (U).

- Lower bound: extracting spectral complexity from large resolvent norms. Suppose that the resolvent–based quantity

$$\gamma := \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, \Phi_\varepsilon(T,t,R)$$

is strictly positive (the case $\gamma = 0$ is trivial and already compatible with $h_{spec}(T) \geq 0$). By the definition of lim sup, there exists $\varepsilon_n \downarrow 0$ and times $t_n \to \infty$ such that

$$\frac{1}{t_n}\, \Phi_{\varepsilon_n}(T, t_n, R) \to \gamma, \text{as } n \to \infty$$

Equivalently,

$$M_{\varepsilon_n}(t_n,R)=\sup_{|\lambda|<R} \parallel R_{\varepsilon_n}\left(\lambda,T_{t_n}\right) \parallel \geq exp((\gamma-o(1))t_n) \qquad (4\#)$$

For each n, choose $\lambda_n \in B(0,R)$ such that

$$\parallel R_{\varepsilon_n}\left(\lambda_n,T_{t_n}\right) \parallel \geq \frac{1}{2} M_{\varepsilon_n}(t_n,R)$$

We now exploit the fact that a large resolvent norm at (λ_n,T_{tn}) forces the existence of a large-dimensional approximate eigenspace near λ_n. This is an approximate version of the classical relationship between resolvent blow-up and spectral concentration. Indeed, consider the operator $A_n:=T_{tn}-\lambda_{nI}$. The ε–resolvent property implies that $\begin{cases} \parallel A_n R_{\varepsilon_n}\left(\lambda_n,T_{t_n}\right)-I \parallel \leq \varepsilon_n \\ \parallel R_{\varepsilon_n}\left(\lambda_n,T_{t_n}\right)A_n-I \parallel \leq \varepsilon_n \end{cases}$. Let $B_n:=R_{\varepsilon_n}(\lambda_n,T_{t_n})$, and let $\parallel B_n \parallel=\beta_n$, where:

$$\beta_n \geq \frac{1}{2} M_{\varepsilon_n}(t_n,R) \geq \frac{1}{2}\exp((\gamma-o(1))t_n)$$

Consider the unit ball of X, and pick a maximal $1\frac{1}{2\beta_n}$–separated subset F_n in the image of the unit ball under B_n, i.e.

$$F_n \subset B_n(B_X) \text{ such that } \parallel y-z \parallel \geq \frac{1}{2\beta_n} \text{ for distinct } y,z \in F_n$$

and F_n is maximal with this property. By a standard volumetric argument in Banach spaces (or by a cardinality argument in the non-separable context), the cardinality of F_n is at least of order β_n in an exponential sense; in particular there exists a constant $c > 0$ such that

$$| F_n | \geq exp(c\, log\, \beta_n)=\beta_n^c$$

For each $y \in F_n$, choose $x_y \in B_X$ with $B_n x_y=y$. Then,

$$\parallel A_n x_y \parallel = \parallel \left(T_{t_n}-\lambda_n I\right)x_y \parallel \approx \varepsilon_n \parallel x_y \parallel$$

in the sense that, by the approximate inverse property,

$$\parallel A_n x_y \parallel \leq \frac{1+\varepsilon_n}{\beta_n} \parallel y \parallel \leq \frac{2}{\beta_n}$$

Thus, as $\beta_n \to \infty$, the vectors x_y become approximate eigenvectors for T_{tn} at eigenvalue λ_n with error of order β_n-1, which is super–exponentially small compared with $exp(-(\gamma-o(1))t_n)$. In particular, for n sufficiently large we can ensure that

$$\parallel A_n x_y \parallel \leq \varepsilon_n \parallel x_y \parallel$$

Let E_n be the linear span of the family $\{x_y : y \in F_n\}$. The separation of the images $y = B_n x_y$ in the range implies that the vectors x_y are linearly independent and well separated: in particular,

$$\parallel T_{t_n}x-\lambda x \parallel \leq 2\varepsilon_n \parallel x \parallel$$

for some $c' > 0$ depending on γ. Moreover, for all $x \in E_n$ and λ in a small disc U_n around λ_n, one can check (by continuity) that

$$\parallel T_{t_n}x-\lambda x \parallel \leq 2\varepsilon_n \parallel x \parallel$$

for sufficiently small radius of U_n. Thus (U_n,E_n) is an $(2\varepsilon_n)$–spectral cell for T_{tn}). We have therefore constructed a family of ε–spectral cells (U_n,E_n) with: (1) $U_n \subset B(0,R)$; (2) $dim\, E_n \geq exp(c'\, t_n)$. This implies that the spectral complexity function satisfies $d_{2\varepsilon n}(t_n,R) \geq dim\, E_n \geq exp(c'\, t_n)$, so that

$$H_{2\varepsilon_n}(T_{t_n}, R) = log\, d_{2\varepsilon_n}(t_n, R) \geq c' t_n$$

Hence,

$$\frac{H_{2\varepsilon_n}(T_{t_n}, R)}{t_n} \geq c'$$

Passing to $lim\,sup_{(n\to\infty)}$, we obtain

$$\underset{t\to\infty}{lim\,sup}\; \underset{r\to\infty}{lim\,sup} \frac{H_{2\varepsilon_n}(T_t, r)}{t} \geq \underset{n\to\infty}{lim\,sup} \frac{H_{2\varepsilon_n}(T_{t_n}, R)}{t_n} \geq c'$$

Since $\varepsilon_{n\to 0}$, by monotonicity in ε (theorem 3.6.1),

$$h_{spec}(T) = \lim_{\eta\to 0} h_\eta(T) \geq \underset{n\to\infty}{lim\,sup}\, h_{2\varepsilon_n}(T) \geq c' > 0$$

Thus, a positive exponent in the resolvent growth

$$\gamma = \lim_{\varepsilon\to 0} \underset{t\to\infty}{lim\,sup} \frac{1}{t} \Phi_\varepsilon(T, t, R)$$

forces a positive spectral entropy. Equivalently,

$$h_{spec}(T) \geq \lim_{\varepsilon\to 0} \underset{t\to\infty}{lim\,sup} \frac{1}{t} \Phi_\varepsilon(T, t, R)$$

which is precisely (L). Combining (U) and (L), we conclude that for any sufficiently large

$R > 0$,

$$h_{spec}(T) = \lim_{\varepsilon\to 0} \underset{t\to\infty}{lim\,sup} \frac{1}{t} log(\sup_{|\lambda|<R} \parallel R_\varepsilon(\lambda, T_t) \parallel)$$

3.8. Summary of Section 3

We have constructed: (a) the ε-spectral growth $H_\varepsilon(T_{t,r})$, (b) the spectral entropy $h_{spec}(T)$, (c) fundamental properties (stability, monotonicity), and a resolvent–entropy correspondence. This provides a non-measurable, non-topological invariant suitable for non-separable operator dynamics.

4. Structural Decomposition Theorems

In this section we show that, for ε –noncommutative dynamical schemes on non-separable Banach spaces, the growth rate of the approximate resolvent and the generalised spectral entropy control the underlying operator structure in a precise and quantifiable way. The results provide a decomposition of the space and the dynamics into three qualitatively distinct components: (1) Low–entropy (quasi-compact) sector. (2) Intermediate (fragmentable) sector. (3) High–entropy (expansive) sector.

Each component corresponds to a different asymptotic regime for the resolvent behaviour and approximate spectrum, making the decomposition a true generalisation of classical results such as the Jacobs–de Leeuw–Glicksberg decomposition for semigroups, Riesz–Schauder theory, and the classical dichotomy between compact and expansive dynamics. Throughout we assume that $(T_t)_{t\geq 0}$ is an ε-noncommutative scheme on a Banach space *X*, with ε small, unless otherwise stated.

4.1. The Trichotomy Principle

We begin by formalising the intuition that spectral entropy measures the degree of non-compactness or structural spread of the dynamics.

Recall from that:

a. $h_{spec}(T) = 0$ is characteristic of quasi-compact / almost commutative behaviour;

b. $h_{spec}(T) > 0$ reflects high-complexity resolvent growth;

c. intermediate behaviour is possible when the resolvent growth is sub-exponential but non-trivial.

These observations lead to a canonical trichotomy.

Theorem 4.1.1 (Spectral Trichotomy Theorem). Let $(T_t)_{t\geq 0}$ be an ε-noncommutative dynamical scheme. Then, there exists a closed, T_t-invariant decomposition

$$X = X_{LC} \oplus X_{FR} \oplus X_{HE}$$

such that:

(1) Low-complexity sector X_{LC}.

(1.1) $T_t|X_{LC}$ has zero spectral entropy,

(1.2) the resolvent is bounded uniformly in t:

$$\sup_{t\geq 0} \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t|_{X_{LC}}) \| < \infty$$

(1.3) equivalently, $T_t|X_{LC}$ is asymptotically quasi-compact.

(2) Fragmentable sector X_{FR}.

(2.1) spectral entropy is zero, but the resolvent grows sub-exponentially,

(2.2) resolvent norms satisfy

$$\log \| R_\varepsilon(\lambda, T_t|_{X_{FR}}) \| = o(t)(t \to \infty)$$

(2.3) the orbits of T_t avoid dentability collapse and the dual action is fragmented (Fabian et al., 2011).

(3) High-entropy sector X_{HE}.

(3.1) the restriction $T_t|X_{HE}$ has strictly positive entropy,

(3.2) the resolvent grows exponentially on X_{HE}:

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t|_{X_{HE}}) \| = h_{spec}(T) > 0$$

(3.3) $X_{HE} \neq \{0\}$ if and only if the scheme exhibits high-complexity approximate spectrum (theorem 3.7.1).

Moreover, this decomposition is unique and is determined solely by the asymptotic resolvent growth profile (hence by the spectral entropy).

Proof. Firstly, we will construct the low–complexity componen*t XLC*. Define:

$$X_{LC} := \{x \in X : \sup_{t\geq 0} \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t)x \| < \infty\}$$

where $R > 0$ is chosen sufficiently large as in theorem 3.7.1 so that the relevant ε – spectrum is contained in $B(0,R)$. We claim that:

- X_{LC} is a closed, T_t-invariant subspace.
- $T_t|X_{LC}$ has zero spectral entropy.
- X_{LC} is maximal with this property.

Let $(x_n) \subset X_{LC}$ converge to x in norm. Then for all $t \geq 0$ and all $|\lambda| < R$,

$$\| R_\varepsilon(\lambda, T_t)x \| = \lim_{n\to\infty} \| R_\varepsilon(\lambda, T_t)x_n \| < \infty$$

Thus $x \in X_{LC}$, proving closedness. For every $x \in X_{LC}$,

$$R_\varepsilon(\lambda, T_t)(T_s x) = T_s R_\varepsilon(\lambda, T_t)x + \mathcal{O}(\varepsilon)$$

because resolvent multiplication respects approximate commutativity by proposition 2.8.1. Since the right–hand side is uniformly bounded in t and λ, it follows that $T_s x \in X_{LC}$. If the ε–resolvent is uniformly bounded on the whole-time axis, then by theorem 3.7.1,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t |_{X_{LC}}) \| = 0$$

Taking $\varepsilon \to 0$ and using theorem 3.6.1 (monotonicity in ε), we get

$$h_{spec}(T |_{X_{LC}}) = 0$$

Thus, the resolvent is tamely behaved and entropy vanishes.

Let $Y \subset X$ be any closed T_t-invariant subspace with uniformly bounded resolvent.

Then for all $x \in Y$ and all $|\lambda| < R$,

$$h_{spec}(T |_{X_{LC}}) = 0$$

Hence, $Y \subseteq X_{LC}$, proving maximality. This finishes the construction of the low–complexity part.

Secondly, we construct of the fragmentable component X_{FR}. Define the intermediate sector:

$$\sup_{t\geq 0} \| R_\varepsilon(\lambda, T_t)x \| < \infty$$

Equivalently:

$$X_{FR} := \{x \in X: \limsup_{t\to\infty} \frac{1}{t} \log \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t)x \| = 0\}$$

This is the subspace where the resolvent grows sub-exponentially. Let $x_n \to x$ with each $x_n \in X_{FR}$. Using the approximate resolvent identity and uniform boundedness for small t, one shows that the growth estimate transfers to the limit:

$$\| R_\varepsilon(\lambda, T_t)x \| \leq \| R_\varepsilon(\lambda, T_t)(x - x_n) \| + \| R_\varepsilon(\lambda, T_t)x_n \| = e^{o(t)}$$

so $x \in X_{FR}$. Now, for $x \in X_{FR}$,

$$R_\varepsilon(\lambda, T_t)(T_s x) = T_s R_\varepsilon(\lambda, T_t)x + \mathcal{O}(\varepsilon(t, s))$$

and hence

$$\| R_\varepsilon(\lambda, T_t)(T_s x) \| \leq e^{o(t)}$$

Thus, $T_s x \in X_{FR}$. If for each $x \in X_{FR}$,

$$\sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t)x \| = e^{o(t)}$$

Then, the same estimates hold for finite-dimensional subspaces in ε – spectral cells.
Hence, the spectral complexity at scale r satisfies: $d_\varepsilon(t,r) = e^{o(t)}$, yielding zero entropy:

$$h_{spec}(T |_{X_{FR}}) = 0$$

Any $Y \subset X$ closed and invariant with zero entropy but not contained in X_{LC} must satisfy sub-exponential resolvent growth. By theorem 3.7.1, zero entropy implies

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t |_Y) \| = 0$$

Hence, $Y \subseteq X_{FR}$. Thus X_{FR} is maximal among zero-entropy components disjoint from X_{LC}.

Third, we construct the high–entropy component X_{HE} . Define:

$$X_{HE} := (X_{LC} \oplus X_{FR})^{\perp}$$

This is an invariant subspace because both X_{LC} and X_{FR} are invariant and closed. Suppose $X_{HE} \neq \{0\}$. For any nonzero $x \in X_{HE}$, the resolvent grows exponentially:

$$\sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t)x \| \geq e^{\alpha t} \, for\ some\ \alpha > 0$$

Otherwise, if for some vector the resolvent grew sub-exponentially, that vector would lie in X_{FR}, contradicting the definition of X_{HE}. Thus, by theorem 3.7.1 (resolvent–entropy correspondence),

$$h_{spec}\left(T |_{X_{HE}}\right) = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log \sup_{|\lambda|<R} \| R_\varepsilon\left(\lambda, T_t |_{X_{HE}}\right) \| > 0$$

This shows that X_{HE} is precisely the high–entropy sector.
Fourth: uniqueness of the decomposition. Suppose there is another decomposition:

$$X = Y_{LC} \oplus Y_{FR} \oplus Y_{HE}$$

with the same properties.
a. By maximality of X_{LC}, necessarily $Y_{LC} = X_{LC}$.
b. By maximality of X_{FR} among zero-entropy subspaces disjoint from X_{LC}, we get $Y_{FR} = X_{FR}$.
c. Therefore $Y_{HE} = X_{HE}$.

Thus, the decomposition is unique.
Conclusion. We have shown:
- The existence of the decomposition

$$X = X_{LC} \oplus X_{FR} \oplus X_{HE}$$

- Invariance and closedness of each subspace,
- Characterisation of each component by resolvent growth and spectral entropy,
- Maximality and uniqueness of each component.

This completes the proof of Theorem 4.1.1. □

4.2. Decomposition via Approximate Resolvent Growth

We now formalise the decomposition described in theorem 4.1.1. Define the resolvent growth exponent on a closed invariant subspace $Y \subseteq X$ by

$$\gamma(Y) = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log \sup_{|\lambda|<R} \| R_\varepsilon(\lambda, T_t |_Y) \|$$

This exponent coincides with the spectral entropy exponent by theorem 3.7.1.
Definition 4.2.1.

(A) $\mathrm{X_{LC}} = \max\{\mathrm{Y} \subseteq \mathrm{X}: \gamma(\mathrm{Y}) = 0 \text{ and } \sup_t \| \mathrm{R}_\varepsilon(\lambda, \mathrm{T_t} |_\mathrm{Y}) \| < \infty\}$.

(B) $X_{FR} = max\{Y \subseteq X: \gamma(Y) = 0\}$, *subject to* $Y \cap X_{LC}^{\perp} = Y$.

(C) $X_{HE} = (X_{LC} \oplus X_{FR})^{\perp}$.

This definition makes the trichotomy transparent and gives the uniqueness.
The structural implications are deep:

- X_{LC} behaves like a generalised Riesz–Schauder part: its resolvent is tame.
- X_{FR} resembles spaces with Bourgain–Fremlin–Talagrand fragmentability properties.
- X_{HE} captures the exponential spectral explosion.

4.3. Approximate Spectral Radius Decomposition

Let

$$\rho_\varepsilon(T_t) = sup\{| \lambda |: \lambda \in \sigma_\varepsilon(T_t)\}$$

where $\sigma_\varepsilon(T_t)$ is the ε-approximate spectrum. Define the approximate spectral radius growth exponent:

$$\eta(T) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \rho_\varepsilon(T_t)$$

Using the resolvent identity, it follows that:
Theorem 4.3.1 (Radius–Entropy Correspondence). On each component X_{LC}, X_{FR}, X_{HE},

$$\eta(T |_Y) = \gamma(T |_Y) = h_{spec}(T |_Y)$$

In particular, exponential resolvent growth is equivalent to exponential growth of the approximate spectral radius. This generalises classical semigroup theory in finite dimensions.

Proof. The key tool will be lemma 4.4.1 (see the next section). Now, let we see that $\eta(T|_Y) \le \gamma(T|_Y)$. Fix $\varepsilon > 0$ and $R > 0$ large. For any t,

$$\rho_\varepsilon(T_t |_Y) = sup\{| \mu |: \mu \in \sigma_\varepsilon(T_t |_Y)\}$$

Geometrically,

$$dist\big(\lambda, \sigma_\varepsilon(T_t |_Y)\big) \ge \big|\ | \lambda | - \rho_\varepsilon(T_t |_Y)\ \big|, \forall\, \lambda \in \mathbb{C}$$

Take $|\lambda|$=R (on the circle). On the one hand, lemma 4.4.1(2) implies

$$dist(\lambda, \sigma_\varepsilon(T_t |_Y)) \ \ge\ \frac{1}{C_2(1+R)^m} \cdot \frac{1}{\sup_{|\mu| \le R} \| R_\varepsilon(\mu, T_t |_Y) \|}$$

On the other hand, $dist\big(\lambda, \sigma_\varepsilon(T_t |_Y)\big) \le | \rho_\varepsilon(T_t |_Y) - R |$. Combining these,

$$| \rho_\varepsilon(T_t |_Y) - R | \ \ge\ \frac{1}{C_2(1+R)^m} \cdot exp(-\Phi_\varepsilon(T |_Y, t, R))$$

For large t, the right-hand side is tiny compared to any exponential scale, so we can write (absorbing constants into C_ε)

$$log(1+\rho_\varepsilon(T_t\ |_Y)) \le \Phi_\varepsilon(T\ |_Y, t, R) + C_\varepsilon$$

Divide by $t$and take $lim\, sup_{t\to\infty}$:

$$\limsup_{t\to\infty}\frac{1}{t}\,log(1+\rho_\varepsilon(T_t\ |_Y)) \le \limsup_{t\to\infty}\frac{1}{t}\,\Phi_\varepsilon(T\ |_Y, t, R)$$

Now, let ε → 0. By definition of η and γ we get $\eta\,(T\,|_Y) \le \gamma\,(T\,|_Y)$. Let we see that: $\eta\,(T\,|_Y) \ge \gamma\,(T|_Y)$- Suppose $\gamma\,(T\,|_Y) > 0$. By definition of $\gamma\,(T\,|_Y)$, there exist sequences $\varepsilon_n \downarrow 0$ and $t_n \to \infty$ such that $\frac{1}{t_n}\,\Phi_{\varepsilon_n}(T\ |_Y, t_n, R) \to \gamma(T\ |_Y)$, i.e.

$$\sup_{|\lambda|\le R}\ \| R_{\varepsilon_n}\left(\lambda, T_{t_n}\ |_Y\right) \| \ge exp((\gamma(T\ |_Y) - o(1))t_n)$$

Choose λ_n with $|\lambda_n| \le R$ such that

$$\| R_{\varepsilon_n}\left(\lambda_n, T_{t_n}\ |_Y\right) \| \ge \frac{1}{2}exp((\gamma(T\ |_Y) - o(1))t_n)$$

Applying lemma 4.4.1(2) contrapositively, a large resolvent norm forces λ_n to be very close to $\sigma_{\varepsilon_n}(T_{t_n}\ |_Y)$:

$$dist(\lambda_n, \sigma_{\varepsilon_n}(T_{t_n}\ |_Y)) \le C(1+R)^m\,exp(-(\gamma(T\ |_Y) - o(1))t_n)$$

So, there exists $\mu_n \in \sigma_{\varepsilon_n}(T_{t_n}\ |_Y)$ with

$$|\ \mu_n - \lambda_n\ | \le C(1+R)^m\,exp(-(\gamma(T\ |_Y) - o(1))t_n)$$

In particular, $|\mu_n| \le R$ + "(super–exponentially small term)" . Hence, $|\mu_n|$is uniformly bounded.

Now, if the ε–spectral radius $\rho_{\varepsilon_n}(T_{t_n}\ |_Y$ would growing strictly slower (in exponent) than the resolvent growth, then the ε–spectrum would remain confined to a region whose exponential growth exponent is strictly smaller than $\gamma(T\ |_Y)$, contradicting the resolvent–entropy correspondence and the structural trichotomy on the high–entropy part. Formally, the argument can be organised as follows: if

$$\limsup_{n\to\infty}\frac{1}{t_n}log(1+\rho_{\varepsilon_n}(T_{t_n}\ |_Y)) < \gamma(T\ |_Y)$$

then theorem 3.7.1 would force the resolvent growth exponent $\gamma(T|_Y)$ to be at most that smaller limit, a contradiction. Therefore, one must have:

$$\limsup_{n\to\infty}\frac{1}{t_n}log(1+\rho_{\varepsilon_n}(T_{t_n}\ |_Y)) \ge \gamma(T\ |_Y)$$

Taking the outer limits (first in t, then ε), this gives

$$\eta(T\ |_Y) \ge \gamma(T\ |_Y)$$

Now, let we see the equality and restriction to each component, From the former considerations, we have, for every invariant subspace Y,

$$\eta(T\ |_Y) = \gamma(T\ |_Y)$$

By theorem 3.7.1, $\gamma(T\mid_Y) = h_{spec}(T\mid_Y)$, so altogether

$$\eta(T\mid_Y) = \gamma(T\mid_Y) = h_{spec}(T\mid_Y)$$

This holds for any Y, and in particular for the three components

$$Y = X_{LC}, Y = X_{FR}, Y = X_{HE}$$

from the trichotomy theorem (theorem 4.1.1). This proves the statement of Theorem 4.3.1.

4.4. Structural Stability of the Decomposition

Let $Y \subseteq X$ be a closed Tt-invariant subspace. For each $t \geq 0$, define the ε–approximate spectrum $\sigma_\varepsilon(Tt|_Y)$ and its ε–spectral radius

$$\rho_\varepsilon(T_t\mid_Y) = sup\{\mid \lambda \mid : \lambda \in \sigma_\varepsilon(T_t\mid_Y)\}$$

Define the approximate spectral radius growth exponent on Y as

$$\eta(T\mid_Y) := \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, log(1 + \rho_\varepsilon(T_t\mid_Y))$$

Define the resolvent growth exponent (using any sufficiently large $R > 0$) as

$$\gamma(T\mid_Y) := \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, log(\sup_{|\lambda|<R} \parallel R_\varepsilon(\lambda, T_t\mid_Y) \parallel)$$

By theorem 3.7.1 we already know that

$$\gamma(T\mid_Y) = h_{spec}(T\mid_Y)$$

Theorem 4.4.1 (Stability under small perturbations). For each closed invariant subspace $Y \subseteq X$,

$$\eta(T\mid_Y) = \gamma(T\mid_Y) = h_{\text{spec}}(T\mid_Y).$$

In particular, exponential growth of the ε–approximate spectral radius on Y is equivalent to exponential resolvent growth on Y.
Proof. We must show

$$\eta(T\mid_Y) = \gamma(T\mid_Y)$$

The equality $\gamma(T|_Y) = h_{spec}(T|_Y)$ has already been established in theorem 3.7.1, so we focus on the relation between spectral radius growth and resolvent growth. The proof is split into two inequalities:

(1) $\eta(T\mid_Y) \leq \gamma(T\mid_Y)$,

(2) $\eta(T\mid_Y) \geq \gamma(T\mid_Y)$.

The argument uses: the definition of the ε–spectrum $\sigma_\varepsilon(T_t|_Y)$; standard resolvent inequalities adapted to ε–resolvents; and theorem 3.7.1. For clarity, I will omit $|_Y$ in the notation and write simply T_t, $\sigma_\varepsilon(T_t)$, etc., understanding that everything happens inside Y. We first establish a lemma that plays the rôle of an approximate spectral radius formula.

Lemma 4.4.1 (ε – resolvent vs ε–spectral radius). Let $T\in B(X)$ be a bounded operator on a Banach space X, and let $\varepsilon > 0$. Assume the ε–resolvent $R_\varepsilon(\lambda, T)$ exists for some $\lambda \in C$, i.e.,

$$\parallel (T - \lambda I)R_\varepsilon(\lambda, T) - I \parallel \leq \varepsilon, \parallel R_\varepsilon(\lambda, T)(T - \lambda I) - I \parallel \leq \varepsilon. \qquad (4.4.1.1)$$

Define the ε–spectrum $\sigma_\varepsilon(T)$ as the complement of the ε – resolvent set, and the ε–spectral radius as

$$\rho_\varepsilon(T) := sup\{|\ \mu\ |: \mu \in \sigma_\varepsilon(T)\}$$

Then, there exist constants $C_1, C_2 > 0$ and an integer $m \geq 0$, depending only on $\|T\|$, ε and the construction of ε –resolvents, such that:

i. If $|\lambda| > \rho_\varepsilon(T)$, then

$$\| R_\varepsilon(\lambda, T) \| \ \leq \ C_1 \frac{(1+|\ \lambda\ |)^m}{|\ \lambda\ | - \rho_\varepsilon(T)} \qquad (4.4.1.2)$$

ii. Conversely, if $|\lambda| \leq R$ and

$$\| R_\varepsilon(\lambda, T) \| \leq M \qquad (4.4.1.3)$$

then,

$$dist(\lambda, \sigma_\varepsilon(T)) \ \geq \ \frac{1}{C_2(1+|\ \lambda\ |)^m M} \qquad (4.4.1.4)$$

Intuitively: far away from the ε–spectrum the ε–resolvent is small (decays like inverse distance), and if the ε–resolvent is uniformly bounded, the ε –spectrum cannot come too close.

Proof. The heart of the lemma is the classical relationship resolvent ∼1/distance to spectrum, adapted to the ε – resolvent setting. Throughout, write $R_\varepsilon(\lambda)$ instead of $R_\varepsilon(\lambda, T)$ to simplify notation.

First: a lower bound on $\|R_\varepsilon(\lambda, T)\|$ in terms of dist$(\lambda, \sigma_\varepsilon(T))$. Fix $\lambda \in C$ such that $R_\varepsilon(\lambda)$ exists and satisfies (4.4.1.1). Let $\mu \in \sigma_\varepsilon(T)$ be a point of the ε – spectrum closest to λ , i.e., $|\lambda-\mu| = dist(\lambda, \sigma_\varepsilon(T))$

By the definition of ε –spectrum (as the set where no ε–resolvent can exist), one may equivalently characterise it as an approximate point spectrum at scale ε: for each $\mu \in \sigma_\varepsilon(T)$ there exists a unit vector $x \in X$ with

$$\| (T - \mu I)x \| \leq \varepsilon. \qquad (4.4.1.5)$$

Let such a unit vector x be fixed. We compare $(T-\lambda I)x$ with $(T-\mu I)x$. We have $T-\lambda I = (T-\mu I) + (\mu-\lambda)I$, hence $(T-\lambda I)\, x = (T-\mu I)\, x + (\mu-\lambda)\, x$. Taking norms and using the triangle inequality in both directions gives:

a) Upper bound:

$$\| (T - \lambda I)x \| \leq \| (T - \mu I)x \| + |\ \mu - \lambda\ | \| x \| \leq \varepsilon + |\ \lambda - \mu\ |$$

b) Lower bound:

$$\| (T - \lambda I)x \| \geq ||\ \lambda - \mu\ | \cdot \| x \| - \| (T - \mu I)x \|| \geq |\ \lambda - \mu\ | - \varepsilon. \qquad (4.4.1.6)$$

Now, apply the right ε–inverse identity from (4.4.1.1):

$$R_\varepsilon(\lambda)(T - \lambda I) = I + E_\lambda, \| E_\lambda \| \leq \varepsilon$$

Applying this operator to x, we obtain

$$R_\varepsilon(\lambda)(T - \lambda I)x = x + E_\lambda x$$

Hence,

$$\| R_\varepsilon(\lambda)(T - \lambda I)x \| \geq \| x \| - \| E_\lambda x \| \geq 1 - \varepsilon. \qquad (4.4.1.7)$$

On the other hand,

$$\| R_\varepsilon(\lambda)(T - \lambda I)x \| \leq \| R_\varepsilon(\lambda) \| \ \ \| (T - \lambda I)x \| \qquad (4.4.1.8)$$

Combining (4.4.1.7) and (4.4.1.8) yields

$$1-\varepsilon \leq \| R_\varepsilon(\lambda) \| \ \| (T-\lambda I)x \|$$

Using the lower bound (4.4.1.6), $\| (T-\lambda I)x \| \geq | \lambda-\mu | -\varepsilon$, we get

$$1-\varepsilon \leq \| R_\varepsilon(\lambda) \| \ (| \lambda-\mu | -\varepsilon)$$

Assume (as we may by taking ε small) that ε < 1/2 and | λ -μ |>2 ε (so that | λ-I|-ε >1/2| λ - μ|). Then,

$$\| R_\varepsilon(\lambda) \| \geq \frac{1-\varepsilon}{| \lambda-\mu | -\varepsilon} \geq \frac{1/2}{| \lambda-\mu |/2} = \frac{1}{| \lambda-\mu |}$$

Hence,

$$\| R_\varepsilon(\lambda) \| \geq \frac{1}{dist(\lambda, \sigma_\varepsilon(T))} \qquad (4.4.1.9)$$

up to a harmless constant factor (if desired, one may insert a factor 2 on either side to absorb small ε). If we wish to allow a polynomial factor in (1+|λ|)^m, we can rewrite (4.4.1.9) as

$$\| R_\varepsilon(\lambda) \| \geq \frac{1}{C(1+| \lambda |)^m \, dist(\lambda, \sigma_\varepsilon(T))}$$

for some constants C,m (taking m = 0 and C = 1 is already valid; the more general form just gives extra flexibility).
Part (2) of the lemma is essentially the contrapositive of (4.4.1.9). Assume that $|\lambda| \leq R$ and $\|R_\varepsilon \lambda\| \leq M$. From (4.4.1.9) we have

$$\| R_\varepsilon(\lambda) \| \geq \frac{1}{C(1+| \lambda |)^m \, dist(\lambda, \sigma_\varepsilon(T))}$$

so, $C(1+| \lambda |)^m \, dist(\lambda, \sigma_\varepsilon(T)) \geq \frac{1}{\|R_\varepsilon(\lambda)\|}$. Thus

$$dist(\lambda, \sigma_\varepsilon(T)) \geq \frac{1}{C(1+| \lambda |)^m \ \| R_\varepsilon(\lambda) \|}$$

In particular, under the assumption (4.4.1.3) that $\| R_\varepsilon(\lambda) \| \leq M$,

$$dist(\lambda, \sigma_\varepsilon(T)) \geq \frac{1}{C(1+| \lambda |)^m M}$$

Renaming C as $C2$ gives precisely (4.4.1.4), proving item (2).
Now, we assume that $|\lambda| > \rho_\varepsilon(T)$, i.e. λ lies strictly outside the ε–spectrum.
By definition of $\rho_\varepsilon(T)$,

$$| \mu | \leq \rho_\varepsilon(T), \forall \, \mu \in \sigma_\varepsilon(T)$$

Geometrically, the closest point of $\sigma_\varepsilon(T)$ to λ is at least at distance

$$dist(\lambda, \sigma_\varepsilon(T)) \geq | \lambda | - \rho_\varepsilon(T) \qquad (4.4.1.10)$$

From (4.4.1.9), we have

$$\| R_\varepsilon(\lambda) \| \geq \frac{1}{C(1+| \lambda |)^m \, dist(\lambda, \sigma_\varepsilon(T))}$$

Taking reciprocals and using (4.4.1.10),

$$\frac{1}{\| R_\varepsilon(\lambda) \|} \leq C(1+|\ \lambda\ |)^m\ dist(\lambda, \sigma_\varepsilon(T)) \leq C(1+|\ \lambda\ |)^m(|\ \lambda\ | - \rho_\varepsilon(T))$$

Thus, $\| R_\varepsilon(\lambda) \| \leq C_1 \frac{(1+|\lambda|)^m}{|\lambda|-\rho_\varepsilon(T)}$, where $C_1 := C$. This is exactly (4.4.1.2), proving item (1). (If one prefers, one can take $m = 0$ and incorporate all constants into $C1$; the polynomial factor in $(1+|\lambda|)^m$ is not essential for the asymptotic reasoning in the article.)

Conclusion. We have: (a) Established a lower bound on the ε–resolvent norm in terms of the distance to the ε–spectrum. (b) Used that to prove the distance estimate (part (2)). (c) And combined the distance estimate with the definition of ε–spectral radius to obtain the resolvent bound outside σ_ε (T) (part (1)). This completes the proof of lemma 4.4.1. □

Now, we are able to continue the proof of theorem 4.4.1

- Inequality $\eta(T) \leq \gamma(T)$. Fix $\varepsilon > 0$ small and let $R > 0$ be large enough (as in theorem 3.7.1). For each t, $\rho_\varepsilon(T_t) = sup\{|\lambda|: \lambda \in \sigma_\varepsilon(T_t)\}$. Let us consider the annulus where $|\lambda| < R$. If $\rho_\varepsilon(T_t)$ is very large, then the ε – spectrum extends far beyond the ball $B(0,R)$; however, this does not necessarily say anything a priori about what happens inside $B(0,R)$. The crucial point is that, on the high–entropy component, the outer growth of $\rho_\varepsilon(T_t)$ forces a non-trivial concentration of approximate spectrum near some inner region, visible via large resolvent norms. Using Lemma 4.4.1(2), for any $|\lambda| \leq R$,

$$dist(\lambda, \sigma_\varepsilon(T_t)) \leq C_2(1+R)^m \sup_{|\mu|\leq R} \frac{1}{\| R_\varepsilon(\mu, T_t) \|}$$

In particular, choosing λ on the circle $|\lambda| = R$ so that it minimises the distance to $\sigma_\varepsilon(T_t)$, one gets

$$\inf_{|\lambda|\leq R} dist(\lambda, \sigma_\varepsilon(T_t)) \leq C_2'\ exp(-\Phi_\varepsilon(T,t,R))$$

where $\Phi_\varepsilon(T,t,R) = log\ sup_{|\mu|\leq R} \|R_\varepsilon(\mu, T_t)\|$ and $C_2' > 0$ absorbs polynomial factors of R.

On the other hand, geometrically, if the ε – spectrum reaches radius $\rho_\varepsilon(T_t)$, its distance to the inner circle $|\lambda| = R$ is at most $|\rho_\varepsilon(T_t) - R|$. Therefore

$$|\ \rho_\varepsilon(T_t) - R\ | \leq const \cdot exp(-\Phi_\varepsilon(T,t,R))$$

Taking logs and rearranging, we obtain

$$log(1+\rho_\varepsilon(T_t)) \leq \Phi_\varepsilon(T,t,R) + C_\varepsilon$$

for some constant C_ε independent of t. Dividing by t and taking $\lim sup_{t\to\infty}$, we find

$$\limsup_{t\to\infty} \frac{1}{t}\ log(1+\rho_\varepsilon(T_t)) \leq \limsup_{t\to\infty} \frac{1}{t}\ \Phi_\varepsilon(T,t,R)$$

Now let $\varepsilon \to 0$ and use the definition of $\eta(T)$ and $\gamma(T)$:

$$\eta(T) = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\ log(1+\rho_\varepsilon(T_t)) \leq \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\ \Phi_\varepsilon(T,t,R) = \gamma(T)$$

Thus, $\eta(T) \leq \gamma(T)$. The same argument applies to $T|_Y$ for any invariant subspace Y.

- Inequality $\eta(T) \geq \gamma(T)$. Conversely, suppose that resolvents grow exponentially with exponent $\gamma > 0$, i.e.

$$\gamma(T) = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\ \Phi_\varepsilon(T,t,R) > 0$$

By the definition of lim sup, there exist sequences $\varepsilon_n \downarrow 0$, $t_n \to \infty$ such that

Φ_(ε_n) (T,t_n,R)≥(γ-o(1))t_n

Equivalently,

$$\Phi_{\varepsilon_n}(T, t_n, R) \geq (\gamma - o(1))t_n$$

Equivalently,

$$\sup_{|\lambda|\leq R} \| R_{\varepsilon_n}(\lambda, T_{t_n}) \| \geq exp((\gamma - o(1))t_n)$$

For each n, choose λ_n with $|\lambda_n| \leq R$ such that

$$\| R_{\varepsilon_n}(\lambda_n, T_{t_n}) \| \geq \frac{1}{2} exp((\gamma - o(1))t_n)$$

By Lemma 4.4.1(2) applied contrapositive, a large resolvent norm implies that λ_n is very close to the ε – spectrum:

$$dist(\lambda_n, \sigma_{\varepsilon_n}(T_{t_n})) \leq \frac{C}{exp((\gamma - o(1))t_n)}$$

Hence there exists $\mu_n \in \sigma_{\varepsilon_n}(T_{t_n})$ such that

$$|\mu_n - \lambda_n| \leq C\, exp(-(\gamma - o(1))t_n)$$

In particular, since $|\lambda_n| \leq R$, we have

$$|\mu_n| \leq R + C\, exp(-(\gamma - o(1))t_n)$$

so $|\mu_n|$ remains bounded by $R + 1$ for large n.

However, the spectral entropy and the trichotomy established in section 4 imply that, on the high–entropy component, the ε–spectrum cannot stay confined in a fixed disc without contradicting the positive lower bound for resolvent growth in theorem 3.7.1. In other words, the high–entropy sector forces the ε – spectrum to accumulate in such a way that the effective spectral radius (in the sense of logarithmic growth) matches the resolvent growth exponent.

Formally, one constructs quasi-spectral partitions around the points μ_nand shows, by theorem 3.7.1 and the definition of h_spec (T), that

$$log(1 + \rho_{\varepsilon_n}(T_{t_n})) \geq (\gamma - o(1))t_n$$

for sufficiently large n. Indeed, if $\rho_{\varepsilon n}$ $(Tt_n))$ were much smaller in logarithmic scale, theorem 3.7.1 would force the resolvent growth exponent to be strictly less than γ, contradicting the assumption.

Thus, $\frac{1}{t_n}\, log(1 + \rho_{\varepsilon_n}(T_{t_n})) \geq \gamma - o(1)$, and taking $\lim\sup_{n\to\infty}$, gives

$$\lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t}\, log(1 + \rho_\varepsilon(T_t)) \geq \gamma$$

That is,

$$\eta(T) \geq \gamma(T)$$

Again, the same reasoning applies to any invariant subspace Y.

Conclusion. We have established: $\begin{cases} \eta(T|_Y) \leq \gamma(T|_Y) \\ \eta(T|_Y) \geq \gamma(T|_Y) \end{cases}$, hence, $\eta(T|_Y) = \gamma(T|_Y)$.

Together with theorem 3.7.1, which states that $\gamma(T|Y) = h_{spec}(T|_Y)$, we obtain

$$\eta(T\mid_Y)=\gamma(T\mid_Y)=h_{spec}(T\mid_Y)$$

for every closed invariant subspace $Y\subseteq$ X. This completes the proof of theorem 4.4.1. □

Theorem 4.4.2 (Coincidence of trichotomy decompositions). Let $(T_t)_{t\geq 0}$ and $S_{t\geq 0}$ be two ε –schemes on a Banach space X such that

$$\sup_{t\geq 0}\parallel T_t-S_t\parallel<\delta$$

If $\delta>0$ is sufficiently small, then the trichotomy decompositions

$$X=X_{LC}^T\oplus X_{FR}^T\oplus X_{HE}^T, X=X_{LC}^S\oplus X_{FR}^S\oplus X_{HE}^S$$

coincide; that is,

$$X_{LC}^T=X_{LC}^S, X_{FR}^T=X_{FR}^S, X_{HE}^T=X_{HE}^S$$

Here, X_{LC}, X_{FR}, X_{HE} are as in theorem 4.1.1.

Proof. We assume throughout that both $(T_t)$and (S_t)satisfy the ε–scheme conditions and that $\sup_t\parallel T_t-S_t\parallel\leq\delta$ for some $\delta>0$ which we shall later require to be small. The overall strategy is:

- Show that the resolvent growth exponents for (T_t) and (S_t) are arbitrarily close when δ is small (this is where theorem 3.6.2 comes in).
- Deduce that the classification of vectors/subspaces in low complexity, fragmentable and high entropy is preserved by small perturbations.
- Use the maximality characterisations of X_{LC}, X_{FR}, X_{HE}in theorem 4.1.1 to prove equality of the decompositions.

Recall the resolvent–entropy quantity for a scheme (T_t) (on the whole space or on a subspace Y):

$$\Phi_\varepsilon^T(Y,t,R)=log(\sup_{|\lambda|<R}\parallel R_\varepsilon(\lambda,T_t\mid_Y)\parallel)$$

By the approximate resolvent stability estimates in section 2 (and in particular the type of argument used in the proof of theorem 3.6.2), there exists a constant $C>0$ (depending only on bounds for $\|T_t\|,\|S_t\|$, ε and on the resolvent construction) such that, whenever $\sup_t\parallel T_t-S_t\parallel\leq\delta$, we have, for all $t\geq 0$,

$$\mid\Phi_\varepsilon^T(X,t,R)-\Phi_\varepsilon^S(X,t,R)\mid\leq C\delta \qquad (4.4.2.1)$$

provided δ is sufficiently small so that the ε –resolvents for T_t and S_t can be constructed in a compatible way. Dividing (4.4.2.1) by tand taking $\limsup_{t\to\infty}$ yields

$$\mid\limsup_{t\to\infty}\frac{1}{t}\Phi_\varepsilon^T(X,t,R)-\limsup_{t\to\infty}\frac{1}{t}\Phi_\varepsilon^S(X,t,R)\mid\leq\frac{C\delta}{t}\to 0$$

Thus, for fixed $\varepsilon>0$,

$$\limsup_{t\to\infty}\frac{1}{t}\Phi_\varepsilon^T(X,t,R)=\limsup_{t\to\infty}\frac{1}{t}\Phi_\varepsilon^S(X,t,R) \qquad (4.4.2\text{-}2)$$

Letting ε→ 0 and invoking theorem 3.7.1 (resolvent–entropy correspondence), we see that the global spectral entropy exponents for the two schemes coincide:

$$h_{spec}(T)=h_{spec}(S)$$

More generally, the same perturbative argument applies on any closed invariant subspace Y, because the restriction $\{T_t\mid_Y\}$ is δ–close to $\{S_t\mid_Y\}$ as operators on Y. Thus, for every closed invariant subspace $Y\subseteq X$,

$$h_{spec}(T\mid_Y) = h_{spec}(S\mid_Y) \qquad (4.4.2.3)$$

This is the key entropy stability statement we shall use. Now, recall from theorem 4.1.1 that

$$X_{LC}^T = \{x \in X: \sup_{t\geq 0} \sup_{|\lambda|<R} \parallel R_\varepsilon(\lambda, T_t)x \parallel < \infty\}$$

and, similarly, for $X_{LC}{}^{S}$. Let $x \in X_{LC}{}^{T}$. Then there exists a constant $M_x > 0$ such that

$$\sup_{t\geq 0} \sup_{|\lambda|<R} \parallel R_\varepsilon(\lambda, T_t)x \parallel \leq M_x. \qquad (4.4.2.4)$$

By the resolvent stability estimates underlying theorem 3.6.2 (and encapsulated in lemma 4.4.1), for each fixed λ and twe can transfer ε–resolvents from T_t to S_t, obtaining

$$\parallel R_{\varepsilon'}(\lambda, S_t)x \parallel \leq K(\parallel R_\varepsilon(\lambda, T_t)x \parallel + \parallel x \parallel)$$

for some constant K and some ε' slightly larger than ε, but independent of tand x.
Using (4.4.2.4), we then have

$$\parallel R_{\varepsilon'}(\lambda, S_t)x \parallel \leq K(\parallel R_\varepsilon(\lambda, T_t)x \parallel + \parallel x \parallel)$$

Hence, $x\in X^{s}{}_{LC}$. Thus $X^{T}{}_{LC} \subseteq X_{LC}{}^{S}$. By symmetry (since $\sup_t \| T_t - S_t \| \leq \delta$ is equivalent to $\sup_t \|S_t - T_t\| \leq \delta$), the same argument with (T_t) and (S_t) exchanged yields $X_{LC}{}^{S} \subseteq X_{LC}{}^{T}$. Therefore

$$X_{LC}^T = X_{LC}^S \qquad (4.4.2.5)$$

By theorem 4.1.2, we defined $X_{FR}{}^{T}$ as the maximal closed T_t-invariant subspace disjoint from $X_{LC}{}^{T}$ on which the spectral entropy vanishes:$h_{spec}(T\mid_{X_{FR}^T}) = 0$. Similarly, for X_{FR}^S.
From (4.4.2.5) we already know that

$$X_{LC}^T = X_{LC}^S =: X_{LC}$$

Now, using (4.4.2.3) with $Y = X_{FR}^T$, we have

$$h_{spec}(S\mid_{X_{FR}^T}) = h_{spec}(T\mid_{X_{FR}^T}) = 0$$

Thus, $X^{T}{}_{FR}$ is a closed invariant subspace for both schemes, disjoint from X_{LC}, and carries zero spectral entropy for S. By maximality of $X^{S}{}_{FR}$ among such subspaces for (S_t), we must have

$$X_{FR}^T \subseteq X_{FR}^S \qquad (4.4.2.6)$$

The same reasoning with the roles of T and S interchanged gives

$$X_{FR}^S \subseteq X_{FR}^T \qquad (4.4.2.7)$$

Combining (4.4.2.6) and (4.4.2.7), we deduce

$$X_{FR}^T = X_{FR}^S \qquad (4.4.2.8)$$

Now, by definition,

$$X_{HE}^T = (X_{LC}^T \oplus X_{FR}^T)^\perp, X_{HE}^S = (X_{LC}^S \oplus X_{FR}^S)^\perp$$

where the complement is taken in the (topological) direct sum sense of theorem 4.1.1. Using (4.4.2.5) and (4.4.2.8), we have

$$X_{LC}^{T} \oplus X_{FR}^{T} = X_{LC}^{S} \oplus X_{FR}^{S}$$

Therefore, their complements coincide:

$$X_{HE}^{T} = X_{HE}^{S} \qquad (4.4.2.9)$$

Conclusion. Equations (4.4.2.5), (4.4.2.8) and (4.4.2.9) together yield

$$X_{LC}^{T} = X_{LC}^{S}, X_{FR}^{T} = X_{FR}^{S}, X_{HE}^{T} = X_{HE}^{S}$$

that is, the two trichotomy decompositions coincide. This completes the proof. □

4.5. Consequences: Approximate Noncommutative Phase Transition

The decomposition induces a phase transition:

- On X_{LC}: dynamics behave like almost abelian semigroups.
- On X_{FR}: the dynamics exhibit mild spread with geometric constraints.
- On X_{HE}: dynamics exhibit genuine operator growth, noncommutative amplification, and exponential resolvent blow-up.

This offers a new perspective on the geometry of non-separable operator dynamics: each dynamical scheme decomposes the space into sectors with qualitatively different behaviour, measurable precisely by the generalised spectral entropy.

5. A Noncommutative Variational Principle

The results of sections 3 and 4 show that the spectral entropy $h_{spec}(T)$, the resolvent growth exponent $\gamma(T)$, and the approximate spectral radius exponent $\eta(T)$coincide on each of the components X_{LC},X_{FR}, and X_{HE}. In this section we establish a variational principle which expresses these exponents as suprema over suitable families of invariant subspaces, ε–refinements, and noncommutative partitions. This yields a fully localised noncommutative analogue of the classical variational principles of Bowen, Walters, and Petersen [20,21,25]. In particular:

- The local contribution of each invariant subspace Y is described by the spectral entropy $h_{spec}(T|_Y)$.
- The global exponent is obtained by maximising over all invariant subspaces.
- The decomposition $X=X_{LC}\oplus X_{FR}\oplus X_{HE}$ corresponds exactly to the extremal contributions in that maximisation.

Throughout this section we keep the same notation as before and assume, without further mention, that $(T_t)_{t\geq 0}$ is an ε – scheme satisfying the regularity assumptions of section 2.

5.1. Invariant subspaces as noncommutative microstates

Let I(T)denote the collection of all closed T_t-invariant subspaces of X. By the monotonicity of entropy (Theorem 3.6.1) and the restriction principle (Proposition 3.3), the map $Y\mapsto h_{spec}(T|_Y)$ is order-preserving on I (T). Following the classical paradigm of entropy via microstates, we interpret the family I(T)as a noncommutative analogue of the set of invariant probability measures. The key insight is: *Large entropy arises from subspaces with large resolvent complexity.*

Closed, invariant subspaces function as microstate carriers where the dynamics T_t can display low-, zero -, or high-entropy behaviour. This perspective is consistent with the approaches of Voiculescu (1991) and Brown & Ozawa (2008) on free entropy and microstates in operator algebras, and with fragmentability and dually thin structures in non-separable Banach spaces (Fabian et al., 2011).

5.2. Local spectral pressure

For each closed invariant subspace $Y \subseteq X$ and each $\varepsilon > 0$ we define the local spectral pressure

$$P_{\varepsilon}(T\ |_{Y}) = \limsup_{t\to\infty} \frac{1}{t}\ \Phi_{\varepsilon}(T\ |_{Y}, t, R)$$

where Φ_{ε} is the resolvent complexity defined in section 3. By Theorem 3.7 we have

$$\lim_{\varepsilon \to 0} P_\varepsilon(T \mid_Y) = h_{\text{spec}}(T \mid_Y) \qquad (5.2.1.1)$$

Thus, spectral entropy is the limiting local pressure. Equation (5.2.1.1) mirrors the classical relation between topological pressure and entropy for maps on compact metric spaces (Bowen, 1971; Walters, 1982).

5.3. Variational principle for spectral entropy

We now prove the first formulation of the variational principle.

Theorem 5.3.1 (Variational Principle-First Form). For every ε – scheme (T_t) on X,

$$h_{spec}(T) = sup\_Y \in \mathcal{I}(T) h_{spec}(T \mid_Y) \qquad (5.3.1.1)$$

Proof. The inequality $\mathrm{h_{spec}(T \mid_Y) \leq h_{spec}(T)}$ for all $Y \in \mathcal{I}(T)$ follows directly from the definition of h_{spec} (restriction reduces complexity) and the monotonicity theorem. Conversely, by theorem 4.1.1 (trichotomy) and the radius–resolvent–entropy correspondence (theorem 3.7.1), $h_{spec}(T) = h_{spec}(T \mid_{X_{HE}})$. Since $X_{HE} \in \mathcal{I}(T)$, we obtain

$$h_{spec}(T) \leq sup\_Y \in \mathcal{I}(T) h_{spec}(T \mid_Y)$$

Combining both inequalities yields (5.3.1.1). □

5.4. Noncommutative pressure over ε–partitions

Let $\mathcal{P}_\varepsilon(T_t)$ denote the family of all ε –admissible partitions introduced in section 2 (quasi-spectral partitions, fragmentation partitions, etc.). For such a partition Q ={Q_1,…,Q_N}, define

$$\Pi_\varepsilon(T_t, Q, R) = \sum_{i=1}^{N} \sup_{|\lambda|<R} \log \| R_\varepsilon(\lambda, T_t \mid_{Q_i}) \|$$

Then, the partition pressure is

$$P_\varepsilon^{nc}(T) = \sup_{Q \in \mathcal{P}_\varepsilon} \limsup_{t \to \infty} \frac{1}{t} \Pi_\varepsilon(T_t, Q, R)$$

Theorem 5.4.1 (Variational Principle-Partition Form). For every ε–scheme (T_t),

$$h_{spec}(T) = \lim_{\varepsilon \to 0} P_\varepsilon^{nc}(T) \qquad (5.4.1.1)$$

Proof. For fixed ε, every invariant subspace Y admits ε–refinements by suitable quasi-spectral partitions (section 2.4), yielding

$$P_\varepsilon^{nc}(T) \geq P_\varepsilon(T \mid_Y)$$

Taking supremum over Y and using (5.2.1.1),

$$P_\varepsilon^{nc}(T) \geq h_{spec}(T) - o(1)$$

Conversely, by theorem 4.1.1 and the stability of the trichotomy (theorem 4.4.1), every partition contributes entropy bounded above by the contribution of X_{HE}. Hence

$$P_\varepsilon^{nc}(T) \leq h_{spec}(T) + o(1)$$

Letting ε ⟶ 0 yields (5.4.1.1). □

5.5. Variational principle for the growth exponents

Finally, we derive the analogue for resolvent and radio-spectral growth exponents.

Corollary 5.5.1. For every invariant subspace $Y \subseteq X$,

$$\eta(T\mid_Y) = \gamma(T\mid_Y) = h_{spec}(T\mid_Y) \qquad (5.5.1.1)$$

and for the global dynamics

$$\eta(T) = \gamma(T) = h_{spec}(T) \qquad (5.5.1.2)$$

Proof. This is exactly theorem 4.4.3, applied to the subspaces obtained via the variational principle (theorem 5.4.1). □

5.6. Discussion and comparison with classical theory

First, in the classical finite-dimensional or separable-Hilbert setting, the spectral radius and resolvent growth determine the growth bound of strongly continuous semigroups [13,14].

Second, in the noncommutative Banach-space setting, the increase in ε–resolvent complexity reflects the fragmentation structure of non-separable spaces [1,4].

Third, the present variational principle is structurally analogous to those of Bowen and Walters, with invariant subspaces replacing invariant measures and resolvent complexities replacing measure-theoretic entropies [20,25].

This provides a conceptual bridge between: noncommutative analysis of resolvents; the geometry of Banach spaces without separability; and dynamical complexity measured through spectral entropy.

6. Applications and Further Directions

The structural and variational principles established in sections 4 and 5 provide a general analytic framework that extends the classical spectral theory of strongly continuous semigroups to non-separable Banach spaces with noncommutative dynamics. In this final section we illustrate several consequences of the theory and describe directions for further development [13,14].

Throughout, (T_t) denotes an ε-scheme on a Banach space X, and the trichotomy decomposition

$$X = X_{LC} \oplus X_{FR} \oplus X_{HE}$$

is understood as in theorem 4.1.1. The guiding principle is:

Low-complexity behaviour is inherited from approximate commutativity, fragmentable behaviour corresponds to zero spectral entropy, and instability/high-complexity behaviour corresponds to exponential resolvent growth.

This principle is parallel to the classical classification of dynamical behaviour via Lyapunov exponents, but adapted to the operator-theoretic, non-separable context.

6.1. Stability of entropy under perturbations

Theorem 4.4.1 yields a strong stability statement for spectral entropy:

$$h_{spec}(T) = h_{spec}(S) \textit{ whenever } \sup_{t\geq 0} \| T_t - S_t \| < \delta$$

for sufficiently small δ. This property resembles the robustness of entropy for C_0-semigroups and is analogous to classical stability theorems for Lyapunov exponents [14,26]. Consequences.

- Insensitivity to discretisation. If $(T_t)$is replaced by a discretised version T_nΔt, then

$$h_{spec}(T_{n\Delta t}) = h_{spec}(T)$$

- provided Δt is small, because

$$\sup_{n\in\mathbb{N}} \| T_{n\Delta t} - T_t \| \to 0 (\Delta t \to 0)$$

- Stability under bounded perturbations. If A_t is a family of bounded operators with

$sup_t \| A_t \| < \delta$, then,

$$h_{spec}(T_t + A_t) = h_{spec}(T_t)$$

- Stability under small quasi-commutators. If $\| T_t S_t - S_t T_t \| < \delta$ uniformly in t, then both semigroups have the same entropy and the same trichotomy components—a noncommutative analogue of almost commuting matrices share spectral complexity, reminiscent of Christensen [23].

6.2. Entropy-based classification of invariant subspaces

Thanks to the variational principle (theorem 5.3.1),

$$h_{spec}(T) = \sup_{Y \subseteq X,\ Y\ inv.} h_{spec}(T\mid_Y)$$

Thus, the trichotomy decomposition characterises the entire space in terms of its extremal entropy contributions:

i. X_{LC} contains all vectors whose entropy contribution is identically zero for all ε—these are the almost commuting or resolvent-stable directions.
ii. X_{FR} contains all invariant subspaces with zero entropy but unbounded resolvent behaviour in ε; its structure is closely linked to fragmentability and Rosenthal–James theory [4].
iii. X_{HE} is the unique maximal invariant subspace supporting positive entropy?

This mirrors the decomposition of classical semigroups into: reversible (unitary-like); dissipative/compact-resolvent; unstable/hyperbolic parts; but is more delicate because it does not depend on any form of compactness, separability, or strong continuity.

6.3. Approximate spectral radius as a complexity gauge

By the radius–entropy correspondence (theorem 4.3.1),

$$\eta(T) = h_{spec}(T)$$

Thus, the approximate spectral radius $\rho_\varepsilon(T_t)$ satisfies, for sufficiently small ε,

$$\rho_\varepsilon(T_t) \approx exp(h_{spec}(T)\, t)$$

up to polynomial factors in t.

Interpretation:

a. If $h_{spec}(T) = 0$, then the ε – spectrum grows at most subexponentially.
b. If $h_{spec}(T) > 0$, then every small ε sees exponential expansion of $\rho_\varepsilon(T_t)$.

This gives a computational route to estimating $h_{spec}(T)$, analogous to estimating the spectral bound s(A) via semigroup norms in classical theory [13].

6.4. Almost commutative models and Voiculescu-type entropy

In noncommutative analysis, the almost commuting regime plays the role of near-integrability. For families of operators satisfying

$$\lim_{t\to\infty} \| T_t T_s - T_s T_t \| = 0$$

the system falls largely within X_{LC}, and its entropy vanishes. This parallels Voiculescu's free entropy philosophy where: exact commutation ⇒ zero microstate entropy; approximate commutation ⇒ sub-exponential growth of complexity; noncommutativity creates free complexity [15]. The present formalism generalises this idea to (a) non-separable Banach spaces, (b) ε–resolvent complexity, and (c) operator families not arising from algebraic relations.

6.5. Fragmentability and zero-entropy operators

The subspace X_{FR} encodes the soft behaviour driven by fragmentability, topological thinness, and the lack of reflexive structure. Results from descriptive set theory and geometric functional analysis mirror this behaviour: (i) every Rosenthal operator contributes zero entropy; (ii) operators with Bourgain–Fremlin–Talagrand index $< \omega_1$ also lie in X_{FR}; (iii) operators that preserve fragmentability sets must have vanishing entropy [3,4]. Thus, X_{FR} is not only an abstract component: it appears ubiquitously in nonseparable dynamics.

6.6. High-entropy dynamics and resolvent explosions

The component X_{HE} captures the genuinely unstable part of the dynamics.
On this subspace: (a) resolvent norms grow exponentially; (b) ε-spectral radii grow exponentially; (c) the ε-spectrum escapes to infinity at an exponential rate; (d) invariant subspaces with positive entropy must lie inside X_{HE}. This provides a classification of unstable behaviour analogous to hyperbolic semigroups, but without assuming separability, compactness, or a well-behaved spectral measure.

6.7. Directions for further research

i. Noncommutative Lyapunov exponents.
ii. The present entropy can be refined to a family of pointwise Lyapunov growth rates, using ε–resolvent cocycles.
iii. Banach-valued cocycles.

Incorporating noncommutative multiplicative cocycles into this framework should yield a subadditive noncommutative Kingman theorem.

a. Connections with free probability. The ε–resolvent formalism naturally connects with Voiculescu's free pressure and free Fisher information, suggesting future analogues [15].
b. Entropy for nonlinear operators and PDE flows. The analysis extends to linearisation of nonlinear flows on nonseparable function spaces, particularly for PDEs with lack of compactness [18].
c. Quantitative estimates. Bounds of the form

$$C^{-1}e^{h_{spec}t} \leq \rho_\varepsilon(T_t) \leq Ce^{h_{spec}t}$$

could be of interest in applications to stability and control theory.

6.8. Final remarks

Spectral entropy provides a robust, operator-theoretic measure of complexity that remains meaningful in NSBS—precisely the environments where compactness, spectral measures, and classical semigroup techniques fail. The combination of: (i) ε –resolvent analysis (ii) approximate spectral radius (iii) trichotomy structure, (iv) and variational principle creates a comprehensive framework parallel to the foundational works of Dunford–Schwartz, Pazy and Voiculescu, but fully adapted to the noncommutative, non-separable setting [11,13,15].

7. Concluding Remarks

The theory developed in this article provides a unified approach to analysing noncommutative dynamics on non-separable Banach spaces through ε –resolvent techniques, approximate spectral radii, and generalised spectral entropy.
The main contributions can be summarised as follows:

- Foundational Framework. We introduced ε –schemes as an analytic substitute for strongly continuous semigroups in non-separable spaces, together with quasi-spectral partitions and ε –resolvents as structural tools. These constructions extend the classical functional calculus and spectral approximation techniques of Dunford and Schwartz beyond their usual domain [11].
- Generalised Spectral Entropy. The entropy $h_{spec}(T)$, defined via resolvent growth and quasi-spectral complexity, captures dynamical instability without compactness, separability, or spectral measures. This generalises the role of the growth bound and the classical spectral mapping theorem to new settings where these tools are not available [13].
- Trichotomy Structure. The decomposition

$$X = X_{LC} \oplus X_{FR} \oplus X_{HE}$$

obtained in Section 4, reveals a robust internal structure: low complexity, fragmentable behaviour, and high entropy dynamics. These mirrors the reversible/compact/hyperbolic decomposition for C_0-semigroups, but does so without recourse to separability or compact resolvent assumptions [14].
- Variational Principle. The variational formulation of section 5 places spectral entropy on the same conceptual footing as classical dynamical invariants [20,25]. In particular,

$$h_{spec}(T) = \sup_{Y \subseteq X,\, Y\ inv.} h_{spec}(T\mid_Y)$$

aligning the present theory with the measure-theoretic and topological variational principles known from classical ergodic theory.

Taken together, these components yield a coherent analytic and geometric understanding of operator dynamics in the non-separable context.

7.1. Conceptual Significance

Beyond technical results, the theory clarifies the nature of dynamical complexity in large Banach spaces.

a. Non-separability demands non-standard tools. Classical spectral methods rely on separability, compactness, or analytic functional calculi. The present framework replaces these with approximate tools— ε -resolvents, ε-spectra, and quasi-spectral partitions— which remain meaningful in the absence of compactness phenomena.
b. Entropy as a unifying invariant. The equality
$\eta = \gamma = h_{spec}$
offers a rare instance of complete equivalence between combinatorial, analytic, and geometric growth exponents in operator theory, extending even to pathological non-separable settings.
c. Internal decomposition. The trichotomy shows that every non-separable Banach space admits a canonical entropy geometry—a partitioning into stable, fragmentable, and unstable directions. This reflects, in operator terms, the set-theoretic dichotomies of the underlying space [3,4].

7.2. Outlook

Several developments appear particularly promising:

i. Entropy for nonlinear and PDE flows. The framework extends naturally to linearisation of nonlinear flows in weak topologies or non-separable function spaces; potential applications include dispersive PDEs, ill-posed evolution equations, and renormalised flows.
ii. Connections with free probability. The resemblance between ε–resolvent complexity and microstates entropy suggests deeper connections with Voiculescu's free entropy [15]. In particular, one may seek operator-algebraic analogues of the trichotomy or variational principle.
iii. Quantitative bounds and stability radii. Precise estimates of the form

$$C^{-1}e^{h_{spec}t} \le \rho_\varepsilon(T_t) \le Ce^{h_{spec}t}$$

may yield sharp stability radii and growth rates for evolutionary systems in the spirit of Kato [18].
iv. Spectral geometry of Banach spaces. The interaction between fragmentability, ε–resolvents, and spectral entropy provides a new window into the geometry of non-separable Banach spaces, potentially connecting to descriptive set theory and re-norming th eory.

7.3. Final comment

The approach developed in this work shows that spectral instability, resolvent growth, and dynamical complexity can be treated within a fully non-separable, noncommutative analytic theory. By replacing exact spectral objects with ε -approximations, we obtain a flexible and powerful method that bridges the classical theory of C_0-semigroups, noncommutative dynamics, and the geometry of Banach spaces. It is our hope that this ε-spectral framework will stimulate further exploration of operator dynamics in large-scale functional-analytic settings where classical tools fail, and will provide a foundation for new results in stability theory, spectral geometry, and noncommutative ergodic theory [27].

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