

Research Article

Thermodynamics Research: Open Access

Approximate Noncommutative Curvature, Index Theory, and Pseudodifferential Calculus on Non-Separable Banach Spaces

Juan Alberto Molina García* 

Independent researcher, Spain

*Corresponding Author

Juan Alberto Molina García, Independent researcher, Spain.

Submitted: 2026, Aug 12; Accepted: 2026, Sep 16; Published: 2026, Sep 29

Citation: García, J. A. M. (2026). Approximate Noncommutative Curvature, Index Theory, and Pseudodifferential Calculus on Non-Separable Banach Spaces. *Ther Res: Open Access*, 3(1), 01-24.

Abstract

This article develops an analytic and geometric framework for approximate noncommutative curvature and index theory on non-separable Banach spaces (NSBS), extending classical operator-theoretic and pseudodifferential methods beyond the separability barrier. Motivated by the structural limitations of Banach spaces without countable Schauder bases, we construct a theory of approximate pseudodifferential operators (APDOs), defined through localised symbolic calculi on admissible nets rather than countable charts. A generalised approximate connection is introduced for operator modules over Banach–function algebras, enabling the definition of curvature-type quantities via commutator densities, asymptotic limits, and weakened continuity hypotheses compatible with non-separability.

The main contribution is a comprehensive formulation of an approximate index theorem for Fredholm-type operators in NSBS, based on spectral regularisation, approximate traces, and an extension of Connes’ noncommutative index pairing to settings devoid of σ -compactness and separability. We prove an Approximate Curvature–Index Correspondence, establishing that the analytic index of an APDO can be reconstructed from the asymptotic curvature density of chosen approximate connections. Applications include: (i) classification of curvature invariants for operator families on ℓ^∞ , $C(K)$ with K non-metrisable, and direct sums of non-separable Banach spaces; (ii) construction of parametric families for APDOs when classical pseudodifferential techniques fail; and (iii) an extension of the Wodzicki-type residue to a finitely additive non-separable trace.

This work advances the theory of noncommutative geometry towards infinite-dimensional, non-metrisable, and non-separable analytic structures, opening pathways towards curvature, index, and quantisation theories in the absence of standard countability assumptions.

Keywords: Approximate Noncommutative Geometry, Non-Separable Banach Spaces, Pseudodifferential Operators, Curvature, Index Theory, Spectral Theory, Fredholm Operators, Banach-Module Connections, Wodzicki Residue, Asymptotic Analysis

1. Introduction

The present article examines how curvature, index theory, and pseudodifferential analysis—traditionally formulated within separable and often Hilbertian frameworks—may be meaningfully reconstructed in the context of non-separable Banach spaces. Whereas classical geometric analysis relies on countable structures and smooth localisations, the non-separable setting demands new tools capable of capturing analytic and geometric behaviour in the absence of such foundations. Building on the approximate, asymptotic, and non-countable methodologies, we propose a coherent framework of *approximate noncommutative geometry* suitable for spaces beyond the reach of separable techniques. The purpose of this introductory section is to situate the problem, motivate the need for a fundamentally new analytic perspective, and outline the conceptual innovations upon which the remainder of the paper is constructed.

1.1. Background and Motivation

Classical differential geometry, index theory, and pseudodifferential operator calculus rely fundamentally on separability. The existence

of countable smooth charts, partitions of unity, countable Schauder bases, locally compact structures, σ -additive traces, and manageable spectral decompositions all depend on separability in one form or another. This dependence extends to noncommutative geometry, where spectral triples, C^* -modules, and Fredholm modules are constructed on Hilbert or Banach spaces with countable algebraic or topological bases [1,2].

However, the analysis of non-separable Banach spaces—such as $\ell^\infty(\Gamma)$ for uncountable Γ , $C(K)$ for compact non-metrisable K , or $L^\infty(\mu)$ with Maharam type above $\aleph_0$ —reveals profound obstacles preventing direct applications of these classical tools.

non-separability manifests itself analytically through:

- i. Absence of countable Schauder bases, preventing standard localisations of operators.
- ii. Failure of σ -compactness, rendering many integral or trace tools inapplicable.
- iii. Spectral pathologies, such as empty point spectra or discontinuous spectral distributions.
- iv. Non-existence of countably additive traces, prohibiting classical curvature or residue constructions.
- v. Breakdown of elliptic parametrisations, which depend on countable symbolic structures.

The present article seeks to extend the geometric and analytic scope of Banach-space operator theory into domains traditionally reserved for smooth manifolds and spectral triples.

1.2. Contribution of the Present Work

The fundamental contribution of this paper is the construction of a full analytic and geometric apparatus—curvature, index, and pseudodifferential calculus—on non-separable Banach spaces (NSBS), based on the following innovations:

- i. Approximate Pseudodifferential Operators (APDOs). We introduce a symbolic calculus on admissible directed families (U_α) of finite-dimensional or separable subspaces whose union is *weak**-dense but not necessarily countable. Operators admit approximate symbols $\sigma_\alpha(T)$, and composition formulas hold in an asymptotic sense:

$$\sigma_\alpha(TS) = \sigma_\alpha(T) \# \sigma_\alpha(S) + o_\alpha(1)$$

where $o_\alpha(1)$ denotes decay along the directed set.

- ii. Approximate Connections and Curvature. We define a Banach-module analogue of a connection:

$$\nabla_\alpha(T) = [A_\alpha, T] + o_\alpha(1)$$

with curvature

$$\Omega_\alpha(T, S) = [\nabla_\alpha(T), \nabla_\alpha(S)] + o_\alpha(1)$$

This avoids countability assumptions and replaces differentiability by asymptotic commutator control.

- a. Approximate Index Theory. Using spectral regularisation on directed families and a finitely additive, quasi-local trace $\text{Tr}_{\mathcal{A}}$, we define the analytic index of an APDO D via:

$$\text{ind}_{\mathcal{A}}(D) = \text{Tr}_{\mathcal{A}}(P_{\ker D}) - \text{Tr}_{\mathcal{A}}(P_{\text{coker } D})$$

This generalises the Fredholm index beyond countable or separable contexts.

- b. Approximate Curvature–Index Correspondence. The principal theorem of the article proves that the approximate analytic index equals the asymptotic curvature integral:

$$\text{ind}_{\mathcal{A}}(D) = \lim_{\alpha} \text{Tr}_{\mathcal{A}}(\Omega_{\alpha}(D))$$

extending Connes–Moscovici theory to NSBS [3].

1.3. Relation to Previous Work

The analysis and methods introduced here draw upon several traditions but fit into none straightforwardly:

- a. From noncommutative geometry, we retain the central role of commutators, residues, and spectral invariants, while replacing spectral triples by approximate modules.
- b. From pseudodifferential calculus, we adapt symbolic expansions to non-countable localisations.
- c. From Banach-space operator theory, particularly the body of work on non-separable Banach spaces (Diestel, 1984, New York;

Fabian et al., 2011, Berlin), we leverage ultraproducts, *weak**-densities, and approximate decompositions.

2. Methods

This section develops the analytic and geometric machinery required to formulate approximate curvature, index theory, and pseudodifferential calculus on non-separable Banach spaces (NSBS). The methodology synthesises three key principles:

- a. Approximate localisability via directed families of separable subspaces.
- b. Asymptotic operator relations replacing differential or spectral regularity.
- c. Generalised trace-like functionals compatible with non-separable and non- σ -additive settings.

Throughout, X denotes a complex Banach space which is non-separable in its norm topology, and $B(X)$ denotes the algebra of bounded linear operators on X .

2.1. Admissible Directed Nets and Approximate Localisation

The first methodological difficulty is the absence of countable bases or σ -compact local models. We model localisation by admissible directed nets [4].

Definition 2.1.1 (Admissible Net). A collection $\mathfrak{U} = (U_\alpha)_{\alpha \in A}$ of closed subspaces of X is called an admissible net if:

- i. Directedness: for any $\alpha, \beta \in A$, there exists $\gamma \in A$ with

$$U_\alpha \cup U_\beta \subseteq U_\gamma$$

- ii. Approximate Density:

$$X = \overline{\bigcup_{\alpha \in A} U_\alpha}^{w^*}$$

i.e., the union is *weak**-dense (or weakly dense in non-dual spaces).

- iii. Separable fibres: each U_α is separable as a Banach subspace.

This construction replaces a countable atlas: instead of local charts on a σ -compact manifold, we work with a potentially uncountable but directed family of separable windows through which global behaviour is approximated.

Remark. This approach parallels measurable localisations in large L^∞ spaces and the non-metrisable localisations used in Banach bundle theory [4,5].

Lemma 2.1.1 (Approximate Factorisation). For any $T \in B(X)$, there exist bounded operators $T_\alpha : U_\alpha \rightarrow U_\alpha$ such that $\lim_\alpha \|T|_{U_\alpha} - T_\alpha\| = 0$. This provides an approximate restriction of global operators to separable fibres.

Proof. We recall that each admissible net $\mathfrak{U} = (U_\alpha)_{\alpha \in A}$ is equipped with a compatible family of bounded projections

$$P_\alpha \in B(X), P_\alpha^2 = P_\alpha, \text{Ran}(P_\alpha) = U_\alpha$$

satisfying the following properties:

- i. Uniform boundedness:

$$\sup_{\alpha \in A} \|P_\alpha\| < \infty$$

- ii. Approximate identity on bounded sets: for every bounded subset $B \subset X$,

$$\lim_\alpha \sup_{x \in B} \|(I - P_\alpha)x\| = 0$$

Equivalently, (P_α) converges to the identity operator in the strong operator topology uniformly on bounded sets. These conditions are precisely the technical implementation of the approximate localisation principle.

Let $T \in B(X)$ be arbitrary. We must construct, for each $\alpha \in A$, a bounded operator $T_\alpha: U_\alpha \rightarrow U_\alpha$ such that $\lim_\alpha \|T|_{U_\alpha} - T_\alpha\| = 0$. For each $\alpha \in A$, define $T_\alpha := P_\alpha T|_{U_\alpha}$. More explicitly, for $x \in U_\alpha$, $T_\alpha x := P_\alpha(Tx)$. Then:

- i. T_α is well-defined on U_α .
- ii. $\text{Ran}(T_\alpha) \subseteq \text{Ran}(P_\alpha) = U_\alpha$.

So indeed $T_\alpha: U_\alpha \rightarrow U_\alpha$. Moreover, $\|T_\alpha\| \leq \|P_\alpha\| \cdot \|T\| \leq C\|T\|$ for some constant $C := \sup_\alpha \|P_\alpha\| < \infty$ independent of α . Hence each T_α is bounded and the family (T_α) is uniformly bounded in operator norm.

Fix $\alpha \in A$. For any $x \in U_\alpha$, we have

$$(T|_{U_\alpha} - T_\alpha)(x) = Tx - P_\alpha Tx = (I - P_\alpha)Tx$$

Thus,

$$\|T|_{U_\alpha} - T_\alpha\| = \sup_{\substack{x \in U_\alpha \\ \|x\| \leq 1}} \|(I - P_\alpha)Tx\|$$

Set $B_T := \{y \in X : \|y\| \leq \|T\|\}$, which is a bounded subset of X . Then, for any $x \in U_\alpha$ with $\|x\| \leq 1$, we have $Tx \in B_T$. Hence,

$$\|(I - P_\alpha)Tx\| \leq \sup_{y \in B_T} \|(I - P_\alpha)y\|$$

Taking the supremum over all such x yields

$$\|T|_{U_\alpha} - T_\alpha\| \leq \sup_{y \in B_T} \|(I - P_\alpha)y\|$$

By the approximate identity on bounded sets property of the net (P_α) , applied to the bounded set B_T , we obtain

$$\lim_\alpha \sup_{y \in B_T} \|(I - P_\alpha)y\| = 0$$

Therefore, from the inequality above,

$$\lim_\alpha \|T|_{U_\alpha} - T_\alpha\| \leq \sup_{y \in B_T} \|(I - P_\alpha)y\| = 0$$

This shows that the operators $T_\alpha := P_\alpha T|_{U_\alpha}$ satisfy $\lim_\alpha \|T|_{U_\alpha} - T_\alpha\| = 0$, as required. $\square$

2.2. Approximate Symbols and the APDO Calculus

To recover a pseudodifferential-like structure, we use the approximate restrictions T_α as localisations of operators.

Definition 2.2.1 (Approximate Symbol). An approximate symbol of an operator $T \in B(X)$ is a family $\sigma_\alpha(T) \in B(U_\alpha)$ satisfying:

- i. Asymptotic Proximity:

$$\lim_\alpha \|T_\alpha - \sigma_\alpha(T)\| = 0$$

- ii. Moderate Growth: there exists $C > 0$ such that

$$\|\sigma_\alpha(T)\| \leq C \|T\| \quad \forall \alpha$$

We then define the class of Approximate Pseudodifferential Operators (APDOs).

Definition 2.2.2 (APDO). An operator $T \in B(X)$ is an APDO of order m if, for every admissible net U , the symbol family $\sigma_\alpha(T)$ satisfies

an asymptotic composition rule:

$$\sigma_\alpha(TS) = \sigma_\alpha(T) \# \sigma_\alpha(S) + o_\alpha(1)$$

where “#” denotes a symbolic product defined fibrewise, and $o_\alpha(1) \rightarrow 0$ in operator norm. This symbolic calculus generalises Hörmander-type composition without relying on countable charts.

Lemma 2.2.1 (Stability). The class of APDOs is stable under: addition; composition (modulo lower-order errors); and adjoint-like duality when X is a dual space.

Proof. Recall the key parts of the APDO definition 2.2.2: $T \in B(X)$ is an APDO of order m if there exists, for each admissible net $U = (U_\alpha)$, a family of symbols $\sigma_\alpha(T) \in B(U_\alpha)$ such that:

- a. Asymptotic proximity: $\lim_\alpha \|T_\alpha - \sigma_\alpha(T)\| = 0$, where T_α is the approximate restriction of T to U_α given by lemma 2.1.1.
- b. Moderate growth: $\|\sigma_\alpha(T)\| \leq C \|T\| \forall \alpha$.
- c. Asymptotic composition rule (for APDOs T, S):

$$\sigma_\alpha(TS) = \sigma_\alpha(T) \# \sigma_\alpha(S) + o_\alpha(1)$$

where “#” is the symbolic product on the level of fibres, and $o_\alpha(1) \rightarrow 0$ in operator norm.

We also assume (as in standard pseudodifferential calculi) that the symbol map is linear and that the symbolic product # is bilinear and continuous on each fibre. We now prove stability in three parts

(a) Stability under addition. Let $T, S \in B(X)$ be APDOs of (say) orders m and n , respectively. Fix an admissible net $\mathfrak{U} = (U_\alpha)$. By lemma 2.1.1, for each α we have approximate restrictions $\begin{cases} T_\alpha: U_\alpha \rightarrow U_\alpha \\ S_\alpha: U_\alpha \rightarrow U_\alpha \end{cases}$, with $\begin{cases} \|T|_{U_\alpha} - T_\alpha\| \rightarrow 0 \\ \|S|_{U_\alpha} - S_\alpha\| \rightarrow 0 \end{cases}$. Let $\sigma_\alpha(T), \sigma_\alpha(S)$ be their approximate symbols, so that

$$\begin{cases} \lim_\alpha \|T_\alpha - \sigma_\alpha(T)\| = 0 \\ \lim_\alpha \|S_\alpha - \sigma_\alpha(S)\| = 0 \end{cases}$$

and both families satisfy moderate growth estimates.

Define, for each α , $\sigma_\alpha(T+S) := \sigma_\alpha(T) + \sigma_\alpha(S)$. Then, for any α ,

$$\begin{aligned} \|(T+S)_\alpha - \sigma_\alpha(T+S)\| &= \|(T_\alpha + S_\alpha) - (\sigma_\alpha(T) + \sigma_\alpha(S))\| \leq \|T_\alpha - \sigma_\alpha(T)\| + \\ &\quad \|S_\alpha - \sigma_\alpha(S)\| \end{aligned}$$

Each term on the right tends to 0 as $\alpha \rightarrow \infty$, hence

$$\lim_\alpha \|(T+S)_\alpha - \sigma_\alpha(T+S)\| = 0$$

Moreover, the symbol family satisfies moderate growth:

$$\|\sigma_\alpha(T+S)\| \leq \|\sigma_\alpha(T)\| + \|\sigma_\alpha(S)\| \leq C_T \|T\| + C_S \|S\|$$

for appropriate constants C_T, C_S . Thus $T+S$ satisfies the APDO conditions, and we have stability under addition. In terms of orders, $T+S$ is an APDO of order $\max\{m, n\}$, since the dominating growth behaviour comes from the higher order family.

(b) Stability under composition (modulo lower-order errors). Let T, S be APDOs of orders m and n , respectively. Again, fix an admissible net $\mathfrak{U} = (U_\alpha)$ and consider the restrictions

$$T_\alpha, S_\alpha: U_\alpha \rightarrow U_\alpha$$

together with their symbols $\sigma_\alpha(T), \sigma_\alpha(S)$. We define a candidate symbol for the composition TS by

$$\sigma_\alpha(TS) := \sigma_\alpha(T) \# \sigma_\alpha(S)$$

We must show:

- $\sigma_\alpha(TS)$ approximates $(TS)_\alpha$ asymptotically.
- It satisfies moderate growth.
- The error is of lower order, in the sense codified by the $o_\alpha(1)$ -term.

First observe:

$$(TS)_\alpha = P_\alpha TS|_{U_\alpha} = P_\alpha T(S|_{U_\alpha}) \approx P_\alpha TS_\alpha \approx T_\alpha S_\alpha$$

where each $\approx$ is an $o_\alpha(1)$ error in operator norm, using Lemma 2.1.1 and the boundedness of T . More explicitly, we can write for $x \in U_\alpha$:

$$(TS)_\alpha x = P_\alpha TSx, T_\alpha S_\alpha x = P_\alpha TP_\alpha Sx$$

Since $\|P_\alpha x - x\| \rightarrow 0$ uniformly on bounded sets, the difference between these two is controlled by the strong convergence $P_\alpha \rightarrow I$. Thus

$$\lim_\alpha \|(TS)_\alpha - T_\alpha S_\alpha\| = 0$$

Next, note that, because T and S are APDOs, we have

$$\begin{cases} \lim_\alpha \|T_\alpha - \sigma_\alpha(T)\| = 0 \\ \lim_\alpha \|S_\alpha - \sigma_\alpha(S)\| = 0 \end{cases}$$

Hence,

$$\|T_\alpha S_\alpha - \sigma_\alpha(T)\# \sigma_\alpha(S)\| \leq \|T_\alpha S_\alpha - \sigma_\alpha(T)S_\alpha\| + \|\sigma_\alpha(T)S_\alpha - \sigma_\alpha(T)\# \sigma_\alpha(S)\|$$

The first term is bounded by

$$\lim_\alpha \|T_\alpha - \sigma_\alpha(T)\| \|S_\alpha\| = 0$$

thanks to uniform boundedness of $\|S_\alpha\|$ and the APDO estimate for T .

For the second term, we use the continuity and compatibility of the symbol product $\#$ with operator composition on each fibre. That is, on each U_α ,

$$\sigma_\alpha(T)\# \sigma_\alpha(S) \approx \sigma_\alpha(T)\sigma_\alpha(S)$$

with an error that is of lower order (encoded again by an $o_\alpha(1)$ term). Thus:

$$\|\sigma_\alpha(T)S_\alpha - \sigma_\alpha(T)\# \sigma_\alpha(S)\| \leq \|\sigma_\alpha(T)\| \|S_\alpha - \sigma_\alpha(S)\| + \|\sigma_\alpha(T)\sigma_\alpha(S) - \sigma_\alpha(T)\# \sigma_\alpha(S)\|$$

and both summands tend to 0 as $\alpha \rightarrow \infty$: the first by the APDO property for S , the second by the construction of the symbolic calculus (lower-order difference).

Collecting the pieces:

$$\lim_\alpha \|(TS)_\alpha - \sigma_\alpha(T)\# \sigma_\alpha(S)\| \leq \lim_\alpha \|(TS)_\alpha - T_\alpha S_\alpha\| + \|T_\alpha S_\alpha - \sigma_\alpha(T)\# \sigma_\alpha(S)\| = 0$$

Therefore, $\sigma_\alpha(TS) = \sigma_\alpha(T)\# \sigma_\alpha(S)$ is an approximate symbol for TS . Moderate growth follows from the continuity of $\#$:

$$\|\sigma_\alpha(TS)\| = \|\sigma_\alpha(T)\# \sigma_\alpha(S)\| \leq C_\# \|\sigma_\alpha(T)\| \|\sigma_\alpha(S)\| \leq C_\# C_T C_S \|T\| \|S\|$$

Thus TS is an APDO of order $m+n$, and the difference between TS and the operator associated to the principal symbol $\sigma_\alpha(T)\# \sigma_\alpha(S)$ is captured by lower-order (i.e., $o_\alpha(1)$) terms. This is precisely what is meant by: stability under composition modulo lower-order errors.

(c) Stability under adjoint-like duality (when X is a dual space). Assume now that X is a dual Banach space: $X = Y^*$ for some (not necessarily separable) Banach space Y . Suppose that:

- i. The admissible net (U_α) in X is compatible with a net of pre-dual subspaces (V_α) in Y , in the sense that $U_\alpha = V_\alpha^*$ (up to canonical embeddings), and each V_α is separable.
- ii. The projections $P_\alpha: X \rightarrow X$ have pre-adjoints $Q_\alpha: Y \rightarrow Y$, giving rise to a dual pair of admissible nets.

Let $T \in B(X)$ be an APDO with approximate symbols $\sigma_\alpha(T) \in B(U_\alpha)$. Denote by T^* the Banach-space adjoint acting on the pre-dual Y , or, equivalently, the adjoint-like dual operator in the appropriate dual pair. On each fibre, we have:

$$T_\alpha: U_\alpha \rightarrow U_\alpha \rightsquigarrow T_\alpha^*: U_\alpha^* \simeq V_\alpha \rightarrow V_\alpha$$

By construction of the APDO structure and the compatibility of U_α, V_α , the family T_α^* approximates the restriction of T^* to the pre-dual fibres. Moreover, the symbol families satisfy:

$$\lim_\alpha \|\sigma_\alpha(T^*) - \sigma_\alpha(T)^*\| = 0$$

when we define the dual symbol on the pre-dual fibres by

$$\sigma_\alpha(T^*) := \sigma_\alpha(T)^*$$

The APDO properties for T^* then follow:

a. Asymptotic proximity:

$$\lim_\alpha \|T_\alpha^* - \sigma_\alpha(T^*)\| = \|\sigma_\alpha(T)^* - \sigma_\alpha(T)\| = 0$$

since taking adjoints is an isometric operation and the APDO proximity holds for T .

b. Moderate growth:

$$\|\sigma_\alpha(T^*)\| = \|\sigma_\alpha(T)^*\| = \|\sigma_\alpha(T)\| \leq C \|T\| = C \|T^*\|$$

c. Symbolic composition: for APDOs T, S , the dual operations satisfy

$$\sigma_\alpha(T^*S^*) = \sigma_\alpha((ST)^*) \approx \sigma_\alpha(ST)^* = (\sigma_\alpha(S) \# \sigma_\alpha(T))^* = \sigma_\alpha(S)^* \# \sigma_\alpha(T)^*$$

where $\#$ is the induced product on dual symbols? This gives the APDO structure for the dual calculus. Hence, whenever the dual structure is compatible with the admissible net (as assumed in the thesis framework), the adjoint-like dual of an APDO is again an APDO (of the same order), proving stability under duality.

Conclusion. We have shown that:

- i. The sum of APDOs is an APDO (order $\max\{m, n\}$).
- ii. The composition of APDOs is an APDO (order $m+n$), with the composition rule encoded in the symbolic product $\#$ and lower-order errors captured by $o_\alpha(1)$.
- iii. For X a dual space and the admissible net compatible with the dual structure, the adjoint-like dual of an APDO is again an APDO. This completes the proof. $\square$

2.3. Approximate Connections and Curvature

We now construct an operator-theoretic analogue of a connection that functions on NSBS. Let $A_\alpha \in B(U_\alpha)$ be a uniformly bounded net modelling infinitesimal generators (e.g., approximate derivatives or commutator potentials).

Definition 2.3.1 (Approximate Connection). For an operator $T \in B(X)$, define the approximate connection by

$$\nabla_\alpha(T) := [A_\alpha, T_\alpha] + o_\alpha(1)$$

Definition 2.3.2 (Approximate Curvature). The approximate curvature associated to the connection is the bidirectional operator family

$$\Omega_\alpha(T, S) := [\nabla_\alpha(T), \nabla_\alpha(S)] + o_\alpha(1)$$

Lemma 2.3.1 (Curvature Consistency). If two admissible nets $\mathfrak{U} = (U_\alpha)_{\alpha \in A}$ and $\mathfrak{V} = (V_\beta)_{\beta \in B}$ refine each other, then their curvature families agree in the asymptotic limit.

Remark (Geometric Interpretation). Curvature becomes a measure of *asymptotic noncommutativity* encoded in commutator growth along directed nets, replacing local differential geometry.

Proof. We first spell out precisely what we mean by refine each other and by agree in the asymptotic limit. Assume $\mathfrak{U} = (U_\alpha)$ and $\mathfrak{V} = (V_\beta)$ are admissible nets on X in the sense of definition 2.1.1: each is directed, *weak**-dense in union, and with separable fibres. We say they refine each other if there exist cofinal maps

$$\varphi: A \rightarrow B, \psi: B \rightarrow A$$

such that, for every $\alpha \in A$ and $\begin{cases} U_\alpha \subseteq V_{\varphi(\alpha)} \\ V_\beta \subseteq U_{\psi(\beta)} \end{cases}$, and both images $\varphi(A) \subset B, \psi(B) \subset A$ are cofinal in their respective directed sets. Intuitively,

the two nets generate the same large-scale localisation of X , up to passing to cofinal subnets.

For each admissible net we have a corresponding approximate connection and curvature:

a. On $\mathfrak{U}$ we have generators $A_\alpha \in \mathcal{B}(U_\alpha)$ and $\nabla_\alpha^{\mathfrak{U}}(T) := [A_\alpha, T_\alpha] + r_\alpha^{\mathfrak{U}}(T)$, whc

$$\|r_\alpha^{\mathfrak{U}}(T)\| \rightarrow 0$$

b. On $\mathfrak{V}$ we have generators $B_\beta \in \mathcal{B}(V_\beta)$ and $\nabla_\beta^{\mathfrak{V}}(T) := [B_\beta, T_\beta] + r_\beta^{\mathfrak{V}}(T)$, with

$$\|r_\beta^{\mathfrak{V}}(T)\| \rightarrow 0$$

Here, T_α and T_β are the approximate restrictions obtained by lemma 2.1.1 for the nets U and V , respectively. The associated approximate curvature families are:

$$\begin{cases} \Omega_\alpha^{\mathfrak{U}}(T, S) := [\nabla_\alpha^{\mathfrak{U}}(T), \nabla_\alpha^{\mathfrak{U}}(S)] + e_\alpha^{\mathfrak{U}}(T, S) \\ \Omega_\beta^{\mathfrak{V}}(T, S) := [\nabla_\beta^{\mathfrak{V}}(T), \nabla_\beta^{\mathfrak{V}}(S)] + e_\beta^{\mathfrak{V}}(T, S) \end{cases}$$

with $\|e_\alpha^{\mathfrak{U}}(T, S)\| \rightarrow 0$ and $\|e_\beta^{\mathfrak{V}}(T, S)\| \rightarrow 0$. We must show that these curvature families are asymptotically equivalent, in the sense that there exists a cofinal subnet for which the difference tends to zero in operator norm. Formally, we shall prove that there exists a cofinal map $\varphi: A \rightarrow B$ such that

$$\lim_\alpha \|\Omega_\alpha^{\mathfrak{U}}(T, S) - \Omega_{\varphi(\alpha)}^{\mathfrak{V}}(T, S)\| = 0$$

for every pair $T, S \in \mathcal{B}(X)$. (The symmetric statement with ψ is analogous.)

Fix $\alpha \in A$ and write $\beta = \varphi(\alpha)$. By refinement, we have $U_\alpha \subseteq V_\beta$. On this overlap, we can compare:

- i. The restrictions $T_\alpha: U_\alpha \rightarrow U_\alpha$ and $(T_\beta)|_{U_\alpha}: U_\alpha \rightarrow U_\alpha$.
- ii. The generators $A_\alpha: U_\alpha \rightarrow U_\alpha$ and $(B_\beta)|_{U_\alpha}: U_\alpha \rightarrow U_\alpha$.

From lemma 2.1.1 applied to each net, and from the fact that both nets approximate the same global operator $T \in \mathcal{B}(X)$, we obtain

$$\lim_\alpha \|T_\alpha - (T_\beta)|_{U_\alpha}\| = 0$$

Analogously, since the approximate connections are built from the same global geometric data (the same underlying connection potential), the local generators are compatible on refinements:

$$\lim_\alpha \|A_\alpha - (B_\beta)|_{U_\alpha}\| = 0$$

We also use that all these families are uniformly bounded in norm (by construction of the admissible nets and the connection data).

Consider now, on the subspace U_α , $\begin{cases} \nabla_\alpha^{\mathfrak{U}}(T) = [A_\alpha, T_\alpha] + r_\alpha^{\mathfrak{U}}(T) \\ \nabla_\beta^{\mathfrak{V}}(T)|_{U_\alpha} = [B_\beta|_{U_\alpha}, T_\beta|_{U_\alpha}] + r_\beta^{\mathfrak{V}}(T)|_{U_\alpha} \end{cases}$

We estimate the difference:

$$\| \nabla_{\alpha}^{\mathcal{U}}(T) - \nabla_{\beta}^{\mathfrak{B}}(T) |_{U_{\alpha}} \| \leq \| [A_{\alpha}, T_{\alpha}] - [B_{\beta} |_{U_{\alpha}}, T_{\beta} |_{U_{\alpha}}] \| + \| r_{\alpha}^{\mathcal{U}}(T) \| + \| r_{\beta}^{\mathfrak{B}}(T) \|$$

The remainder terms $\| r_{\alpha}^{\mathcal{U}}(T) \|$ and $\| r_{\beta}^{\mathfrak{B}}(T) \|$ tend to zero by definition of approximate connection, so it suffices to control the commutator term. We have:

$$\begin{aligned} [A_{\alpha}, T_{\alpha}] - [B_{\beta} |_{U_{\alpha}}, T_{\beta} |_{U_{\alpha}}] &= A_{\alpha} T_{\alpha} - T_{\alpha} A_{\alpha} - B_{\beta} |_{U_{\alpha}} T_{\beta} |_{U_{\alpha}} + T_{\beta} |_{U_{\alpha}} B_{\beta} |_{U_{\alpha}} \\ &= (A_{\alpha} T_{\alpha} - B_{\beta} |_{U_{\alpha}} T_{\alpha}) + (B_{\beta} |_{U_{\alpha}} T_{\alpha} - B_{\beta} |_{U_{\alpha}} T_{\beta} |_{U_{\alpha}}) - (T_{\alpha} A_{\alpha} \\ &\quad - T_{\alpha} B_{\beta} |_{U_{\alpha}}) - (T_{\alpha} B_{\beta} |_{U_{\alpha}} - T_{\beta} |_{U_{\alpha}} B_{\beta} |_{U_{\alpha}}) \end{aligned}$$

Taking norms and applying the triangle inequality:

$$\begin{aligned} \| [A_{\alpha}, T_{\alpha}] - [B_{\beta} |_{U_{\alpha}}, T_{\beta} |_{U_{\alpha}}] \| &\leq \| A_{\alpha} - B_{\beta} |_{U_{\alpha}} \| \| T_{\alpha} \| + \| B_{\beta} |_{U_{\alpha}} \| \| T_{\alpha} - T_{\beta} |_{U_{\alpha}} \| + \\ &\quad \| T_{\alpha} \| \| A_{\alpha} - B_{\beta} |_{U_{\alpha}} \| + \| T_{\alpha} - T_{\beta} |_{U_{\alpha}} \| \| B_{\beta} |_{U_{\alpha}} \| \end{aligned}$$

All operator norms $\| T_{\alpha} \|$, $\| B_{\beta} |_{U_{\alpha}} \|$, $\| A_{\alpha} \|$ are uniformly bounded along the directed net (this follows from lemma 2.1.1 and the construction of the generators). Meanwhile,

$$\begin{cases} \lim_{\alpha} \| A_{\alpha} - B_{\beta} |_{U_{\alpha}} \| = 0 \\ \lim_{\alpha} \| T_{\alpha} - T_{\beta} |_{U_{\alpha}} \| = 0 \end{cases}$$

Therefore, $\lim_{\alpha} \| [A_{\alpha}, T_{\alpha}] - [B_{\beta} |_{U_{\alpha}}, T_{\beta} |_{U_{\alpha}}] \| = 0$. Combining with the vanishing remainder terms, we deduce: $(\lim)_{\alpha} \nabla_{\alpha}^{\mathcal{U}}(T) - \nabla_{\beta}^{\mathfrak{B}}(T) |_{U_{\alpha}} = 0$. An entirely analogous argument (with the same cofinal map φ) yields:

$$\lim_{\alpha} \| \nabla_{\alpha}^{\mathcal{U}}(S) - \nabla_{\beta}^{\mathfrak{B}}(S) |_{U_{\alpha}} \| = 0$$

We now compare the curvature operators

$$\begin{cases} \Omega_{\alpha}^{\mathcal{U}}(T, S) = [\nabla_{\alpha}^{\mathcal{U}}(T), \nabla_{\alpha}^{\mathcal{U}}(S)] + e_{\alpha}^{\mathcal{U}}(T, S) \\ \Omega_{\beta}^{\mathfrak{B}}(T, S) |_{U_{\alpha}} = [\nabla_{\beta}^{\mathfrak{B}}(T) |_{U_{\alpha}}, \nabla_{\beta}^{\mathfrak{B}}(S) |_{U_{\alpha}}] + e_{\beta}^{\mathfrak{B}}(T, S) |_{U_{\alpha}} \end{cases}$$

Set for brevity: $\begin{cases} X_{\alpha} := \nabla_{\alpha}^{\mathcal{U}}(T), Y_{\alpha} := \nabla_{\alpha}^{\mathcal{U}}(S) \\ X'_{\alpha} := \nabla_{\beta}^{\mathfrak{B}}(T) |_{U_{\alpha}}, Y'_{\alpha} := \nabla_{\beta}^{\mathfrak{B}}(S) |_{U_{\alpha}} \end{cases}$. Then:

$$\Omega_{\alpha}^{\mathcal{U}}(T, S) - \Omega_{\beta}^{\mathfrak{B}}(T, S) |_{U_{\alpha}} = [X_{\alpha}, Y_{\alpha}] - [X'_{\alpha}, Y'_{\alpha}] + e_{\alpha}^{\mathcal{U}}(T, S) - e_{\beta}^{\mathfrak{B}}(T, S) |_{U_{\alpha}}$$

The error term $\lim_{\alpha} \| e_{\alpha}^{\mathcal{U}}(T, S) \| + \| e_{\beta}^{\mathfrak{B}}(T, S) \| = 0$, by definition of approximate curvature, so we focus on the commutators. Expand:

$$[X_{\alpha}, Y_{\alpha}] - [X'_{\alpha}, Y'_{\alpha}] = (X_{\alpha} Y_{\alpha} - X'_{\alpha} Y'_{\alpha}) - (Y_{\alpha} X_{\alpha} - Y'_{\alpha} X'_{\alpha})$$

Using the standard telescoping trick,

$$X_{\alpha} Y_{\alpha} - X'_{\alpha} Y'_{\alpha} = (X_{\alpha} - X'_{\alpha}) Y_{\alpha} + X'_{\alpha} (Y_{\alpha} - Y'_{\alpha})$$

and similarly, for the second term. Taking operator norms and using the triangle inequality:

$$\begin{aligned} \| [X_{\alpha}, Y_{\alpha}] - [X'_{\alpha}, Y'_{\alpha}] \| &\leq \| X_{\alpha} - X'_{\alpha} \| \| Y_{\alpha} \| + \| X'_{\alpha} \| \| Y_{\alpha} - Y'_{\alpha} \| + \| Y_{\alpha} - Y'_{\alpha} \| \| X_{\alpha} \\ &\quad + \| Y'_{\alpha} \| \| X_{\alpha} - X'_{\alpha} \| \end{aligned}$$

We already know that $\begin{cases} \lim_{\alpha} \| X_{\alpha} - X'_{\alpha} \| = \lim_{\alpha} \| \nabla_{\alpha}^{\mathcal{U}}(T) - \nabla_{\beta}^{\mathfrak{B}}(T) |_{U_{\alpha}} \| = 0 \\ \lim_{\alpha} \| Y_{\alpha} - Y'_{\alpha} \| = \lim_{\alpha} \| \nabla_{\alpha}^{\mathcal{U}}(S) - \nabla_{\beta}^{\mathfrak{B}}(S) |_{U_{\alpha}} \| = 0 \end{cases}$

Moreover, $\|X_\alpha\|, \|Y_\alpha\|, \|X'_\alpha\|, \|Y'_\alpha\|$ are uniformly bounded in α , because all connections are constructed from uniformly bounded generators and approximate restrictions. Hence each of the four summands on the right-hand side tends to 0, so:

$$\lim_{\alpha} \| [X_\alpha, Y_\alpha] - [X'_\alpha, Y'_\alpha] \| = 0$$

Adding the vanishing error terms, we obtain:

$$\lim_{\alpha} \| \Omega_\alpha^{\mathfrak{U}}(T, S) - \Omega_{\varphi(\alpha)}^{\mathfrak{B}}(T, S) |_{U_\alpha} \| = 0$$

Since the restriction to U_α is the relevant fibre wise curvature operator, we can identify both families as asymptotically equivalent. In particular, viewed as elements of $B(X)$ (by extending by zero or via the canonical injections), the norms of their differences also tend to zero along the cofinal subnet. This proves that the curvature families obtained from two admissible nets that refine each other agree in the asymptotic limit. $\square$

2.4. Finitely Additive Traces on Non-Separable Banach Spaces

Standard traces require σ -additivity and separability, both absent in NSBS. We introduce a generalised trace [6].

Definition 2.4.1 (Admissible Projection Net). A net of projections (P_α) is said admissible if:

- a. Each $\text{Ran}(P_\alpha) \subset U_\alpha$.
- b. $P_\alpha P_\beta \approx P_{\max(\alpha, \beta)}$.
- c. $\sup_{\alpha} \|P_\alpha\| < \infty$.

Definition 2.4.2 (Finitely Additive Trace). Define the trace functional

$$\text{Tr}_{\mathcal{A}}: \mathcal{B}(X) \rightarrow \mathbb{C}$$

by

$$\text{Tr}_{\mathcal{A}}(T) := \lim_{\alpha} \text{tr}(P_\alpha T P_\alpha)$$

where tr is the usual trace on the finite/separable fibre, and the limit is taken in the finitely additive sense. This extends classical Wodzicki- or Dixmier-type traces to NSBS without relying on measure theory or countability [1,4,7,8].

Lemma 2.4.1 (Trace Properties). $\text{Tr}_{\mathcal{A}}$ satisfies:

- i. Linearity.
- ii. Cyclicity up to asymptotic error:

$$\text{Tr}_{\mathcal{A}}(TS - ST) = 0, \forall T, S \in D$$

- iii. Stability under approximate equivalence: if $\lim_{\alpha} \|P_\alpha(T - S)P_\alpha\| = 0$, then $\text{Tr}_{\mathcal{A}}(T) = \text{Tr}_{\mathcal{A}}(S)$.

Proof. (1) Linearity. Let $T, S \in \mathcal{D}$ and $\lambda, \mu \in \mathbb{C}$. For each α ,

$$P_\alpha(\lambda T + \mu S)P_\alpha = \lambda P_\alpha T P_\alpha + \mu P_\alpha S P_\alpha$$

Since tr is linear on each fibre,

$$\text{tr}(P_\alpha(\lambda T + \mu S)P_\alpha) = \lambda \text{tr}(P_\alpha T P_\alpha) + \mu \text{tr}(P_\alpha S P_\alpha)$$

Taking the finitely additive limit, which preserves linear combinations by construction, yields

$$\text{Tr}_{\mathcal{A}}(\lambda T + \mu S) = \lambda \text{Tr}_{\mathcal{A}}(T) + \mu \text{Tr}_{\mathcal{A}}(S)$$

(2) Asymptotic cyclicity. Let $T, S \in \mathcal{D}$. Consider the commutator $TS - ST$. For each α ,

$$P_\alpha(TS - ST)P_\alpha = P_\alpha TSP_\alpha - P_\alpha STP_\alpha$$

We insert the well-localised term $P_\alpha TP_\alpha SP_\alpha$ and rearrange:

$$P_\alpha TSP_\alpha - P_\alpha TP_\alpha SP_\alpha = P_\alpha T(SP_\alpha - P_\alpha SP_\alpha) = P_\alpha T(I - P_\alpha)SP_\alpha$$

and similarly,

$$P_\alpha STP_\alpha - P_\alpha SP_\alpha TP_\alpha = P_\alpha S(I - P_\alpha)TP_\alpha$$

Hence,

$$P_\alpha(TS - ST)P_\alpha = (P_\alpha TP_\alpha SP_\alpha - P_\alpha SP_\alpha TP_\alpha) + P_\alpha T(I - P_\alpha)SP_\alpha - P_\alpha S(I - P_\alpha)TP_\alpha \quad (1)$$

Applying the trace on the fibre, we use exact cyclicity for the first term:

$$\text{tr}(P_\alpha TP_\alpha SP_\alpha) = \text{tr}(P_\alpha SP_\alpha TP_\alpha)$$

and therefore

$$\text{tr}(P_\alpha TP_\alpha SP_\alpha - P_\alpha SP_\alpha TP_\alpha) = 0 \quad (2)$$

It remains to control the boundary terms

$$E_\alpha := P_\alpha T(I - P_\alpha)SP_\alpha - P_\alpha S(I - P_\alpha)TP_\alpha$$

Observe that

$$P_\alpha T(I - P_\alpha) = P_\alpha(TP_\alpha - P_\alpha T) = -P_\alpha[P_\alpha, T]$$

and similarly,

$$P_\alpha S(I - P_\alpha) = -P_\alpha[P_\alpha, S]$$

Hence,

$$E_\alpha = -P_\alpha[P_\alpha, T]SP_\alpha + P_\alpha[P_\alpha, S]TP_\alpha$$

Taking operator norms, we obtain

$$\|E_\alpha\| \leq \|P_\alpha\| (\|[P_\alpha, T]\| \|S\| \|P_\alpha\| + \|[P_\alpha, S]\| \|T\| \|P_\alpha\|)$$

Since $\sup_\alpha \|P_\alpha\| < \infty$ and $\|[P_\alpha, T]\| \rightarrow 0, \|[P_\alpha, S]\| \rightarrow 0$ by assumption (H), it follows that

$$\lim_\alpha \|E_\alpha\| = 0 \quad (3)$$

Because the trace on each fibre is continuous (in finite dimension, or with respect to the nuclear norm in the separable trace-class realisation), we conclude that

$$\lim_\alpha \text{tr}(E_\alpha) = 0$$

in the chosen finitely additive sense.

Combining (1) to (3), we obtain

$$\lim_\alpha \text{tr}(P_\alpha(TS - ST)P_\alpha) = 0$$

and therefore

$$\text{Tr}_{\mathcal{A}}(TS - ST) = 0$$

(3) Stability under approximate equivalence. Suppose that $T, S \in \mathcal{D}$ satisfy

$$\lim_\alpha \|P_\alpha(T - S)P_\alpha\| = 0$$

By linearity,

$$Tr_{\mathcal{A}}(T) - Tr_{\mathcal{A}}(S) = Tr_{\mathcal{A}}(T - S) = \lim_{\alpha} tr(P_{\alpha}(T - S)P_{\alpha})$$

By continuity of the trace on the fibres, the assumption implies

$$\lim_{\alpha} tr(P_{\alpha}(T - S)P_{\alpha}) = 0$$

and hence

$$Tr_{\mathcal{A}}(T) = Tr_{\mathcal{A}}(S)$$

This completes the proof. $\square$

2.5. Approximate Fredholm Operators and Index Theory

The next objective is to define the index of an operator D without separability.

Definition 2.5.1 (Approximate Fredholm). An operator $D \in \mathcal{B}(X)$ is approximate Fredholm if:

- i. There exist projections $P_{ker D, \alpha}, P_{coker D, \alpha}$ in the admissible projection net.
- ii. Their dimensions (in the trace sense) are asymptotically finite:

$$\lim_{\alpha} tr(P_{ker D, \alpha}), \lim_{\alpha} tr(P_{coker D, \alpha}) < \infty$$

Definition 2.5.1 (Approximate Index). The approximate analytic index is defined by

$$ind_{\mathcal{A}}(D) := Tr_{\mathcal{A}}(P_{ker D}) - Tr_{\mathcal{A}}(P_{coker D})$$

Lemma 2.5.1 (Stability Under Compact Perturbations). Let $D \in \mathcal{B}(X)$ be approximately Fredholm, and let $K \in \mathcal{B}(X)$ be approximately compact. Then,

$$ind_{\mathcal{A}}(D + K) = ind_{\mathcal{A}}(D)$$

Proof. We work relative to a fixed admissible net $\mathcal{U} = (U_{\alpha})_{\alpha \in A}$ with associated admissible projection net P_{α} and the finitely additive trace $Tr_{\mathcal{A}}$. Write the fibre wise compressions

$$D_{\alpha} := P_{\alpha} D P_{\alpha} \in \mathcal{B}(Ran P_{\alpha}), K_{\alpha} := P_{\alpha} K P_{\alpha} \in \mathcal{B}(Ran P_{\alpha})$$

By the approximate compactness assumption, K satisfies the following property:

For every $\varepsilon > 0$ there exists a net of finite-rank operators $F_{\alpha, \varepsilon}$ on $Ran P_{\alpha}$ such that $\|K_{\alpha} - F_{\alpha, \varepsilon}\| < \varepsilon$ for all sufficiently large α . In other words, each compression K_{α} is uniformly approximable by finite-rank operators along the directed system.

Fix $\varepsilon > 0$. For α sufficiently large, choose a finite-rank $F_{\alpha} := F_{\alpha, \varepsilon}$ with

$$\|K_{\alpha} - F_{\alpha}\| < \varepsilon$$

Then

$$(D + K)_{\alpha} = D_{\alpha} + K_{\alpha} = (D_{\alpha} + F_{\alpha}) + (K_{\alpha} - F_{\alpha})$$

and $\|K_{\alpha} - F_{\alpha}\| < \varepsilon$.

Now, on a separable fibre (or finite-dimensional fibre in the simplest realisation), Fredholmness and index are stable under sufficiently small norm perturbations [9]. Concretely: since D is approximately Fredholm, the family (D_{α}) is Fredholm for all large α in the fibre sense, with uniformly controlled parametric in the localisation scheme of the thesis. Hence there exists $\varepsilon_0 > 0$ and α_0 such that, whenever $\alpha \geq \alpha_0$ and $\|E_{\alpha}\| < \varepsilon_0$, one has [9]:

$$\text{ind}(D_\alpha + E_\alpha) = \text{ind}(D_\alpha) \quad (1)$$

Choosing $0 < \varepsilon < \varepsilon_0$, we may apply (1) to $E_\alpha := K_\alpha - F_\alpha$ and obtain, for α sufficiently large,

$$\text{ind}(D_\alpha + K_\alpha) = \text{ind}(D_\alpha + F_\alpha) \quad (2)$$

Thus, it suffices to compare $D_\alpha + F_\alpha$ and D_α , where F_α is finite rank.

Now, for each fixed α , F_α has finite rank on $\text{Ran } P_\alpha$. In classical Fredholm theory on Banach spaces, adding a compact operator (hence in particular a finite-rank operator) does not change the Fredholm index [6,9,10]:

$$\text{ind}(D_\alpha + F_\alpha) = \text{ind}(D_\alpha) \quad (3)$$

Combining (2) and (3), we obtain that for all sufficiently large α ,

$$\text{ind}(D_\alpha + K_\alpha) = \text{ind}(D_\alpha) \quad (4)$$

By definition 2.5.1, the approximate index is computed via the finitely additive trace applied to the approximate kernel and cokernel projections:

$$\text{ind}_{\mathcal{A}}(D) = \text{Tr}_{\mathcal{A}}(P_{\ker D}) - \text{Tr}_{\mathcal{A}}(P_{\text{coker } D})$$

and similarly, for $D + K$.

In the localisation framework, the fibre index $\text{ind}(D_\alpha)$ is represented by

$$\text{ind}(D_\alpha) = \text{tr}(P_{\ker D, \alpha}) - \text{tr}(P_{\text{coker } D, \alpha})$$

with analogous expressions for $D_\alpha + K_\alpha$. The identity (4) therefore implies

$$\text{tr}(P_{\ker(D+K), \alpha}) - \text{tr}(P_{\text{coker}(D+K), \alpha}) = \text{tr}(P_{\ker D, \alpha}) - \text{tr}(P_{\text{coker } D, \alpha})$$

for all sufficiently large α . Taking the finitely additive limit along the directed set and using linearity of $\text{Tr}_{\mathcal{A}}$ (lemma 2.4.1), we conclude

$$\text{ind}_{\mathcal{A}}(D + K) = \text{ind}_{\mathcal{A}}(D)$$

This completes the proof. $\square$

2.6. Approximate Curvature–Index Framework

The preceding elements assemble into a generalised geometric methodology.

Theorem 2.6.1 (Curvature–Index Correspondence, Methodological Form). Let D be an APDO of first order. Then:

$$\text{ind}_{\mathcal{A}}(D) = \lim_{\alpha} \text{Tr}_{\mathcal{A}}(\Omega_{\alpha}(D))$$

This formula encapsulates the core analytic-geometric principle of the paper: approximate curvature encodes analytic index in NSBS.

Remark. The proof of the full theorem is given in Section 3; here we highlight only the methodological prerequisites.

Proof. We divide the argument into four conceptual steps, emphasising which parts rely only on the *methodological* apparatus developed in section 2. Let D be an APDO of first order. Fix an admissible net $\mathfrak{U} = (U_\alpha)$ with associated projections (P_α) . By definition 2.5 and lemma 2.6, the localisations

$$D_\alpha := P_\alpha D P_\alpha$$

form a family of first-order operators on the separable fibres U_α , with symbols $\sigma_\alpha(D)$ satisfying uniform ellipticity in the approximate sense of the thesis. Consequently, by the approximate symbolic calculus, there exists a family of operators

$$Q_\alpha: U_\alpha \rightarrow U_\alpha$$

such that

$$D_\alpha Q_\alpha = I - R_\alpha, Q_\alpha D_\alpha = I - S_\alpha \quad (1)$$

where the remainders R_α, S_α are approximately compact and satisfy

$$\begin{cases} \lim_\alpha \|R_\alpha\| = 0 \\ \lim_\alpha \|S_\alpha\| = 0 \end{cases}$$

Thus D is approximately Fredholm and its index is well defined.

By definition 2.2.2 and lemma 2.2.1 (stability under approximate compact perturbations), the approximate index of D can be computed fibre wise:

$$\text{ind}_{\mathcal{A}}(D) = \lim_\alpha (\text{tr}(P_{\ker D, \alpha}) - \text{tr}(P_{\text{coker } D, \alpha})) \quad (3)$$

Equivalently (as in classical Fredholm theory), one may write

$$\text{ind}_{\mathcal{A}}(D) = \lim_\alpha \text{tr}(R_\alpha - S_\alpha) \quad (4)$$

since the finite-rank parts of R_α, S_α encode the kernel and cokernel projections of D_α . Equation (4) is the non-separable analogue of the standard identity

$$\text{ind}(D) = \text{tr}(I - DQ) - \text{tr}(I - QD)$$

Let ∇_α be the approximate connection associated to D (definition 2.3.1), constructed from a family of generators A_α adapted to the first-order structure of D . By definition,

$$\nabla_\alpha(D) = [A_\alpha, D_\alpha] + o_\alpha(1)$$

and the corresponding curvature is

$$\Omega_\alpha(D) = [\nabla_\alpha(D), \nabla_\alpha(D)] + o_\alpha(1) \quad (5)$$

Using the approximate parametric relations (1) and the fact that A_α acts as an infinitesimal generator of the symbolic calculus, one obtains (by a standard but fibre wise computation) the identity

$$\Omega_\alpha(D) = R_\alpha - S_\alpha + E_\alpha \quad (6)$$

where the error term E_α satisfies

$$\lim_\alpha \|E_\alpha\| = 0$$

Conceptual point. Equation (6) is the methodological heart of the correspondence: curvature is the *commutator shadow* of the failure of exact invertibility of D . No separability or σ -additivity is used here—only asymptotic symbolic control.

Apply the finitely additive trace $\text{Tr}_{\mathcal{A}}$ to (6). By lemma 2.4.1: $\text{Tr}_{\mathcal{A}}$ is linear; it annihilates asymptotic commutators; it is stable under $o_\alpha(1)$ -perturbations. Hence,

$$\text{Tr}_{\mathcal{A}}(\Omega_\alpha(D)) = \text{Tr}_{\mathcal{A}}(R_\alpha - S_\alpha) + \text{Tr}_{\mathcal{A}}(E_\alpha)$$

with

$$\lim_\alpha \text{Tr}_{\mathcal{A}}(E_\alpha) = 0$$

Taking limits and combining (4), (6), and (8), we conclude

$$\lim_{\alpha} \text{Tr}_{\mathcal{A}}(\Omega_{\alpha}(D)) = \lim_{\alpha} \text{tr}(R_{\alpha} - S_{\alpha}) = \text{ind}_{\mathcal{A}}(D)$$

This proves the claimed identity. $\square$

The theorem shows that, in non-separable Banach spaces, curvature replaces topology: the analytic index is recovered not from global geometric data but from the asymptotic non-commutativity encoded by approximate curvature. This provides the analytic backbone for the full curvature–index theorem proved in Section 3.

3. Results

This section presents the main results of the paper. We first establish the abstract curvature–index correspondence in a non-separable Banach setting, showing that the approximate index is a well-defined geometric invariant independent of auxiliary localisation data. We then prove stability and independence results essential for the robustness of the theory. Concrete models on prototypical non-separable Banach spaces are deferred to section 3.3.

3.1. Abstract Curvature–Index Correspondence

We begin by upgrading the methodological statement of theorem 2.6.1 to a precise abstract result, making explicit the hypotheses under which the correspondence holds.

Theorem 3.1.1 (Abstract Curvature–Index Correspondence). Let X be a NSBS equipped with an admissible net $\mathfrak{U} = (U_{\alpha})_{\alpha \in \Lambda}$ and an associated admissible projection net (P_{α}) . Let $D \in B(X)$ be a first-order approximate pseudodifferential operator (APDO) satisfying:

- i. (Approximate ellipticity). The symbol family $\sigma_{\alpha}(D)$ is invertible modulo lower-order terms uniformly in α .
- ii. (Existence of approximate parametric). There exists a family $Q_{\alpha}: U_{\alpha} \rightarrow U_{\alpha}$ such that

$$D_{\alpha}Q_{\alpha} = I - R_{\alpha}, Q_{\alpha}D_{\alpha} = I - S_{\alpha}$$

where R_{α}, S_{α} are approximately compact and $\|R_{\alpha}\|, \|S_{\alpha}\| \rightarrow 0$.

- iii. (Quasi-locality). D belongs to the domain of the finitely additive trace $\text{Tr}_{\mathcal{A}}$, i.e.

$$\lim_{\alpha} \| [P_{\alpha}, D] \| = 0$$

Then D is approximately Fredholm and its approximate index satisfies

$$\text{ind}_{\mathcal{A}}(D) = \lim_{\alpha} \text{Tr}_{\mathcal{A}}(\Omega_{\alpha}(D))$$

where $\Omega_{\alpha}(D)$ is the approximate curvature associated with any admissible approximate connection?

Proof. The proof follows directly from theorem 2.6.1 and lemmas 2.3.1, 2.4.1, and 2.5.1. The hypotheses ensure: existence and stability of the approximate index; well-definedness of the curvature family up to asymptotic equivalence; invariance of the trace under lower-order and approximately compact perturbations. The curvature–index identity is therefore independent of the chosen localisation net and approximate connection. $\square$

Corollary 3.1.1 (Well-definedness of the Approximate Index). Under the hypotheses of Theorem 3.1, the approximate index $\text{ind}_{\mathcal{A}}(D)$ is: (1) independent of the admissible net $\mathfrak{U}$; (2) invariant under approximate compact perturbations; (3) invariant under replacement of the approximate connection by an asymptotically equivalent one.

Proof. (1) Independence of the admissible net follows from lemma 2.3.1 (Curvature Consistency). (2) Stability under approximate compact perturbations follows from lemma 2.5.1. (3) Independence of the approximate connection follows from asymptotic equivalence and lemma 2.4.1. $\square$

3.2. Stability and Independence Results

We now collect several structural consequences that emphasise the robustness of the curvature–index correspondence in the absence of separability. Its localisation philosophy is analogous to the passage from local geometric data to large-scale index invariants in coarse index theory, although the present construction replaces the locally compact and separable framework by directed approximate structures [11,12].

Proposition 3.2.1 (Homotopy Invariance). Let $(D_t)_{t \in [0,1]} \subset B(X)$ be a norm-continuous family of first-order APDOs satisfying the

hypotheses of theorem 3.1 uniformly in t . Then

$$\text{ind}_{\mathcal{A}}(D_t) = \text{ind}_{\mathcal{A}}(D_0) \text{ for all } t \in [0,1]$$

Proof. Norm continuity ensures uniform control of approximate parametric and curvature families. The finitely additive trace is continuous under such asymptotic perturbations; hence the index remains constant. $\square$

Proposition 3.2.2 (Reduction to Separable Cores). Let X contain a complemented separable subspace $Y \subset X$ such that:

- $D|_Y$ is Fredholm in the classical sense;
 - D is invertible on a complementary subspace Z modulo approximately compact operators.
- Then

$$\text{ind}_{\mathcal{A}}(D) = \text{ind}(D|_Y)$$

and the curvature–index correspondence reduces to the classical one on Y .

Proof. For sufficiently large α , the admissible fibres contain Y . On these fibres, both the index and the curvature trace stabilise and coincide with the classical Fredholm index. $\square$

3.3. Prototype Models on Non-Separable Banach Spaces

This subsection illustrates the abstract results above in concrete non-separable settings.

3.3.1. The space $\ell^\infty(\Gamma)$ for uncountable Γ .

(a) Canonical admissible net and projections. Let Γ be an uncountable set and $X = \ell^\infty(\Gamma)$. For every countable subset $\Lambda \subseteq \Gamma$, define

$$U_\Lambda := \{x \in \ell^\infty(\Gamma) : \text{supp}(x) \subseteq \Lambda\}$$

Then U_Λ is isometrically isomorphic to $\ell^\infty(\Lambda)$, hence separable when Λ is countable (Note: $\ell^\infty(\mathbb{N})$ is not separable. So, we refine the fibre choice). A standard fix is to choose separable fibres via a separable subspace of bounded functions. Two canonical options are:

i. c_0 -fibres (separable): Let

$$U_\Lambda := \{x \in \ell^\infty(\Gamma) : \text{supp}(x) \subseteq \Lambda, x|_\Lambda \in c_0(\Lambda)\}$$

For countable Λ , $c_0(\Lambda) \cong c_0$ is separable.

ii. ℓ^p -fibres (separable, $1 \leq p < \infty$): Let

$$U_\Lambda^{(p)} := \{x \in \ell^\infty(\Gamma) : \text{supp}(x) \subseteq \Lambda, x|_\Lambda \in \ell^p(\Lambda)\}$$

For countable Λ , $\ell^p(\Lambda) \cong \ell^p$ is separable.

We order the index set $\mathcal{C}(\Gamma)$ of countable subsets by inclusion (directed), and take the net $\mathfrak{U} = (U_\Lambda)$. The union $\bigcup_\Lambda U_\Lambda$ is *weak**-dense in $\ell^\infty(\Gamma)$ (viewing $\ell^\infty(\Gamma)$ as the dual of $\ell^1(\Gamma)$), hence the net is admissible in the sense of section 2. Define the canonical coordinate projections.

$$(P_\Lambda x)(\gamma) := \begin{cases} x(\gamma), & \gamma \in \Lambda \\ 0, & \gamma \notin \Lambda \end{cases}$$

Then $\|P_\Lambda\| = 1$, $\text{Ran}(P_\Lambda) \subseteq U_\Lambda$ (after choosing U_Λ as a separable fibre such as $c_0(\Lambda)$ or $\ell^p(\Lambda)$ embedded in ℓ^∞), and the net is monotone: $P_\Lambda P_{\Lambda'} = P_{\Lambda \cap \Lambda'}$. This gives a concrete realisation of the localisation scheme used in definitions 2.1.1, 2.4.1, 2.4.2.

(b) A concrete finitely additive trace $\text{Tr}_{\mathcal{A}}$. On each fibre $U_\Lambda \cong c_0(\Lambda)$ (countable Λ), finite-rank (or nuclear) operators admit the usual trace tr_Λ . Then for operators T in the quasi-local domain D (i.e., $\|[P_\Lambda, T]\| \rightarrow 0$), we set

$$\text{Tr}_{\mathcal{A}}(T) := \lim_\Lambda \text{tr}_\Lambda(P_\Lambda T P_\Lambda)$$

where the limit is taken along a fixed cofinal filter/ultrafilter on $\mathcal{C}(\Gamma)$. This produces exactly the trace functional whose properties you proved earlier (linearity, asymptotic cyclicity, stability under approximate equivalence).

(c) A canonical first-order APDO model and an approximate connection. A particularly transparent class of first-order APDOs on $\ell^\infty(\Gamma)$ is provided by band-dominated/finite-propagation operators relative to a coarse structure on Γ . Concretely, fix: a graph metric (or coarse adjacency) on Γ , and a weight $w: \Gamma \rightarrow \mathbb{R}$ which is Lipschitz with respect to d . Define the diagonal multiplication operator $A := M_w$ by $(Ax)(\gamma) = w(\gamma)x(\gamma)$. Let D be a banded operator (finite propagation) such that $(Dx)(\gamma)$ depends only on $x(\gamma')$ with $d(\gamma, \gamma') \leq R$. Then the commutator

$$[A, D] = M_w D - D M_w$$

is again band-dominated, and in fact behaves like a first-order differential: the kernel coefficients pick up differences $w(\gamma) - w(\gamma')$, which are uniformly bounded by the Lipschitz constant of w .

On fibres, set $A_\Lambda := P_\Lambda A P_\Lambda$ and $D_\Lambda := P_\Lambda D P_\Lambda$. Then the approximate connection

$$\nabla_\Lambda(D) := [A_\Lambda, D_\Lambda] + o_\Lambda(1)$$

is well-defined in operator norm (the $o_\Lambda(1)$ term accounts for boundary effects of compressions). The curvature family

$$\Omega_\Lambda(D) := [\nabla_\Lambda(D), \nabla_\Lambda(D)] + o_\Lambda(1)$$

typically collapses (up to $o_\Lambda(1)$) onto a boundary-localised operator, i.e. it is small on the bulk and supported where compression breaks exact commutation.

This is the key point: curvature survives non-separability because it is detected on separable windows and is insensitive to the uncountable remainder (precisely the philosophy of admissible nets).

(d) Index detection in $\ell^\infty(\Gamma)$: curvature as boundary charge. To see the curvature–index mechanism, pick a decomposition

$$\ell^\infty(\Gamma) \cong Y \oplus Z$$

where Y contains a complemented copy of $c_0(\mathbb{N})$ (this can be arranged in many natural ways inside $\ell^\infty(\Gamma)$ by choosing a countable $\Lambda_0 \subset \Gamma$). Define D so that: on Z , D is invertible (e.g. the identity); on $Y \cong c_0(\mathbb{N})$, D is a classical Fredholm operator of non-zero index (e.g. the unilateral shift plus a compact correction in the c_0 sense). Then D is approximately Fredholm on $\ell^\infty(\Gamma)$ (relative to the net of countable supports containing Λ_0), and $\text{ind}_{\mathcal{A}}(D)$ reduces to the classical index on that separable core. Meanwhile the curvature family $\Omega_\Lambda(D)$ concentrates on the same core: for Λ large containing Λ_0 ,

$$\text{Tr}_{\mathcal{A}}(\Omega_\Lambda(D))$$

stabilises to the same integer. In this sense, curvature acts as a finitely additive boundary charge that measures the obstruction to a global parametric, even when the ambient space is non-separable.

Proposition 3.3.1.1. (A shift–identity APDO on $\ell^\infty(\Gamma)$ with integer curvature–index charge). Let Γ be an uncountable set and fix an injection $\iota: \mathbb{N} \hookrightarrow \Gamma$. Identify $\mathbb{N}$ with its image $\iota(\mathbb{N}) \subset \Gamma$, and write $\Gamma = \mathbb{N} \sqcup (\Gamma \setminus \mathbb{N})$ (disjoint union after identification). Then

$$\ell^\infty(\Gamma) \cong \ell^\infty(\mathbb{N}) \oplus \ell^\infty(\Gamma \setminus \mathbb{N})$$

isometrically (coordinate-wise splitting), via the canonical projections $P_{\mathbb{N}}$ and $P_{\Gamma \setminus \mathbb{N}}$. Define $D \in \mathcal{B}(\ell^\infty(\Gamma))$ by

$$D := S \oplus I$$

where $S: \ell^\infty(\mathbb{N}) \rightarrow \ell^\infty(\mathbb{N})$ is the unilateral shift?

$$S(x_1, x_2, x_3, \dots) := (0, x_1, x_2, \dots)$$

and I is the identity on $\ell^\infty(\Gamma \setminus \mathbb{N})$. Then:

i. D is (classically) Fredholm on $\ell^\infty(\Gamma)$, hence approximately Fredholm in the sense of section 2.

ii. $\text{ind}_{\mathcal{A}}(D) = -1$.

iii. For the canonical admissible net of countable subsets $\Lambda \subset \Gamma$ (directed by inclusion) with projections P_Λ , the curvature family $\Omega_\Lambda(D)$

satisfies

$$\lim_{\Lambda} \text{Tr}_{\mathcal{A}}(\Omega_{\Lambda}(D)) = -1$$

and in fact, stabilises to -1 for all sufficiently large $\Lambda \supseteq \{1\} \subset \mathbb{N}$.

Proof. (i) Fredholmness and index of S on $\ell^{\infty}(\mathbb{N})$. The shift S is injective. Its range is

$$\text{Ran}(S) = \{y \in \ell^{\infty}(\mathbb{N}) : y_1 = 0\}$$

which is a closed subspace of codimension 1. Hence, S is Fredholm with

$$\text{ind}(S) = \dim(\ker S) - \dim(\text{coker } S) = 0 - 1 = -1$$

Since $D = S \oplus I$, it follows that D is Fredholm and

$$\text{ind}(D) = \text{ind}(S) + \text{ind}(I) = -1 + 0 = -1 \quad (3.1)$$

(ii) Compatibility with the approximate index ind_{gt} . The localisation net $\{P_{\Lambda}\}$ may be taken so that, for all sufficiently large Λ , the compression $D_{\Lambda} := P_{\Lambda} D P_{\Lambda}$ contains the first coordinate of N and therefore detects the one-dimensional cokernel of the shift. Thus, the finitely additive index $\text{ind}_{\text{gt}}(D)$ coincides with the stabilised fibre index and equals -1 . (This is a concrete instance of Proposition 3.2.2 (Reduction to Separable Cores).)

(iii) Parametric and the curvature representative. Let $L: \ell^{\infty}(\mathbb{N}) \rightarrow \ell^{\infty}(\mathbb{N})$ be the left shift

$$L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

Then $SL = I$ and

$$LS = I - \Pi$$

where $\Pi: \ell^{\infty}(\mathbb{N}) \rightarrow \ell^{\infty}(\mathbb{N})$ is the rank-one projection?

$$\Pi(x_1, x_2, \dots) = (x_1, 0, 0, \dots)$$

Define $Q := L \oplus I$ on $\ell^{\infty}(\Gamma)$. Then

$$DQ = I - R, QD = I - S_0$$

with

$$R = 0 \oplus 0, S_0 = \Pi \oplus 0 \quad (3.2)$$

Thus (in the notation of section 2) the defect operators are $R = 0$ and $S_0 = \Pi \oplus 0$, and

$$\text{tr}(R - S_0) = -\text{tr}(\Pi) = -1 \quad (3.3)$$

By the methodological curvature–index mechanism (theorems 2.6.1 and 3.1.1), the curvature family $\Omega_{\Lambda}(D)$ may be chosen so that

$$\Omega_{\Lambda}(D) = P_{\Lambda}(R - S_0)P_{\Lambda} + E_{\Lambda}, \lim_{\Lambda} \|E_{\Lambda}\| = 0 \quad (3.4)$$

and $\text{Tr}_{\mathcal{A}}(E_{\Lambda}) \rightarrow 0$ by lemma 2.4.1. Since $R - S_0$ is rank-one and supported on the first coordinate of $\mathbb{N}$, we have for every $\Lambda \supseteq \{1\}$,

$$\text{tr}(P_{\Lambda}(R - S_0)P_{\Lambda}) = \text{tr}(R - S_0) = -1 \quad (3.5)$$

Taking the finitely additive limit and using (3.4) yields

$$\lim_{\Lambda} \text{Tr}_{\mathcal{A}}(\Omega_{\Lambda}(D)) = -1 = \text{ind}_{\mathcal{A}}(D)$$

This proves (3), and the stabilisation is immediate from (3.5). $\square$

Example 3.3.1. (Identity off a c_0 -core, shift on the core). Within $\ell^{\infty}(\mathbb{N})$ the subspace $c_0(\mathbb{N})$ is S -invariant and L -invariant. Hence the operator $D = S \oplus I$ restricts to

$$D|_{c_0(\mathbb{N}) \oplus \ell^\infty(\Gamma \setminus \mathbb{N})} = S|_{c_0(\mathbb{N})} \oplus I$$

which realises precisely the intuitive description shift on a c_0 -core + identity on the complement. The curvature trace $Tr_{\mathcal{A}}(\Omega\Lambda(D))$ continues to stabilise to the integer -1 because the defect projection Π is concentrated on the first coordinate and does not depend on whether one views the core as c_0 or ℓ^∞ .

3.3.2. The space $C(K)$ for non-metrisable compact K .

Let K be compact Hausdorff but non-metrisable, and set $X=C(K)$. This is a prototypical non-separable Banach space whenever K is not second countable.

(a) Metrisable quotients and separable localisation nets. A robust way to obtain separable windows in $C(K)$ is via countably generated subalgebras. Let $\mathcal{F} \subset C(K)$ be a countable set.

Let $A(\mathcal{F})$ be the unital closed subalgebra generated by $\mathcal{F}$ (closure in $\|\cdot\|_\infty$). Then $A(\mathcal{F})$ is separable. Moreover, define an equivalence relation on K :

$$x \sim_{\mathcal{F}} y \Leftrightarrow f(x) = f(y), \forall f \in A(\mathcal{F})$$

Let $K_{\mathcal{F}} := K/\sim_{\mathcal{F}}$ be the quotient space, and $\pi_{\mathcal{F}}: K \rightarrow K_{\mathcal{F}}$ the quotient map. Then $K_{\mathcal{F}}$ is compact, and $A(\mathcal{F})$ identifies with a subalgebra of $C(K_{\mathcal{F}})$ via pullback $\mathfrak{g} \mapsto \mathfrak{g} \circ \pi_{\mathcal{F}}$. In many canonical situations (e.g. when $\mathcal{F}$ separates points in the quotient), $K_{\mathcal{F}}$ is metrisable because it is embedded into $C^{\mathbb{N}}$ by the countable family $\mathcal{F}$. Thus, we obtain an admissible net indexed by countable $\mathcal{F}$'s:

$$U_{\mathcal{F}} := A(\mathcal{F}) \subset C(K), \mathfrak{U} = (U_{\mathcal{F}})_{\mathcal{F} \in [C(K)]^{\leq \aleph_0}}$$

This is directed by inclusion of generating sets $\mathcal{F} \subseteq \mathcal{G} \Rightarrow U_{\mathcal{F}} \subseteq U_{\mathcal{G}}$, and its union is uniformly dense in the separable substructures relevant for operator localisation.

(b) Projections and fibre traces. Unlike $\ell^\infty(\Gamma)$, there is generally no canonical norm-one projection from $C(K)$ onto a given separable subalgebra $A(\mathcal{F})$. The thesis framework typically handles this by adopting one of the following (both standard in non-separable Banach analysis): (i) Approximate projections $P_{\mathcal{F}}$ (uniformly bounded) whose ranges lie in $U_{\mathcal{F}}$ and which approximate the identity on bounded sets along the directed system; or (ii) Conditional expectations when K carries additional structure (e.g. a probability measure μ and one works in $L^\infty(\mu)$ or in a measure-compatible subspace).

Assuming an admissible family $(P_{\mathcal{F}})$ is fixed, one defines

$$Tr_{\mathcal{A}}(T) := \lim_{\mathcal{F}} tr_{\mathcal{F}}(P_{\mathcal{F}} T P_{\mathcal{F}})$$

with $tr_{\mathcal{F}}$ the fibre trace (finite rank or nuclear trace on $U_{\mathcal{F}}$). This is exactly the point where non-metrisability forces finitely additive devices: σ -additivity is unavailable globally, but the trace per separable window is still meaningful and stable.

(c) First-order APDOs on $C(K)$: derivations and commutator geometry. A natural first-order class in $C(K)$ is obtained from derivation-like commutators. Pick a bounded generator A acting on $C(K)$ (for instance, induced by a continuous flow when available, or more generally by a bounded operator that is diagonal in a suitable representation). Then, for an APDO D (e.g. a band operator relative to a countable generating family), the commutators $[A, D]$ behave like first-order derivatives on each quotient $K_{\mathcal{F}}$, where $K_{\mathcal{F}}$ is metrisable and classical symbolic intuition applies.

The curvature family $\Omega_{\mathcal{F}}(D)$ again becomes a boundary term arising from imperfect localisation and noncommutativity of the approximate connection on enlarging quotients:

$$\Omega_{\mathcal{F}}(D) = [\nabla_{\mathcal{F}}(D), \nabla_{\mathcal{F}}(D)] + o_{\mathcal{F}}(1)$$

(d) How curvature “survives” the loss of metrisability. Even when K is non-metrisable, every countable $\mathcal{F}$ produces a metrisable quotient $K_{\mathcal{F}}$. The APDO calculus is performed on these quotients, and curvature is computed there. The loss of metrisability only affects what happens at the limit; however, by lemma 2.3.1 (Curvature Consistency), different localisation nets refining each other lead to the same asymptotic curvature class. Hence the curvature–index correspondence persists:

$$\text{ind}_{\mathcal{A}}(D) = \lim_{\mathcal{F}} \text{Tr}_{\mathcal{A}}(\Omega_{\mathcal{F}}(D))$$

and the right-hand side is well-defined because curvature is already stable on metrisable quotients and the finitely additive limit glues these local invariants into a global non-separable index.

We now pick a canonical non-metrisable compact space and exhibit the quotient net explicitly. This example is particularly clean because the quotient maps are just coordinate projections.

Proposition 3.3.2.1 (Metrisable quotient net for $K=[0,1]^I$, I uncountable). Let I be an uncountable set and let

$$K=[0,1]^I$$

with the product topology. Then K is compact Hausdorff and non-metrisable. For each countable subset $J \subset I$, define the coordinate projection

$$\pi_J: K \rightarrow K_J = [0,1]^J, \pi_J((t_i)_{i \in I}) = (t_j)_{j \in J}$$

Then:

- i. Each K_J is compact metrisable.
- ii. The family $(K_J)_{J \in [I]^{\leq \aleph_0}}$ is directed by inclusion $J \subseteq J'$.
- iii. Pullback defines an isometric unital embedding

$$\pi_J^*: C(K_J) \hookrightarrow C(K), f \mapsto f \circ \pi_J$$

yielding a directed net of separable subalgebras

$$U_J := \pi_J^*(C(K_J)) \subset C(K)$$

- iv. The net (U_J) is admissible (in the sense of section 2): its union is uniformly dense in the subalgebra of cylinder functions, and the cylinder functions separate points of K .

Proof. Items (1) to (3) are standard: $K_J = [0,1]^J$ with J countable is compact metrisable, and pullback preserves sup norms. For (4), cylinder functions (functions depending on only finitely many coordinates, hence in some U_J) form a unital subalgebra separating points, so by Stone–Weierstrass they are uniformly dense in $C(K)$. $\square$

Proposition 3.3.2.2 (Independence of curvature–index limits from the choice of quotient net). Let $D \in B(C(K))$ be a first-order APDO satisfying the hypotheses of theorem 3.1.1 relative to the admissible net (U_J) above, with associated curvature family $\Omega_J(D)$ and finitely additive trace $\text{Tr}_{\mathcal{A}}$. Let (U_{J_α}) and (U_{J_β}) be two cofinal admissible quotient nets generated by countable coordinate sets $J_\alpha, J_\beta \subset I$. Then

$$\lim_{\alpha} \text{Tr}_{\mathcal{A}}(\Omega_{J_\alpha}(D)) = \lim_{\beta} \text{Tr}_{\mathcal{A}}(\Omega_{J_\beta}(D))$$

and both limits equal $\text{ind}_{\mathcal{A}}(D)$.

Proof. Because both nets are cofinal in the directed set of countable subsets of I , they refine each other: for every J_α there exists $J_\beta \supseteq J_\alpha$, and conversely. Hence, by lemma 2.3.1 (Curvature Consistency), the curvature families $\Omega_{J_\alpha}(D)$ and $\Omega_{J_\beta}(D)$ are asymptotically equivalent after passing to appropriate cofinal subnets. By lemma 2.4.1 (stability of $\text{Tr}_{\mathcal{A}}$ under asymptotic equivalence), their traced limits coincide. Finally, theorem 3.1.1 identifies this common limit with $\text{ind}_{\mathcal{A}}(D)$. $\square$

Example 3.3.3 (Reduction to βN and recovery of the l^∞ model). If $K = \beta N$, then $C(K) \cong \ell^\infty(N)$. Choosing $I = N$ in the product model reduces the quotient net to the familiar coordinate compressions, and proposition 3.3.1.1 becomes a special case of the $C(K)$ framework. In particular, for the shift–identity operator $D = S \oplus I$ (viewed now in $C(\beta N)$), the curvature trace stabilises to -1 and equals the approximate index.

3.3.3. Extension to genuinely pseudodifferential APDOs

We now show that the preceding model constructions are not artefacts of simple block operators (shift $\oplus$ identity), but persist for broad classes of operators that admit a genuinely pseudodifferential interpretation within the approximate calculus developed in section 2. In particular, we consider: (i) band-dominated APDOs on coarse discrete spaces Γ ; and (ii) cylinder-symbol APDOs on $C([0,1]^I)$ with uncountable I . In both cases, we prove that the curvature–index correspondence remains valid and that the curvature trace establishes to

an integer equal to the approximate index.

Proposition 3.3.3.1 (Band-dominated APDOs on coarse spaces). Let Γ be an uncountable discrete set equipped with a coarse structure given by a bounded-geometry graph metric d . Let

$$X = \ell^\infty(\Gamma)$$

and let $D \in \mathcal{B}(X)$ be a band-dominated operator, i.e. the norm-limit of operators $D^{(R)}$ with finite propagation:

$$(D^{(R)}x)(\gamma) = \sum_{\gamma' : d(\gamma, \gamma') \leq R} a_{\gamma, \gamma'} x(\gamma'), \sup_{\gamma, \gamma'} |a_{\gamma, \gamma'}| < \infty$$

Assume further that:

i. (Approximate ellipticity). There exists $c > 0$ such that, for all sufficiently large $\Lambda \subset \Gamma$, the compressed operator $D_\Lambda = P_\Lambda D P_\Lambda$ is Fredholm and $\|(D_\Lambda x) - c\|x\|$ modulo finite-rank errors.

ii. (Quasi-locality). $\|[P_\Lambda, D]\| \rightarrow 0$ along the admissible net of countable subsets $\Lambda \subset \Gamma$.

Then D is a first-order APDO and satisfies

$$\text{ind}_{\mathcal{A}}(D) = \lim_{\Lambda} \text{Tr}_{\mathcal{A}}(\Omega_{\Lambda}(D))$$

with the limit stabilising to an integer.

Proof. Band-dominated operators admit a natural approximate symbol calculus indexed by finite propagation radius. On each separable fibre $\ell^\infty(\Lambda)$, D_Λ behaves as a discrete first-order pseudodifferential operator, with ellipticity controlled uniformly by assumption (1). Approximate parametric exist by standard band-dominated Fredholm theory, and the defect operators are finite-rank modulo $\mathfrak{o}_{\Lambda}(1)$.

Let $A = M_w$ be a Lipschitz weight multiplication operator on Γ . Then $[A, D]$ is again band-dominated and defines an approximate connection. The resulting curvature operator $\Omega_{\Lambda}(D)$ localises near the boundary of Λ and coincides, up to $\mathfrak{o}_{\Lambda}(1)$, with the difference of the parametric defects. Applying the finitely additive trace yields an integer equal to the stabilised Fredholm index of D_{Λ} , which is independent of Λ for large Λ . $\square$

Example 3.3.3.1 (Discrete Dirac-type operators). Let $\Gamma = \mathbb{Z} \sqcup \Gamma'$ with Γ' uncountable, and define D to act as a discrete Dirac operator on $\ell^\infty(\mathbb{Z})$ (finite-difference first order) and as the identity on $\ell^\infty(\Gamma')$. Then D is band-dominated and approximately elliptic. Its curvature trace stabilises to the integer given by the classical index of the discrete Dirac operator on $\mathbb{Z}$, illustrating the robustness of the curvature–index mechanism under coarse perturbations.

Proposition 3.3.3.2 (Cylinder-symbol APDOs on $C([0,1]^I)$). Let I be uncountable and let $K = [0,1]^I$. Let $X = C(K)$. Let $D \in \mathcal{B}(X)$ be an operator such that there exists a countable family of coordinates $J_0 \subset I$ with the following property:

For every countable $J \supseteq J_0$, the compression of D to

$$U_J = \pi_J^*(C([0,1]^J))$$

corresponds, under the identification $U_J \simeq C([0,1]^J)$, to a classical first-order pseudodifferential operator with elliptic symbol on the compact manifold with corners $[0,1]^J$. Then D is a first-order APDO on $C(K)$, and

$$\text{ind}_{\mathcal{A}}(D) = \lim_J \text{Tr}_{\mathcal{A}}(\Omega_J(D))$$

with stabilisation to an integer independent of the chosen cofinal net of countable coordinate sets.

Proof. For each countable J , the operator $D_J := P_J D P_J$ admits a classical parametric Q_J on $C([0,1]^J)$, and the index $\text{ind}(D_J)$ is given by the Atiyah–Singer formula (see Atiyah & Singer, 1968) on the finite-dimensional cube $[0,1]^J$. Because the symbol depends only on coordinates in J_0 , this index is constant for all $J \supseteq J_0$.

The approximate connection may be taken to be induced by a bounded derivation acting on cylinder functions. The resulting curvature $\Omega_j(D)$ coincides, up to $\mathfrak{o}_j(1)$, with the local index density on $[0,1]^j$. Applying the finitely additive trace extracts precisely the stabilised integer index. Independence of the net follows from lemma 2.1 and proposition 3.3.2.2. $\square$

Example 3.3.3.2 (Cylinder Dirac operators). Let D act as a Dirac-type operator on finitely many coordinates of $[0,1]^I$ and trivially on the remaining coordinates. Then the curvature trace $Tr_{\mathfrak{A}}(\Omega_j(D))$ stabilises to the classical Dirac index on the finite-dimensional cube determined by those coordinates, independently of how the remaining uncountably many directions are treated.

3.4. Relation with Classical Noncommutative Geometry and the Separable Limit

This subsection clarifies how the framework developed in this article relates to classical noncommutative geometry and index theory in the separable setting, and how the present results should be understood as a genuine extension rather than an alternative formulation.

3.4.1. Recovery of the classical index in the separable case

Suppose that the underlying Banach space X is separable (or, more restrictively, a Hilbert space), and that the admissible net $\mathfrak{U} = (U_\alpha)$ may be chosen countable and increasing, with U_α exhausting X in norm. In this case, the finitely additive trace $Tr_{\mathfrak{A}}$ reduces to a σ -additive trace on an appropriate ideal of operators (trace class, Dixmier trace, or Wodzicki residue, depending on the order of the operator). Under these assumptions, approximate pseudodifferential operators coincide with classical pseudodifferential operators, approximate compactness coincides with compactness, and approximate Fredholmness reduces to classical Fredholmness. The curvature family $\Omega_\alpha(D)$ converges (in the appropriate weak sense) to the usual curvature density associated with the operator D .

Consequently, the curvature–index correspondence established in theorem 3.1.1 collapses to the familiar formula expressing the analytic index in terms of local curvature data, recovering classical results from Atiyah–Singer theory and, in the noncommutative setting, from the Connes–Moscovici local index formula. In this sense, the present framework is fully consistent with the classical theory in the separable limit [13].

3.4.2. Why classical spectral triples fail beyond separability

The necessity of the present approach becomes apparent when separability is lost. In non-separable Banach spaces, several foundational ingredients of classical noncommutative geometry fail simultaneously: (i) compact resolvents typically do not exist; (ii) trace-class and Schatten ideals are no longer meaningful globally; (iii) σ -additive traces cannot be defined on sufficiently large operator algebras; (iv) spectral growth and metric dimension become ill-defined or degenerate.

As a result, the standard notion of a spectral triple cannot be extended in any canonical way to non-separable Banach spaces such as $\ell^\infty(\Gamma)$ for uncountable Γ or $C(K)$ for non-metrisable compact spaces. The obstruction is not technical but structural: it arises from the absence of countable localisation and σ -additivity.

The framework developed in this article replaces these missing structures by directed localisation, asymptotic symbolic control, and finitely additive traces, which are precisely the minimal tools that survive in the non-separable context.

3.4.3. Conceptual shift: from spectral data to asymptotic curvature

From a conceptual standpoint, the main shift introduced here is the replacement of global spectral data by asymptotic curvature data. In the separable theory, curvature is ultimately recovered from spectral asymptotic via trace formulas. In the non-separable setting, spectral asymptotic are unavailable, but curvature remains accessible through commutator defects of approximate parametric.

The results of Section 3 show that this asymptotic curvature is not merely a formal substitute: it encodes the analytic index completely, is integer-valued, and is stable under refinement of localisation nets and under approximate compact perturbations. Thus, curvature—understood in this asymptotic, noncommutative sense—emerges as the fundamental invariant that survives the loss of separability.

4. Discussion

This article develops a framework for curvature, index theory, and pseudodifferential calculus on non-separable Banach spaces, based on approximate localisation, asymptotic symbolic control, and finitely additive traces. The results of sections 2 and 3 demonstrate that, despite the collapse of classical countability assumptions, a robust analytic–geometric theory persists. In this final section we synthesise the conceptual implications of the work, discuss its limitations, and outline several promising directions for future research.

4.1. Conceptual Significance: Curvature without Separability

A central message of the paper is that curvature is more fundamental than spectrum in highly non-separable settings. Classical index theory extracts curvature information from spectral asymptotic using σ -additive traces and compactness assumptions. In contrast, non-

separable Banach spaces typically lack compact resolvents, trace-class ideals, and meaningful global spectral growth.

The present framework shows that curvature can instead be recovered from asymptotic commutator defects of approximate parametric. These defects are local in nature, detectable on directed families of separable windows, and stable under refinement. The curvature–index correspondence established here confirms that curvature remains an integer-valued invariant encoding analytic index even when spectral data are unavailable or ill-defined.

From a geometric standpoint, this represents a shift from global spectral geometry to local asymptotic geometry, where directed localisation replaces atlases and finitely additive traces replace integrals.

4.2. Robustness and Independence Phenomena

An important feature of the theory is its robustness. The approximate index and curvature trace are shown to be: (i) independent of the chosen admissible localisation net; (ii) invariant under approximate compact perturbations; (iii) stable under homotopies of APDOs; (iv) compatible with reductions to separable cores.

These properties suggest that the invariants constructed here are intrinsic to the operator and the ambient Banach space, rather than artefacts of auxiliary choices. In particular, the independence results indicate that the finitely additive nature of the trace is not a weakness but a necessary adaptation to the non-separable context.

4.3. Relation to Classical and Noncommutative Geometry

In the separable limit, the theory collapses to the classical picture: approximate pseudodifferential operators become genuine pseudodifferential operators, approximate compactness becomes compactness, finitely additive traces become σ -additive, and the curvature–index correspondence recovers the familiar local index formulas of Atiyah–Singer and Connes–Moscovici type.

Beyond separability, however, classical spectral triples fail structurally. The present work suggests that approximate curvature triples—based on directed localisation, commutators, and asymptotic traces—are a more flexible and natural replacement. This reframing opens a path towards a noncommutative geometry that is not tied to Hilbert space representations or spectral growth conditions.

4.4. Limitations and Open Technical Questions

While the framework is deliberately general, several limitations and open questions remain:

- i. Trace non-uniqueness. Finitely additive traces are not unique in general. Understanding canonical choices, uniqueness criteria, or equivalence classes of traces is an important problem.
- ii. Higher-order and secondary invariants. The present paper focuses on first-order APDOs and integer-valued indices. Extensions to higher-order operators, secondary invariants, or torsion-type phenomena remain open.
- iii. Analytic regularity. The theory relies on asymptotic control rather than precise regularity. Developing sharper regularity notions compatible with non-separability would strengthen the analytic foundations.

4.5. Perspectives and Future Directions

The results presented here suggest several promising directions for further research:

- a. Non-separable index theory in mathematical physics. Large quantum systems, mean-field limits, and algebras of observables with uncountably many degrees of freedom naturally give rise to non-separable Banach spaces.
- b. Coarse and large-scale geometry. The band-dominated and coarse-space examples indicate strong connections with coarse index theory and large-scale geometry.
- c. Operator algebras beyond C^* -contexts. Extending approximate curvature methods to Banach algebras not admitting C^* -completions could broaden the scope of noncommutative geometry.
- d. Foundations of geometry and analysis. Conceptually, the work contributes to a broader programme in which geometric invariants are reformulated in terms of approximation, asymptotic, and localisation, rather than exact structures.

5. General Conclusions

This article has developed a coherent framework for approximate noncommutative curvature, index theory, and pseudodifferential calculus on non-separable Banach spaces. By systematically replacing countability-based structures with directed localisation, asymptotic symbolic control, and finitely additive traces, we have shown that core analytic–geometric invariants survive well beyond the classical separable setting.

The principal achievement is the establishment of a curvature–index correspondence valid for first-order approximate pseudodifferential operators on non-separable Banach spaces. The analytic index, defined via approximate Fredholm theory, is recovered as the asymptotic

trace of an approximate curvature operator. This correspondence is robust: it is independent of the chosen admissible net, stable under approximate compact perturbations and homotopies, and reduces to the classical Atiyah–Singer and Connes–Moscovici formulas in the separable limit.

A key conceptual insight of the work is that curvature, rather than spectral data, constitutes the fundamental invariant in highly non-separable contexts. While classical noncommutative geometry relies on spectral triples, compact resolvents, and σ -additive traces, these structures fail structurally in spaces such as $\ell^\infty(\Gamma)$ for uncountable Γ or $C(K)$ for non-metrisable compact spaces. The present framework demonstrates that curvature can instead be encoded in commutator defects of approximate parametrices and extracted through finitely additive traces, yielding integer-valued invariants with genuine geometric meaning.

The theory has been illustrated through explicit and genuinely non-trivial models: block-type operators and band-dominated APDOs on $\ell^\infty(\Gamma)$, as well as cylinder-symbol pseudodifferential operators on $C([0,1]^I)$ for uncountable index sets. In all cases, the curvature trace stabilises to an integer coinciding with the approximate index, confirming that the loss of separability does not destroy index-theoretic information but merely changes the mechanisms through which it is accessed.

Beyond the specific results proved here, the work suggests a broader shift in perspective. Geometry in large analytic settings should be understood as asymptotic and localisable, rather than spectral and globally integrable. Directed nets, approximate symbols, and finite additivity are not technical compromises but essential structural tools for extending geometry to its largest possible domain.

It is hoped that the methods introduced in this article will provide a foundation for further developments in non-separable analysis, coarse geometry, mathematical physics, and the foundations of noncommutative geometry. In particular, they point towards a future theory in which curvature and index remain meaningful even when traditional spectral and measure-theoretic frameworks are no longer available.

References

1. Connes, A. (1994). Noncommutative geometry. Cambridge: *Academic Press*.
2. Higson, N., & Roe, J. (2000). Analytic K-homology, ser.
3. Connes, A., & Moscovici, H. (1995). The local index formula in noncommutative geometry. *Geometric & Functional Analysis GFA*, 5(2), 174-243.
4. Diestel, J. (1984). Sequences and series in Banach spaces. New York, NY: *Springer*.
5. Fabian, M., Habala, P., Hájek, P., Santalucía, V. M., Pelant, J., & Zizler, V. (2001). Functional analysis and infinite-dimensional geometry (Vol. 8). New York: *Springer*.
6. Simon, B. (1979). Trace ideals and their applications.
7. Lord, S., Sukochev, F., & Zanin, D. (2012). Singular traces: theory and applications (Vol. 46). *Walter de Gruyter*.
8. Wodzicki, M. (1987). Noncommutative residue. I. Fundamentals. In S. Gindikin & V. Guillemin (Eds.), *Functional analysis on the eve of the 21st century*, Vol. 1 (pp. 320–399). Boston, MA: Birkhäuser.
9. Kato, T. (1995). Perturbation theory for linear operators (2nd ed.). Berlin: *Springer*.
10. Taylor, A. E., & Davis, P. J. (1959). Introduction to functional analysis. *Physics Today*, 12(6), 48-50.
11. Roe, J. (1993). Coarse cohomology and index theory on complete Riemannian manifolds (Vol. 497). American Mathematical Soc.
12. Roe, J. (2003). Lectures on coarse geometry (No. 31). *American Mathematical Soc.*
13. Atiyah, M. F., & Singer, I. M. (1968). The index of elliptic operators: I. *Annals of mathematics*, 87(3), 484-530.

Copyright: ©2026 Juan Alberto Molina García. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.