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Approximate C^*- and W^*-Algebras on Non-Separable Banach Spaces: Structure, Positivity, and Representation Theory

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Abstract

This paper develops a comprehensive algebraic framework for extending the classical theory of C^*- and W^*-algebras to non-separable Banach spaces. By constructing Approximate C^*- and $W*$-algebras, defined as directed inductive limits of local operator algebras acting on separable subspaces, we obtain a rigorous structure that preserves the essential algebraic, topological, and spectral properties of classical operator algebras while overcoming the limitations imposed by separability. The first part of the article establishes the fundamental definitions of approximate C^*-algebras, including approximate positivity, involution stability, and norm consistency under directed limits of projections. We then formulate and prove an Approximate Gelfand–Naimark theorem, showing that every approximately self-adjoint algebra can be faithfully represented as an approximate norm-closed-subalgebra of $B(X)$ for some non-separable Banach space X. The second part introduces Approximate W^*-algebras, defined through projective limits of preduals corresponding to the local separable components. We demonstrate that these algebras retain weak - compactness and support approximate normal states, extending the classical duality between $W*$-algebras and Banach pre-duals to the non-separable setting. An Approximate Bicommutant Theorem is also established, identifying the approximate double commutant with the closure of the algebra under the approximate weak*-topology. The third part develops an Approximate Representation Theory. By generalising the Gelfand– Naimark–Segal (GNS) construction, we show that every approximate positive functional induces a cyclic approximate representation, and that factor decompositions (of types I, II, and III) extend naturally to this setting through local coherence conditions. Several explicit examples—particularly within ℓ^∞, L^∞, and $C(\beta\mathbb{N})$—illustrate the structural behaviour of approximate $C*$- and $W*$-algebras, their spectra, and their reflexive properties. This framework provides a unified foundation for analysing operator algebras on non-separable Banach spaces, reconciling algebraic and topological aspects in the absence of countable bases. The results obtained not only generalise classical operator algebra theory but also open the door to future developments in noncommutative geometry and infinite-dimensional mathematical physics.*

Keywords: Non-Separable Banach Spaces, C^*-Algebras; W^*-Algebras, Approximate Operator Algebras, Gelfand–Naimark Theorem, *Weak**-Topology, Reflexivity, Representation Theory, Functional Analysis

1. Introduction

The classical theory of operator algebras, inaugurated by von Neumann. and subsequently formalised through the works of Sakai, Kadison and Ringrose, and Takesaki, provides a profound connection between algebraic structure and topological duality. C^* -algebras and W^* -algebras (or von Neumann algebras) constitute the analytic core of modern functional analysis and mathematical physics, offering a unified language for bounded operators on Hilbert and Banach spaces. Their success relies heavily on two assumptions: completeness and separability [1-5]. While completeness guarantees closure under operator topologies, separability ensures that spectral and dual constructions can be expressed through sequences and countable bases.

However, separability, though convenient, is not intrinsic to the algebraic essence of operator theory. Many natural and physically relevant settings—such as ℓ^∞, $L^\infty(\mu)$ with non-σ-finite measures, and $C(\beta\mathbb{N})$ —are non-separable Banach spaces that nonetheless host rich operator structures [6]. The restriction to separable environments thus obscures an entire landscape of analytic phenomena: operators defined on non-metrizable duals, *weak**-compact but non-sequential limits, and algebras lacking countable dense subsets. In these contexts, the classical notions of spectrum, positivity, and weak operator topology must be re-examined.

Recent advances in the theory of approximate operator algebras have provided a pathway to overcoming these limitations. The key idea, originally inspired by constructions in non-separable Banach space theory, is to replace sequential arguments by net-based consistency and to reconstruct algebraic and topological properties through directed systems of separable subspaces.

Given a non-separable Banach space X, one considers a directed family of separable, weakly dense subspaces $(X_F)_{F\in\mathcal{F}}(I)$ with associated projections $P_F: X \to X_F$. The corresponding family of local operator algebras $\mathcal{A}_F = P_F B(X) P_F$ yields an inductive system whose limit defines the approximate algebra

$$\mathcal{A}^{approx} = \lim_{\substack{\to \\ F\in\mathcal{F}(I)}} \mathcal{A}_F$$

This construction captures the local behaviour of operators on separable components while maintaining global coherence across the non-separable space. The objective of this paper is to develop, within this framework, the approximate analogues of C^*- and W^*-algebras, denoted respectively by $\mathcal{A}^{approx}_{C^*}$ and $\mathcal{A}^{approx}_{W^*}$. These algebras preserve the essential features of their classical counterparts—norm closure, involution, positivity, and duality—while extending them to spaces beyond the reach of sequential compactness. In particular, we establish:

i. The notion of *approximate positivity*, ensuring that the algebra remains closed under approximate *-operations and positive functional evaluation.

ii. An *Approximate Gelfand–Naimark Theorem*, proving that every approximately self-adjoint algebra can be faithfully represented as an approximate-subalgebra of $B(X)$ for some non-separable X.

iii. The definition and analysis of *Approximate* W^* -Algebras, characterised by local pre-duals and global *weak**-compactness, including an *Approximate Bicommutant Theorem*.

iv. A complete *Approximate Representation Theory*, generalising the Gelfand– Naimark–Segal (GNS) construction to directed limits of local states.

These results extend the structural unity of operator algebra theory to nonseparable Banach spaces, offering a consistent description of algebraic duality, positivity, and reflexivity without relying on countability. They also provide a theoretical foundation for further developments in noncommutative geometry and the mathematical formulation of infinite-dimensional physical systems.

The remainder of the paper is organised as follows. Section 2 reviews the classical background on C^* - and W^* -algebras, operator topologies, and representation theory. Section 3 introduces approximate C^* -algebras, defines approximate positivity, and establishes the *Approximate Gelfand–Naimark Theorem*. Section 4 develops the theory of approximate W^* -algebras and proves the *Approximate Bicommutant Theorem*. Section 5 presents the representation theory and GNS framework for approximate positive functionals. Section 6 contains illustrative examples and structural results, while Section 7 discusses the broader implications and future directions of this research.

2. Preliminaries and Classical Background

The purpose of this section is to review the fundamental concepts of C^*-algebras, W^* -algebras, and their representations, which form the basis for the approximate generalizations developed in later sections. We recall the essential definitions, duality properties, and topologies that govern the structure of operator algebras on Banach and Hilbert spaces, with emphasis on the role of separability.

2.1. C*-Algebras

A C^*-algebra is a complex Banach algebra $\mathcal{A}$ equipped with an involution $T \mapsto T^*$ satisfying the $C*$-identity

$$\|T^* T\| = \|T\|^2, \forall \ T \in \mathcal{A}.$$

Every *closed**-subalgebra of $B(\mathcal{H})$, the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$, is a C^* -algebra [2,3]. Conversely, by the Gelfand–Naimark Theorem, every C^* -algebra is *-isomorphic to a norm-closed *-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

An element $T \in \mathcal{A}$ is self-adjoint if $T = T^*$, and positive if there exists $S \in \mathcal{A}$ such that $T = S^* S$.

The spectrum of T, denoted $\sigma(T)$, is the set of complex numbers λ for which $T-\lambda I$ is not invertible.

The spectral radius is $r(T) = \sup \{|\lambda|:\lambda \in \sigma(T)\}$, and it satisfies $r(T) = \lim_{n\to\infty} \|T^n\| \, 1/n$.
A state on $\mathcal{A}$ is a positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(I) = 1$. Every state defines a cyclic representation through the Gelfand–

Naimark–Segal construction [7-9].

Given φ , one defines a pre-Hilbert space $\mathcal{H}_\varphi^{(0)} = \mathcal{A}/\mathcal{N}_\varphi$, where $\mathcal{N}_\varphi = \{T \in \mathcal{A}: \varphi(T^*T) = 0\}$, with inner product

$$\langle [T],[S] \rangle_\varphi = \varphi(S^*T)$$

Completing $\mathcal{H}_\varphi^{(0)}$ yields a Hilbert space $\mathcal{H}_\varphi$, and the representation $\pi_\varphi : \mathcal{A} \to B(\mathcal{H}_\varphi)$ is defined by

$$\pi_\varphi(T)[S] = [TS], T, S \in \mathcal{A}$$

Then, $\varphi(T) = \langle \pi_\varphi(T)\xi_\varphi, \xi_\varphi \rangle$, where $\xi_\varphi = [I]$ is cyclic.

2.2. W^*-Algebras (von Neumann Algebras)

A W^*-algebra or von Neumann algebra is a C^*-algebra $\mathcal{M} \subseteq B(\mathcal{H})$ that is closed in the weak operator topology (WOT) or, equivalently, in the strong operator topology (SOT) [2,5].

Equivalently, $\mathcal{M}$ satisfies the Bicommutant Theorem:

$$\mathcal{M} = \mathcal{M}'' = \{T \in B(\mathcal{H}): TS = ST \ \forall S \in \mathcal{M}'\}$$

A von Neumann algebra $\mathcal{M}$ always possesses a pre-dual $\mathcal{M}_*$, that is, a Banach space such that $\mathcal{M} = (\mathcal{M}_*)^*$. This property distinguishes W^*-algebras from general C^*-algebras.

The weak operator topology on $B(\mathcal{H})$ is the weakest topology for which all functionals $T \mapsto \langle Tx, y \rangle$ (with $x, y \in \mathcal{H}$) are continuous.

The strong operator topology corresponds to pointwise convergence on vectors. Both topologies are weaker than the norm topology and coincide on bounded subsets when restricted to reflexive subspaces.

States on W^*-algebras are classified as normal, singular, or pure depending on their continuity with respect to the *weak**-topology induced by the pre-dual $\mathcal{M}_*$. Normal states are precisely those arising from trace-class operators or density matrices on $\mathcal{H}$.

2.3. Positivity, Order, and Duality

The positive cone of a C^*-algebra $\mathcal{A}$ is the set

$$\mathcal{A}_+ = \{T^*T : T \in \mathcal{A}\}$$

It induces a partial order: $S \leq T$ if $T - S \in \mathcal{A}_+$. Every positive functional φ satisfies $\varphi(T^*T) \geq 0$. If $\mathcal{A}$ is unital , then $\| T \| = sup \ \{| \varphi(T) |: \varphi \text{ is a state on } \mathcal{A}\}$. This dual characterization of the norm establishes the self-duality of C^* -algebras [3].

In the W^*-setting, the duality is refined: the dual space $\mathcal{M}_*$ decomposes as $\mathcal{M}_* \oplus \mathcal{M}_*^\perp$, where $\mathcal{M}_*$ is the predual and $\mathcal{M}_*^\perp$ consists of singular functionals. Weak*-compactness of the unit ball in $\mathcal{M}_*$ follows from the Banach–Alaoglu theorem.

2.4. Operator Topologies

For completeness, we recall the following classical operator topologies on $B(\mathcal{H})$:

a. Norm topology ($\|\cdot\|$):

$$T_\alpha \to T \text{ if and only if } \|T_\alpha - T\| \to 0$$

b. Strong operator topology (SOT):

$$T_\alpha \to T \text{ if and only if } T_\alpha x \to Tx \text{ for all } x \in \mathcal{H}$$

c. Weak operator topology (WOT):

$$T_\alpha \to T \ \ if\ and\ only\ if\ \langle T_\alpha x\ , y\rangle \to \langle T_x , y\rangle \text{ for all } x, y \in \mathcal{H}$$

d. Ultraweak topology:

The weak*-topology induced by the pre-dual $\mathcal{M}_*$ on a W_* -algebra $\mathcal{M}$ .These topologies satisfy the inclusions:

$$Norm \Rightarrow SOT \Rightarrow WOT \Rightarrow Ultraweak$$

In separable settings, the weak and strong topologies can often be described sequentially, but in non-separable spaces, nets must replace sequences to capture convergence.

2.5. Motivation for the Approximate Framework

The need to extend these notions beyond separability arises from both analytic and physical motivations. Analytically, many Banach spaces of interest—such as $L^\infty(\mu)$ when μ is not σ-finite—fail to be separable but remain the natural domains for measurable operators. Physically, non-separable Hilbert or Banach spaces appear in infinite tensor product systems, quantum field theory, and ergodic systems with uncountable degrees of freedom [10]. In such cases, classical results like the Gelfand– Naimark or Bicommutant theorems must be reformulated.

The approximate approach replaces sequential compactness by directed weak compactness: algebraic operations are preserved locally on separable components, and global coherence is ensured by inductive or projective limits.

The next section formalizes this approach by introducing *approximate* C^* algebras, developing their algebraic properties, positivity, and spectral structure in the absence of separability.

3. Approximate C*-Algebras on Non-Separable Banach Spaces

In this section we develop the notion of *approximate* C^* *-algebras* on nonseparable Banach spaces (NSBS). The guiding principle is to reconstruct the algebraic and topological features of classical C^*-algebras by using directed systems of separable subspaces and **locally coherent -operations*. This approach generalizes the standard framework of functional analysis to settings where sequential compactness and countable bases fail.

3.1. Approximate Structures and Directed Systems

Let X be a Banach space (not necessarily separable). Denote by $\mathcal{F}(X)$ the directed set of all separable, closed subspaces $X_F \subseteq X$, ordered by inclusion. For each $F \in \mathcal{F}(X)$, let P_F: $X \to X_F$ be the canonical projection satisfying $P_F P_G = P_{\min}(F, G)$. Moreover,

$$\mathcal{A}_F = P_F B(X) P_F \subseteq B(X_F)$$

denotes the algebra of bounded linear operators acting on the separable component X_F.

Definition 3.1.1 (*Directed inductive system*). The family $(\mathcal{A}_F, \iota_{FG})_{F, G \in \mathcal{F}(X)}$, where ι_{FG}: $\mathcal{A}_F \to \mathcal{A}_G$ is the canonical embedding $\iota_{FG}(T) = P_G T |_{X_F} PG$ for $F \subseteq G$, forms a directed inductive system of Banach*-algebras.

Definition 3.1.2 (*Approximate* C^* *-algebra*). The *approximate* C^* *-algebra* associated with X is defined as the inductive limit

$$\mathcal{A}^{\text{approx}}(X) = \lim_{\substack{\rightarrow \\ F \in \mathcal{F}(X)}} \mathcal{A}_F$$

An element $T \in \mathcal{A}^{approx}(X)$ is an equivalence class $[T_F]$ of compatible families $(T_F)_{F \in \mathcal{F}(X)}$, satisfying

$$\iota_{FG}(T_F) = T_G\ |_{X_F} \text{ ,for all } F \subseteq G$$

Addition, multiplication, involution, and norm are defined component wise:

$$(T+S)_F = T_F + S_F, (TS)_F = T_F S_F, (T^*)_F = (T_F)^*, \| T \| = \sup_F \ \| T_F \|$$

Proposition 3.1.1 (C^**-identity*). The norm and involution in $\mathcal{A}^{approx}(X)$ satisfy the C^*-identity:

$$\left||T * T|\right| = \left||T|\right|^2, \forall T \in \mathcal{A}^{approx}(X)$$

Proof. For each $F \in \mathcal{F}(X)$, $\| T_F^* T_F \| = \| T_F \|^2$ in $B(X_F)$, because $\mathcal{A}_F$ is a C^*-algebra. Taking the supremum over all F, we obtain

$$\| T^* T \| = \sup_F \| T_F^* T_F \| = \sup_F \| T_F \|^2 = \| T \|^2$$

Hence $\mathcal{A}^{\text{approx}}(X)$ satisfies the C^*-identity.

3.2. Approximate Positivity and Self-Ad Jointness

Definition 3.2.1 (*Approximate positivity*). An element $T \in \mathcal{A}^{\text{approx}}(X)$ is said to be approximately positive if each representative T_F is positive in $B(X_F)$, i.e.

$$\langle T_F x, x \rangle \geq 0, \forall x \in X_F$$

The cone of positive elements is denoted $\mathcal{A}_+^{approx}(X)$.

Definition 3.2.2 (*Approximate order*). For $S, T \in \mathcal{A}^{\text{approx}}$ (X), we write $S \leq_{approx} T$ if $T_F - S_F \geq 0$ for all $F \in \mathcal{F}(X)$.

Proposition 3.2.1. $\mathcal{A}_+^{approx}(X)$. is a convex cone and is closed under approximate limits.

Proof. If $T, S \in \mathcal{A}_+^{approx}(X)$. and $\alpha, \beta \geq 0$, then $(\alpha T + \beta S)_F = \alpha T_F + \beta S_F \geq 0$ in $B(X_F)$ for all F. Closure under inductive limits follows from the continuity of positivity in the operator norm topology.

3.3. Approximate Spectral Theory

For each $T = [T_F] \in \mathcal{A}^{approx}(X)$, define the approximate spectrum as

$$\sigma_{approx}(T) = \overline{\bigcup_{F \in \mathcal{F}(X)} \sigma(T_F)}$$

This definition ensures spectral coherence between local and global components.

Proposition 3.3.1 (*Spectral mapping*). If p is a polynomial with complex coefficients, then

$$\sigma_{approx}(p(T)) = p(\sigma_{approx}(T))$$

Proof. Since the spectral mapping theorem holds in each $B(X_F)$,

$$\sigma(p(T_F)) = p(\sigma(T_F)), \forall\ F$$

Taking the union and closure over F yields the desired equality.

3.4. Approximate Gelfand–Naimark Theorem

We now establish the central result of this section.

Theorem 3.4.1 (*Approximate Gelfand–Naimark*). Every approximate C^*-algebra $\mathcal{A}^{approx}(X)$ admits a faithful *-representation as an approximate norm-closed *-subalgebra of $B(X)$.

Proof. For each $F \in \mathcal{F}(X)$, the canonical embedding

$$\iota_F : \mathcal{A}_F \hookrightarrow B(X)$$

is defined by $\iota_F(T_F) = P_F T_F P_F$. These embeddings are compatible with the inductive system:

$$\iota_G(\iota_{FG}(T_F)) = \iota_F(T_F), F \subseteq G$$

Hence, they induce a *-homomorphism

$$\iota: \mathcal{A}^{approx}(X) \longrightarrow B(X), \iota([T_F]) = \lim_F P_F T_F P_F$$

where the limit is taken in the strong operator topology. Faithfulness follows from the norm relation

$$\| \iota(T) \| = \sup_F \| T_F \| = \| T \|$$

Therefore, $\mathcal{A}^{approx}(X)$ is *-isomorphic to its image under ι , which is norm-closed in $B(X)$. □

Corollary 3.4.1. Let $\mathcal{A}^{approx}(X)$ be the approximately self-adjoint algebra associated with a (possibly non-separable) Banach space X, constructed as the inductive limit of local algebras

$$\mathcal{A}_F = P_F B(X) P_F \subseteq B(X_F)$$

where $X_F = P_F(X)$ ranges over the family $\mathcal{F}(X)$ of separable closed subspaces of X . Then $\mathcal{A}^{approx}(X)$ is completely determined by its local restrictions to these separable components. In particular, for any

$$T, S \in \mathcal{A}^{approx}(X)$$

one has

$$T = S \Leftrightarrow T_F = S_F, \forall\, F \in \mathcal{F}(X)$$

where $T_F, S_F \in \mathcal{A}_F$ denote the local representatives of T and S.

Proof. Recall that an element $T \in \mathcal{A}^{approx}(X)$ is represented by a compatible family

$$T = [T_F]_{F \in \mathcal{F}(X)}$$

where each $T_F \in \mathcal{A}_F$ satisfies the compatibility condition

$$\iota_{FG}(T_F) = T_G \text{ whenever } F \subseteq G$$

and the norm is given by

$$\| T \| = \sup_{F \in \mathcal{F}(X)} \| T_F \|$$

From theorem 3.1 (Approximate Gelfand–Naimark) we know that the canonical map

$$\iota: \mathcal{A}^{approx}(X) \longrightarrow B(X), \iota([T_F]) = \lim_F P_F T_F P_F$$

is an isometric *-monomorphism. Hence ι is injective.

(⇒) If $T = S$, then $T_F = S_F$ for all F. Suppose that

$$T = S \text{ in } \mathcal{A}^{approx}(X)$$

Then $T - S = 0$. Writing

$$T - S = [T_F - S_F]_{F \in \mathcal{F}(X)}$$

we have

$$0 = \| T - S \| = \sup_{F \in \mathcal{F}(X)} \| T_F - S_F \|$$

Therefore, for each F,

$$\| T_F - S_F \| = 0 \Rightarrow T_F = S_F \; in \; \mathcal{A}_F$$

(⇐) If $T_F = S_F$ for all F, then T=S. Conversely, suppose that

$$T_F = S_F \; for \; all \; F \in \mathcal{F}(X)$$

Then the components of $T-S$ satisfy

$$(T - S)_F = T_F - S_F = 0, \forall F$$

Hence,

$$\| T - S \| = \sup_{F \in \mathcal{F}(X)} \| (T - S)_F \| = \sup_{F} \| 0 \| = 0$$

Thus, $T-S$=0, and therefore T = S in $\mathcal{A}^{approx}(X)$. Alternatively, applying the faithful representation

$$\iota : \mathcal{A}^{approx}(X) \to B(X)$$

we have, for every F,

$$P_F \, \iota(T) \, P_F = P_F \, \iota(S) \, P_F$$

because these compressions coincide with the operators T_F and S_F, respectively. Since the family (P_F) determines the operator $\iota(T)$ uniquely via directed limits in the strong operator topology, we conclude that

$$\iota(T) = \iota(S)$$

Injectivity of ι then implies

$$T = S$$

Conclusion. Both implications are verified, and thus

$$T = S \Leftrightarrow T_F = S_F \; \; for \; all \; F \in \mathcal{F}(X)$$

Hence the approximately self-adjoint algebra $\mathcal{A}^{approx}(X)$ is completely determined by its local restrictions to separable components of X, as claimed.

3.5. Approximate Functional Calculus

Given the local spectral structure, a functional calculus can be extended to $\mathcal{A}^{approx}(X)$.

Definition 3.5.1 (*Approximate functional calculus*). Let T=$[T_F]$ ∈ $\mathcal{A}^{approx}(X)$ be normal. For every continuous function f on $\sigma_{approx}(T)$, define

$$f(T) = [f(T_F)]$$

The norm satisfies

$$\| f(T) \| = \sup_{F} \| f(T_F) \|$$

Proposition 3.5.1. Let $T = [T_F] \in \mathcal{A}^{approx}(X)$ be a normal element, and let

$$\sigma_{approx}(T) = \overline{\bigcup_{F\in\mathcal{F}(X)} \sigma(T_F)}$$

denote its approximate spectrum. For every continuous function $f \in C(\sigma_{approx}(T))$, define

$$f(T) := [f(T_F)]$$

where $f(T_F)$ is given by the usual continuous functional calculus for the normal operator T_F on the separable space X_F. Then, the mapping

$$\Phi_T: C(\sigma_{approx}(T)) \longrightarrow \mathcal{A}^{approx}(X), f \longmapsto f(T)$$

is a well-defined, continuous *-homomorphism. In particular, it satisfies:

1. $\Phi_T(f + g) = \Phi T(f) + \Phi_T(g)$ and $\Phi_T(fg) = \Phi_T(f)\,\Phi_T(g)$ for all $f, g \in \boldsymbol{C(\sigma_{approx}(T))}$;
2. $\Phi_T(\bar{f}) = \Phi_T(f)^*$ for all $f \in C(\sigma_{approx}(T))$;
3. $\|\Phi_T(f)\| = \|f(T)\| = sup\ F \in \mathcal{F}(X) \|f(T_F)\| \le \|f\|_\infty$, so that Φ_T is contractive and therefore continuous.

Proof. Fix $F \in \mathcal{F}(X)$. Since T_F is normal in the C^* -algebra $\mathcal{A}_F \subseteq B(X_F)$, the classical continuous functional calculus provides a unique *-homomorphism

$$\Phi_F: C(\sigma(T_F)) \longrightarrow C^*(T_F) \subseteq \mathcal{A}_F, f_F \longmapsto f_F(T_F)$$

satisfying:

a) $(f_F + g_F)(T_F) = f_F(T_F) + g_F(T_F)$,

b) $(f_F g_F)(T_F) = f_F(T_F)\, g_F(T_F)$,

c) $\bar{f}_F(T_F) = f_F(T_F)^*$,

d) $\| f_F(T_F) \| \le \| f_F \|_{\infty,\sigma(T_F)}$.

This is the standard continuous functional calculus for normal elements in a C^*-algebra [2,3].

Now let $f \in C(\sigma_{approx}(T))$. For each F, the restriction

$$f_F := f\,|_{\sigma(T_F)} \in C(\sigma(T_F))$$

is continuous. We then define

$$f(T_F) := f_F(T_F) := \Phi_F(f_F) \in \mathcal{A}_F$$

We must check that the family $(f(T_F))\ F \in \mathcal{F}(X)$ is compatible with the inductive system $(\mathcal{A}_F, \iota_{FG})$ used to define $\mathcal{A}^{approx}(X)$.

Fix $F, G \in \mathcal{F}(X)$ with $F \subseteq G$. Recall that the connecting map

$$\iota_{FG}: \mathcal{A}_F \longrightarrow \mathcal{A}_G$$

is given by

$$\iota_{FG}(S_F) = P_G S_F|_{X_F} P_G, S_F \in \mathcal{A}_F$$

and that the local representatives of T satisfy

$$T_G = \iota_{FG}(T_F) whenever\ F \subseteq G$$

The continuous functional calculus for normal operators is functorial with respect to *-homomorphisms: if $\psi : \mathcal{A}_F \to \mathcal{A}_G$ is a *-homomorphism and T_F is normal, then

$$\psi(f_F(T_F)) = f_F(\psi(T_F))$$

In our setting, ι_{FG} is a *-homomorphism from $\mathcal{A}_F$ into $\mathcal{A}_G$, and $T_G = \iota_{FG}(T_F)$. Moreover, on spectra we have

$$\sigma(T_F) \subseteq \sigma(T_G) \subseteq \sigma_{approx}(T)$$

so, the restrictions satisfy

$$f_F = f|_V, f_G = f\ |_{\sigma(T_G)}\ , and\ f_G|_{\sigma(T_F)} = f_F$$

Therefore,

$$\iota_{FG}(f(T_F)) = \iota_{FG}(f_F(T_F)) = f_F(\iota_{FG}(T_F)) = f_F(T_G) = f_G(T_G) = f(T_G)$$

Hence the family $(f(T_F))_F$ is compatible with the inductive system and defines an element

$$f(T) := [f(T_F)]_{F\in\mathcal{F}(X)} \in \mathcal{A}^{approx}(X)$$

Thus, the mapping

$$\Phi_T : C(\sigma_{approx}(T)) \to \mathcal{A}^{approx}(X), f \mapsto f(T) = [f(T_F)]$$

is well-defined.

Let $f, g \in C\,(\sigma_{\text{approx}}(T))$. For each F, by the local continuous functional calculus we have:

a. Linearity: $(f + g)(T_F) = f(T_F) + g(T_F)$. Thus,

$$(f + g)(T) = [(f + g)(T_F)] = [f(T_F) + g(T_F)] = f(T) + g(T)$$

b. Multiplicativity: $fg)\ (T^F)\ = f(T_F)\ \ g(T_F)$. Hence,

$$(fg)(T) = [(fg)(T_F)] = [f(T_F)g(T_F)] = f(T)\,g(T)$$

c. Involution: Let $\bar{f}$ denote the complex conjugate function on $\sigma_{approx}(T)$. Then,

$$\bar{f}(T_F) = f(T_F)^*$$

So,

$$\bar{f}(T) = [\bar{f}(T_F)] = [f(T_F)^*] = f(T)^*$$

Therefore, Φ_T is a *-homomorphism from $C(\sigma_{approx}(T))$ into $\mathcal{A}a^{pprox}(X)$.

For each F, the continuous functional calculus satisfies

$$\| f(T_F) \| \leq \| f_F \|_{\infty,\sigma(T_F)} \leq \| f \|_{\infty,\sigma_{approx}(T)}$$

since $\sigma\,(T_F) \subseteq \sigma_{approx}(T)$. Taking the supremum over all F, we obtain

$$\| f(T) \| = \sup_{F\in\mathcal{F}(X)} \| f(T_F) \| \leq \| f \|_{\infty,\sigma_{approx}(T)}$$

Thus, Φ_T is contractive:

$$\| \Phi_T(f) \| \leq \| f \|_\infty$$

In particular, Φ_T is continuous with respect to the sup norm on $C(\sigma_{approx}(T))$ and the C^*-norm on $\mathcal{A}^{approx}(X)$.

Conclusion. We have shown that for each normal element $T \in \mathcal{A}^{approx}(X)$, the mapping

$$\Phi_T: C(\sigma_{approx}(T)) \to \mathcal{A}^{approx}(X), f \mapsto f(T)$$

is a well-defined, continuous *-homomorphism. This proves Proposition 3.4.

3.6. Summary of Results

We have shown that the class of approximate C^* -algebras retains the essential analytical and algebraic properties of classical C^* -algebras, including norm and *structure coherence, positivity and order preservation, spectral mapping and functional calculus, and a faithful representation as approximate norm-closed subalgebras of $B(X)$. This provides the foundation for the *approximate* W^*-algebra theory, developed in the next section.

4. Approximate W*-Algebras and Duality

In this section, we extend the construction of approximate C*-algebras to the dual framework of *approximate* W^* -algebras. The aim is to capture the duality, weak*compactness, and bicommutant structure characteristic of von Neumann algebras, while avoiding any reliance on separability. We achieve this by introducing approximate preduals and establishing a generalised version of the *Bicommutant Theorem*.

4.1. Motivation and Preliminary Concepts

In classical theory, a W^*-algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is a C^*-algebra that is weak*-closed in the dual pairing with its unique predual $\mathcal{M}_*$ [2,5]. The weak*compactness of the unit ball in $\mathcal{M}_*$ (by Banach–Alaoglu) ensures reflexivity of states and the existence of normal functionals. However, in NSBS, sequential compactness and countable *weak**-bases fail. To recover these features, we introduce local pre-duals $(\mathcal{A}_F)_*$ corresponding to the separable components $X_F \subseteq X$, and reconstruct the global dual structure via a projective limit.

4.2. Approximate Pre-duals

Definition 4.2.1 (*Approximate pre-dual*). Let $\mathcal{A}^{approx}(X) = \lim_{\rightarrow F} \mathcal{A}_F$ be the approximate C^*-algebra associated with X. The *approximate pre-dual* is defined as the projective limit

$$(\mathcal{A}^{approx}(X))_* = \lim_{\substack{\leftarrow \\ F \in \mathcal{F}(X)}} (\mathcal{A}_F)_*$$

where the bonding maps $\pi_{GF}:(\mathcal{A}_G)_* \to (\mathcal{A}_F)_*$ are given by restriction of functionals. An element $\varphi = [\varphi_F] \in (\mathcal{A}^{approx}(X))^*$ is a compatible family of functionals such that

$$\varphi_G |_{\mathcal{A}_F} = \varphi_F, F \subseteq G$$

Proposition 4.2.1 (*Dual pairing*). The canonical bilinear form

$$\langle T, \varphi \rangle = \lim_{F \in \mathcal{F}(X)} \varphi_F(T_F)$$

defines a non-degenerate pairing between $\mathcal{A}^{approx}(X)$ and its approximate pre-dual.

Proof. Compatibility of the inductive and projective systems ensures that the limit is well-defined. If $\langle T,\varphi \rangle = 0$ for all φ , then each component T_F vanishes because $(\mathcal{A}_F)_*$ separates points in $\mathcal{A}_F$. Hence, $T = 0$. Non-degeneracy follows. □

Definition 4.2.2 (*Approximate weak- topology**). The *approximate weak- topology** on $\mathcal{A}^{approx}(X)$ is the weakest topology such that all functionals $\varphi \in (\mathcal{A}^{approx}(X))_*$ are continuous. A net (T_α) converges to T in this topology if and only if

$$\lim_{\alpha} \langle T_\alpha, \varphi \rangle = \langle T, \varphi \rangle, \forall\, \varphi \in (\mathcal{A}^{approx}(X))_*$$

4.3. Definition of Approximate W^*-Algebras

Definition 4.3.1. An *approximate* W^* -algebra is an approximate C^* -algebra $\mathcal{A}^{approx}(X)$ that is closed in the approximate *weak**-topology induced by its pre-dual $(\mathcal{A}^{approx}(X))_*$. Equivalently, $\mathcal{A}^{approx}(X)$ is the inductive limit of local W^* -algebras $\mathcal{A}_F \subseteq B(X_F)$, each with its own predual $(\mathcal{A}_F)_*$, such that the restriction maps between them are *weak**-continuous.

4.4. Approximate Normal States and Positivity

Definition 4.4.1 (*Approximate normal functional*). A functional $\varphi = [\varphi_F] \in (\mathcal{A}^{approx}(X))_*$ is *approximately normal* if each φ_F is normal on $\mathcal{A}_F$, i.e.

$$\varphi_F(T_F) = \mathrm{Tr}(D_F T_F), D_F \in (\mathcal{A}_F)_*^+$$

Proposition 4.4.1. The set of approximately normal functionals is convex and *weak**-closed.

Proof. Each component $(\mathcal{A}_F)_*^+$ is convex and *weak**-closed; the projective limit of such sets remains convex and closed under the induced topology. □

Definition 4.4.2 (*Approximate state*). An *approximate state* is an approximately normal positive functional $\varphi = [\varphi_F]$ such that

$$\sup_F \varphi_F(I_F) = 1$$

The set of approximate states is denoted by $S(\mathcal{A}^{approx}(X))$.

Proposition 4.4.2 (*Weak*-compactness of states*). The set $S(\mathcal{A}^{approx}(X))$ is compact in the approximate *weak**-topology.

Proof. Each local state space $S(\mathcal{A}_F)$ is *weak**-compact by Banach–Alaoglu. Since the bonding maps between state spaces are continuous and surjective, the projective limit of compact spaces is compact. Hence, $S(\mathcal{A}^{approx}(X))$ is compact. □

4.5. The Approximate Bicommutant Theorem

Let $\mathcal{A}^{approx}(X) \subseteq B(X)$ be an approximate C^* -algebra acting on X . Define the approximate commutant of a subset $S \subseteq B(X)$ as

$$S'_{\text{approx}} = \{T \in B(X) : T_F S_F = S_F T_F \text{ for all } F \in \mathcal{F}(X)\}$$

The approximate bicommutant is

$$S''_{\text{approx}} = (S'_{\text{approx}})'_{\text{approx}}$$

Theorem 4.5.1 (*Approximate Bicommutant Theorem*). Let $\mathcal{A}^{approx}(X) \subseteq B(X)$ be an approximate C^*-algebra. Then, $\mathcal{A}^{approx}(X)$ is closed in the approximate *weak**-topology if and only if

$$\mathcal{A}^{approx}(X) = (\mathcal{A}^{approx}(X))''_{approx}$$

Proof. For each separable subspace X_F , the classical bicommutant theorem implies $\mathcal{A}_F = (\mathcal{A}_F)''$. The inductive system preserves commutation relations, hence

$$(\mathcal{A}^{\text{approx}}(X))''_{\text{approx}} = \lim_{F}\!{}_{\rightarrow}\ (\mathcal{A}_F)'' = \lim_{F}\!{}_{\rightarrow}\ \mathcal{A}_F = \mathcal{A}^{\text{approx}}(X)$$

Conversely, if $\mathcal{A}^{approx}(X) = (\mathcal{A}^{approx}(X))_{approx}''$, it is necessarily closed under approximate *weak**-limits, since each $(\mathcal{A}_F)''$ is weak*-closed. □

4.6. Approximate Reflexivity and Dual Closure

Definition 4.6.1. An approximate W^*-algebra $\mathcal{M}^{\text{approx}}$ is said to be *approximately reflexive* if the canonical map

$$\mathcal{M}^{approx} \longrightarrow (\mathcal{M}^{approx})_{**}, T \mapsto \hat{T}$$

is isometric and surjective, where $(\cdot)_{**}$ denotes the approximate bi-dual.

Proposition 4.6.1. Every approximate W^*-algebra is approximately reflexive.

Proof. Each local W^*-algebra $\mathcal{A}_F$ is reflexive under the dual pairing with its pre-dual. The projective limit of reflexive dual pairs remains reflexive under the natural topology induced by compatibility of projections. Hence, $\mathcal{M}^{approx}$ is approximately reflexive. □

4.7. Concluding Remarks on Duality

The preceding results show that approximate W^* -algebras inherit the essential dual properties of classical von Neumann algebras:
i. Existence of an approximate pre-dual.
ii. *Weak**-compactness of the unit ball and state space.
iii. Closure under approximate *weak**-topology.
iv. Validity of an approximate bicommutant theorem.

Thus, the category of approximate W^*-algebras generalises the classical theory to non-separable contexts, preserving duality and algebraic self-consistency.

The next section develops the *Approximate Representation Theory*, extending the GNS construction and factor decomposition to approximate algebras.

5. Representation Theory

The representation theory of operator algebras plays a central role in understanding their structure and duality. For approximate operator algebras on nonseparable Banach spaces, representations must preserve not only algebraic operations but also the approximate coherence between separable substructures. This section develops the Approximate GNS Theorem, establishes the existence of cyclic representations, and derives an approximate analogue of the factor decomposition of W^*-algebras.

5.1. Approximate Representations

Let $\mathcal{A}^{approx}(X) = lim \to F\mathcal{A}_F$ be an approximate C^* -algebra. For each separable X_F, denote by $\pi_F : \mathcal{A}_F \to B(\mathcal{H}^F)$ a *-representation on a Hilbert space $\mathcal{H}_F$.

Definition 5.1.1 (*Approximate representation*). An *approximate representation* of $\mathcal{A}^{approx}(X)$ is a compatible family $\pi=[\pi_F]$ such that:
i. $\pi_F : \mathcal{A}_F \to B(\mathcal{H}_F)$ is a *-homomorphism for each F.
ii. $F \subseteq G$ implies $\pi_G | \mathcal{A}_F = \pi_F$ up to restriction via the canonical embedding ι_{FG}.
iii. The induced map

$$\pi: \mathcal{A}^{approx}(X) \longrightarrow \lim_{F} \to \quad B(\mathcal{H}_F)$$

is continuous in the approximate weak operator topology.

Proposition 5.1.1. Every approximate representation $\pi = [\pi_F]$ defines a *-homomorphism satisfying

$$\| \pi(T) \| = \sup_{F} \| \pi_F(T_F) \| = \| T \|$$

Proof. For each F , π_F is contractive by the C^* -identity; hence, $\|\pi_F(T_F)\| \leq \|T_F\|$. The supremum over F preserves equality by directed compatibility

5.2. Approximate Positive Functionals

A key step in the GNS construction is the correspondence between states and cyclic representations. Here we extend the definition of positive functionals to the approximate setting.

Definition 5.2.1 (*Approximate positive functional*). A functional $\varphi=[\varphi_F] \in (\mathcal{A}^{approx}(X))_*$ is approximately positive if

$$\varphi_F(T_F^* T_F) \geq 0,\ \forall T_F \in \mathcal{A}_F, F \in \mathcal{F}(X)$$

Lemma 5.2.1. If $\varphi = [\varphi_F]$ is approximately positive, then $|\varphi(T)|2 \leq \varphi\ (T^* T)\ \varphi(I), \forall\ \ T \in \mathcal{A}^{approx}(X)$.

Proof. The Cauchy–Schwarz inequality holds for each component φ_F. Taking the supremum over F yields the inequality for φ. □

5.3. The Approximate GNS Theorem

We now establish the central representation theorem of this section.

Theorem 5.3.1 (*Approximate Gelfand–Naimark–Segal Theorem*). Let $\varphi=[\varphi_F]$ be an approximately positive functional on $\mathcal{A}^{approx}(X)$. Then, there exists a unique (up to approximate unitary equivalence) triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ satisfying:
i. $\pi_\varphi : \mathcal{A}^{approx}(X) \to B(\mathcal{H}_\varphi)$ is an approximate representation;
ii. $\xi_\varphi \in \mathcal{H}_\varphi$ is cyclic, i.e. $\pi_\varphi(\mathcal{A}^{\text{approx}}(X))\ \xi_\varphi$ is dense in $\mathcal{H}_\varphi$;
iii. $\varphi(T) = \langle \pi_\varphi(T)\ \xi_\varphi, \xi_\varphi \rangle$ for all $T \in \mathcal{A}^{\text{approx}}(X)$.

Proof. For each $F\ \in \mathcal{F}(X)$, define the pre-Hilbert space

$$\mathcal{H}_F^{(0)} = \mathcal{A}_F / \mathcal{N}_F, \mathcal{N}_F = \{T_F : \varphi_F(T_F^* T_F) = 0\}$$

with inner product

$$\langle [T_F], [S_F] \rangle_F = \varphi_F(S_F^* T_F)$$

Let $\mathcal{H}_F$ be its completion. Define $\pi_F(T_F)\ [S_F] = [T_F S_F]$. Then, $(\pi_F, \mathcal{H}_F, [I_F])$ is the standard GNS triple for $(\mathcal{A}_F, \varphi_F)$. If $F \subseteq G$, define the embedding $J_{FG} : \mathcal{H}_F \to \mathcal{H}_G$ by $J_{FG}([T_F])\ = [T_G]$, where $T_G = \iota_{FG}(T_F)$. These maps are isometric and coherent:

$$J_{GH} \circ J_{FG} = J_{FH}$$

Hence, the family $(\mathcal{H}_F, J_{FG})$forms a directed system of Hilbert spaces. Define the inductive limit

$$\mathcal{H}_\varphi = \lim_{\substack{\to \\ F \in \mathcal{F}(X)}} \mathcal{H}_F$$

The representation $\pi_\varphi = [\pi_F]$ acts on $\mathcal{H}_\varphi$ by

$$\pi_\varphi([T_F])[S_F] = [T_F S_F]$$

Finally, set $\xi_\varphi = [I_F]$, which satisfies

$$\langle \pi_\varphi(T) \xi_\varphi, \xi_\varphi \rangle = \lim_F\ \varphi_F(T_F) = \varphi(T)$$

Uniqueness up to approximate unitary equivalence follows by the same argument as in the classical GNS construction, applied component wise and extended via the inductive limit. □

Corollary 5.3.1 (*Approximate cyclicity*). Every approximate positive functional induces a cyclic approximate representation.

Proof. Immediate from the construction of $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$. □

5.4. Approximate Factor Decomposition

In classical theory, a von Neumann algebra decomposes into factors of types I, II, and III. The same phenomenon occurs in the approximate framework when decompositions are taken locally and extended coherently.

Definition 5.4.1 (*Approximate factor*). An approximate W^*-algebra $\mathcal{M}^{approx}$ is a factor if its centre is trivial, i.e.

$$Z(\mathcal{M}^{approx}) = \{T \in \mathcal{M}^{approx}: TS = ST \ \forall S \in \mathcal{M}^{approx}\} = \mathbb{C}I$$

Theorem 5.4.1 (*Approximate factor decomposition*). Every approximate W^* -algebra admits a unique decomposition (up to approximate isomorphism)

$$\mathcal{M}^{approx} = \int_{\Omega}^{\oplus} \mathcal{M}_{\omega}^{approx} \, d\mu(\omega)$$

where each $\mathcal{M}_{\omega}^{approx}$ is an approximate factor of type I, II, or III.

Proof. Each local algebra $\mathcal{A}_F$ decomposes as

$$\mathcal{A}_F = \int_{\Omega_F}^{\oplus} (\mathcal{A}_F)_{\omega} \, d\mu_F(\omega)$$

where the summands are factors. The directed system of measure spaces (Ω_F, μ_F)yields a measurable limit (Ω,μ), and the induced decomposition on the inductive limit algebra provides the stated form. Uniqueness follows from the uniqueness of local decompositions and coherence of embeddings.

Corollary 5.4.1. Let $\mathcal{M}^{approx}$ be an approximate W^* -algebra associated with a NSBS X, constructed as the inductive limit of local W^*-algebras $(\mathcal{A}_F)$ $F \in \mathcal{F}(X)$, where each $\mathcal{A}_F \subseteq B(X_F)$ acts on a separable subspace $X_F \subseteq X$. Then the following are equivalent:

a. $\mathcal{M}^{\text{approx}}$ is of type I (in the approximate sense, i.e. it decomposes into approximate type I factors).
b. Each local component $\mathcal{A}_F$ is of type I (in the classical sense of von Neumann algebra theory).

Proof. We recall that, by construction, an approximate W^* -algebra $\mathcal{M}^{approx}$ is obtained as a directed inductive limit

$$\mathcal{M}^{approx} = \lim_{\substack{\rightarrow \\ F \in \mathcal{F}(X)}} \mathcal{A}_F$$

where each $\mathcal{A}_F$ is a W^* -algebra acting on the separable space X_F , and the connecting maps $\iota_{FG}: \mathcal{A}_F \to \mathcal{A}_G$ (for $F \subseteq G$) are normal *-homomorphisms preserving the von Neumann algebra structure. We also recall theorem 5.4.1, which states that any approximate W^*-algebra admits a decomposition

$$\mathcal{M}^{\text{approx}} = \int_{\Omega}^{\oplus} \mathcal{M}_{\omega}^{approx} \, d\mu(\omega)$$

where each $\mathcal{M}_{\omega}^{approx}$ is an approximate factor of type I, II or III. In particular, $\mathcal{M}^{approx}$ is said to be of type I if almost all its approximate factor components $\mathcal{M}_{\omega}^{approx}$ are of type *I*. We now prove the equivalence.

(⇒) If $\mathcal{M}^{approx}$ is of type I, then each $\mathcal{A}_F$ is of *type I*. Assume that $\mathcal{M}^{approx}$ is of *type I*. By definition, this means that in its approximate factor decomposition

$$\mathcal{M}^{\text{approx}} \cong \int_{\Omega}^{\oplus} \mathcal{M}_{\omega}^{approx} \, d\mu(\omega)$$

each factor $\mathcal{M}_{\omega}^{approx}$ is of *type I*.

Fix $F \in \mathcal{F}(X)$. The local algebra $\mathcal{A}_F$ embeds into $\mathcal{M}^{approx}$ via the canonical inclusion

$$\iota_F: \mathcal{A}_F \hookrightarrow \mathcal{M}^{approx}$$

and its image is a von Neumann subalgebra. The approximate factor decomposition of $\mathcal{M}^{approx}$ restricts to a direct integral decomposition of $\iota_F(\mathcal{A}_F)$:

$$\iota_F(\mathcal{A}_F) = \int_{\Omega}^{\oplus} \mathcal{A}_{F,\omega}\, d\mu(\omega)$$

where each $\mathcal{A}_{F,\omega}$ is a von Neumann subalgebra of the type I factor $\mathcal{M}_{\omega}^{approx}$.

Now we use a standard fact from classical von Neumann algebra theory: Every von Neumann subalgebra of a *type I* von Neumann algebra is again of *type I* (Sakai, 1971; Takesaki, 2002.). Thus, each $\mathcal{A}_{F,\omega}$ is of *type I* Since $\mathcal{A}_F$ is (isomorphic to) a direct integral of type I algebras $\mathcal{A}_{F,\omega}$, it follows that $\mathcal{A}_F$ itself is of *type I*. As F was arbitrary, we conclude that:

$$\mathcal{M}^{approx}\ type\ I \Rightarrow \mathcal{A}_F\ type\ I,\ \forall F \in \mathcal{F}(X)$$

(⇐) If each $\mathcal{A}F$ is of *type I,* then $\mathcal{M}^{approx}$ is of *type I*. Assume now that every local component $\mathcal{A}_F$ is of *type I.* We must show that the approximate W^* -algebra $\mathcal{M}^{approx}$ constructed as

$$\mathcal{M}^{approx} = \lim_{\substack{\rightarrow \\ F}}\ \mathcal{A}_F$$

is itself of *type I.*

By theorem 5.4.1, $\mathcal{M}^{approx}$ admits an approximate factor decomposition

$$\mathcal{M}^{approx} = \int_{\Omega}^{\oplus} \mathcal{M}_{\omega}^{approx}\, d\mu(\omega)$$

We claim that each approximate factor $\mathcal{M}_{\omega}^{approx}$ is of *type I*. To see this, fix $\omega \in \Omega$. For each $F \in \mathcal{F}(X)$, the local algebra $\mathcal{A}_F$ is of *type I*, and therefore admits a factor decomposition

$$\mathcal{A}_F = \int_{\Omega_F}^{\oplus} (\mathcal{A}_F)_\eta\, d\mu_F(\eta)$$

with each $(\mathcal{A}_F)_\eta$ a *type I* factor.

The directed system $(\mathcal{A}_F)$ with connecting maps ι_{FG} induces a coherent system of measure spaces (Ω_F, μ_F) and factor components $(\mathcal{A}_F)$ η . Passing to the inductive (and projective) limit of this system yields:
i) a global measurable parameter space (Ω,μ), and
ii) global factor components $\mathcal{M}_{\omega}^{approx}$ corresponding to inductive limits of compatible families $((\mathcal{A}_F)_{\eta(F)})_F$.

Since each local component $(\mathcal{A}_F)_\eta$ is of *type I*, and the connecting maps are *-homomorphisms preserving the type, the inductive limit of such a coherent family is again of *type I*: the *type I* property is stable under direct integrals and directed inductive limits of *type I* factors.

Thus, for μ -almost all $\omega \in \Omega$, the component $\mathcal{M}_{\omega}^{approx}$ is a *type I* factor. By definition, this means that the approximate □□-algebra $\mathcal{M}^{approx}$ is of type I. Hence, we have shown:

$$\mathcal{A}_F\ of\ type\ I,\ \forall F \Rightarrow \mathcal{M}^{approx}\ of\ type\ I$$

Conclusion. Combining both implications, we obtain the equivalence

$$\mathcal{M}^{approx}\ is\ of\ type\ I \Leftrightarrow \mathcal{A}_F\ is\ of\ type\ I\ for\ every\ F \in \mathcal{F}(X)$$

which proves corollary 5.4.1. □

5.5. Approximate Unitary Equivalence

Two approximate representations π_1, π_2 on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ are approximately unitarily equivalent if there exists a family of partial isometries (U_F) such that $U_F \pi_1, F(T_F)U_F{}^* = \pi_{2,F}(T_F)$, $\forall\, F \in \mathcal{F}(X)$. This defines an equivalence relation consistent with inductive limits, ensuring that approximate representations behave functorially with respect to states.

5.6. Summary

The representation theory of approximate operator algebras extends the classical framework to non-separable settings. Each approximately positive functional defines a *cyclic approximate representation*, the approximate GNS construction ensures faithfulness, and the decomposition into factors recovers the typology of von Neumann algebras. These results complete the analytic foundation of the approximate operator algebra theory.

In the next section, we will present explicit examples and structural results for ℓ^∞ ,$L^\infty(\mu)$,$C(\beta\mathbb{N})$, illustrating how approximate duality and representation manifest in concrete non-separable Banach spaces.

6. Structural Results and Examples

In this section, we illustrate the general theory of approximate operator algebras developed in the previous sections by analysing concrete examples. The spaces ℓ^∞, $L^\infty(\mu)$ (with μ not σ -finite), and $C(\beta\mathbb{N})$ provide canonical models of NSBS whose algebraic and topological structures reveal the key features of approximate C^*- and W^* algebras: approximate duality, reflexivity, and local spectral coherence.

6.1. Example 1: The Space ℓ^∞

The Banach space $\ell\infty$ of bounded scalar sequences, endowed with the supremum norm, is non-separable since it contains c_0 as a separable, dense subspace. Each finitely supported subspace

$$X_F = span\{e_1, \dots, e_n\} \subset \ell^\infty$$

is finite-dimensional and therefore separable.

Let $\mathcal{A}_F = B(X_F) = Mn(\mathbb{C})$. The directed family $(\mathcal{A}_F)\ F \in \mathcal{F}$ under canonical inclusions $\iota_{FG} : M_n(\mathbb{C}) \hookrightarrow M_m(\mathbb{C})$ defines the inductive system

$$\mathcal{A}^{approx}(\ell^\infty) = \lim_{\substack{\rightarrow \\ F}} \ M_n(\mathbb{C})$$

which coincides algebraically with the algebra of finite-rank bounded operators on ℓ^∞.

Proposition 6.1. The approximate C^* -algebra $\mathcal{A}^{approx}(\ell^\infty)$ is dense in $B(\ell^\infty)$ under the approximate strong operator topology.

Proof. Each $T \in B(\ell^\infty)$ can be approximated by its finite-dimensional compressions $P_F T P_F$, where P_F is the projection onto X_F. The net $(P_F T P_F)$ converges to T in the approximate strong operator topology because $\|P_F x - x\| \to 0, \forall\ x \in \ell^\infty$. □

Corollary 6.1. Let $T \in B(\ell^\infty)$, and for each finite subset $F \subset \mathbb{N}$, let

$$P_F: \ell^\infty \longrightarrow \ell^\infty$$

denote the canonical coordinate projection onto the finite-dimensional subspace

$$X_F = span\{e_n : n \in F\} \subset \ell^\infty$$

Consider the finite-dimensional compression

$$T_F := P_F T\ |_{X_F}\ P_F \in B(X_F) \cong M_{|F|}(\mathbb{C})$$

Then, the approximate spectrum of T , defined via the approximate operator algebra $\mathcal{A}^{approx}(\ell^\infty)$, satisfies

$$\sigma_{approx}(T) = \overline{\bigcup_F \sigma(T_F)}$$

where the closure is taken in $\mathbb{C}$ and the union ranges over all finite subsets $F \subset \mathbb{N}$.

Proof. We first recall the structure of the approximate algebra $\mathcal{A}^{approx}(\ell^\infty)$ described in section 6.1. For each finite subset $F \subset \mathbb{N}$, we let

$$X_F = span\{e_n : n \in F\} \subset \ell^\infty$$

which is a finite-dimensional (hence separable) subspace of ℓ^∞. The associated projection

$$P_F : \ell^\infty \to X_F$$

is the canonical coordinate projection (zeroing out all coordinates outside F). The local algebra at level F is taken to be

$$\mathcal{A}_F = B(X_F) \cong M_{|F|}(\mathbb{C})$$

and the directed system $(\mathcal{A}_F, \iota_{FG})$ is indexed by the directed set of all finite subsets of $\mathbb{N}$, ordered by inclusion.

If $F \subseteq G$, the connecting *-homomorphism

$$\iota_{FG} : \mathcal{A}_F \to \mathcal{A}_G$$

is given by $\iota_{FG}(S_F) = P_G S_F | X_F P_G$ for $S_F \in \mathcal{A}_F$.

The approximate C^*-algebra associated with ℓ^∞ is then defined as the inductive limit $\mathcal{A}^{approx}(\ell^\infty) = lim_{F} \to \mathcal{A}_F$ and its elements are represented by compatible families $S = [S_F]_F$, $S_F \in \mathcal{A}_F$, with $\iota_{FG}(S_F) = S_G$ whenever $F \subseteq G$.

In the general setting of section 3, for an element $S = [S_F] \in \mathcal{A}^{\mathrm{approx}}(X)$, the approximate spectrum of S is defined by

$$\sigma_{approx}(S) = \overline{\bigcup_F \sigma(S_F)}$$

where $\sigma(S_F)$ denotes the usual (classical) spectrum of the operator S_F acting on the separable space X_F, and the closure is taken in $\mathbb{C}$. This definition applies to any approximate C^*-algebra built as an inductive limit of C^*-algebras $\mathcal{A}_F$.

Now fix $T \in B(\ell^\infty)$. For each finite subset $F \subset \mathbb{N}$, consider the compression

$$T_F := P_F T|_{X_F} P_F \in B(X_F) = \mathcal{A}_F$$

The family $(T_F)_F$ is compatible with the inductive system: if $F \subseteq G$, then

$$\iota_{FG}(T_F) = P_G T_F|_{X_F} P_G = P_G(P_F T P_F)|_{X_F} P_G$$

Since $P_G P_F = P_F$ (because projecting first onto F and then onto a larger set G leaves the coordinates in F unchanged), we have

$$\iota_{FG}(T_F) = P_G P_F T P_F P_G = P_G T P_G = T_G$$

Thus, $\iota_{FG}(T_F) = T_G$ whenever $F \subseteq G$, and the family $(T_F)_F$ defines an element

$$T^{approx} := [T_F]_F \in \mathcal{A}^{approx}(\ell^\infty)$$

In other words, the operator $T \in B(\ell^\infty)$ determines canonically an element T^{approx} of the approximate C^*-algebra via its finite-dimensional compressions.

By definition of the approximate spectrum in the general theory, the approximate spectrum of T^{approx} is

$$\sigma_{approx}(T^{approx}) = \overline{\bigcup_F \sigma(T_F)}$$

where T_F is the local representative in $\mathcal{A}_F$.

Since in this example we are by definition identifying the approximate spectrum of an operator $T \in B(\ell_\infty)$ with the approximate spectrum of its associated element $T^{approx} \in \mathcal{A}^{approx}(\ell^\infty)$, it follows that

$$\sigma_{approx}(T) = \sigma_{approx}(T^{approx}) = \overline{\bigcup_F \sigma(T_F)}$$

This is precisely the statement of the corollary.

Conclusion. We have shown that, for every $T \in B(\ell^\infty)$, the approximate spectrum defined via the approximate C^*-algebra $\mathcal{A}^{approx}(\ell^\infty)$ coincides with the closure of the union of the spectra of its finite-dimensional compressions $T_F = P_F T P_F$. Thus,

$$\sigma_{approx}(T) = \overline{\bigcup_F \sigma(T_F)}$$

6.2. Example 2: The Space $L^\infty(\mu)$

Let (X,Σ,μ) be a measure space with μ non-σ -finite, so that $L^\infty(\mu)$ is non-separable. For each σ-finite measurable subset $E \subseteq X$, the restriction

$$L^\infty(E, \mu|_E)$$

is separable, and the inclusion $L^\infty(E, \mu|E) \hookrightarrow L^\infty(\mu)$ is isometric. The directed system of separable subspaces $\{L^\infty(E)\}_E$ yields an inductive system of local algebras

$$\mathcal{A}_E = L^\infty(E, \mu\,|_E)$$

as multiplication operators on $L^2(E)$.

Theorem 6.2.1. The approximate algebra $\mathcal{A}^{approx}(L^\infty(\mu))$ is an abelian approximate W^* -algebra, and its pre-dual is isomorphic to the projective limit of $L_1(E,\mu|_E)$ spaces.

Proof. Each local algebra $\mathcal{A}_E = L^\infty(E)$ is abelian and possesses a predual $L_1(E)$. The projective limit $\lim \leftarrow {}_E L^1(E)$defines the approximate predual $(\mathcal{A}^{\text{approx}})_*$.

The pairing

$$\langle f, g\rangle = \int_E f(x)g(x)\, d\mu(x)$$

extends by coherence to $\mathcal{A}^{\text{approx}}(L^\infty(\mu))$. Approximate weak*-compactness follows from the local Banach–Alaoglu theorem applied to each $L_1(E)$. □

Corollary 6.2.1. Let (X,Σ,μ) be a measure space such that $\mu$is not σ-finite, and consider the non-separable Banach space $L^\infty(\mu)$. Let $\mathcal{A}^{approx}(L^\infty(\mu))$ denote the approximate W^*-algebra constructed as the inductive limit

$$\mathcal{A}^{approx}(L^\infty(\mu)) = \lim_{\substack{\rightarrow \\ E}} \; L^\infty(E, \mu\,|_E)$$

where the limit is taken over the directed family of measurable subsets $E \subseteq X$ such that $\mu|_E$ is σ-finite. Then, the approximate state space of $\mathcal{A}^{approx}(L^\infty(\mu))$ is given by

$$S\left(\mathcal{A}^{approx}\left(L^\infty(\mu)\right)\right) = \{[\varphi_E] : \; \varphi_E(f) = \int_E f\, d\nu_E \,, \nu_E \ll \mu|_E, \sup \nu_E(E) = 1, \nu_F$$
$$= \nu_E|_F : \; F \subseteq E\}$$

that is, by compatible families of normalized integral functionals defined by measures νE absolutely continuous with respect to $\mu|_E$, with global normalisation $\sup_E \nu_E(E)$=1.

Proof. We recall the setting and notation of example 2:
i. For each measurable $E \subseteq X$ with $\mu|_E$ σ -finite, we consider the commutative W^*-algebra

$$\mathcal{A}_E = L^\infty(E, \mu|_E)$$

acting as multiplication operators on $L_2(E,\mu|_E)$.

ii. The approximate W^*-algebra $\mathcal{A}^{approx}(L^\infty(\mu))$ is defined as the inductive limit

$$\mathcal{A}^{approx}(L^\infty(\mu)) = \lim_{\substack{\rightarrow \\ E}} \; \mathcal{A}_E$$

where the directed system is indexed by the σ-finite subsets E, ordered by inclusion, and the connecting maps are the natural restriction/extension *-homomorphisms.

iii. By Theorem 6.1, $\mathcal{A}^{approx}(L^\infty(\mu))$ is an abelian approximate W^* -algebra, whose approximate pre-dual is the projective limit of the local preduals:

$$(\mathcal{A}^{approx}(L^\infty(\mu)))_* = \lim_{\substack{\leftarrow \\ E}} \; L^1(E, \mu|_E)$$

We now describe and characterize its approximate states.

Fix a measurable subset $E \subseteq X$ with $\mu|_E$ σ-finite. The W^* -algebra $\mathcal{A}_E = L^\infty(E,\mu|_E)$ is commutative with pre-dual

$$(\mathcal{A}_E)_* \cong L^1(E, \mu|_E)$$

via the pairing

$$\langle f, g\rangle = \int_E f(x)g(x)\, d\mu(x)\,, f \in L^\infty(E), g \in L^1(E)$$

It is classical (commutative W^*-theory) that any normal positive functional φ_E on $L^\infty(E,\mu|_E)$ can be written as

$$\varphi_E(f) = \int_E f\, d\nu_E \,, f \in L^\infty(E, \mu|_E),$$

for some finite positive measure ν_E on E, absolutely continuous with respect to $\mu|_E$.

Equivalently, by Radon–Nikodým, there exists $h_E \in L^1(E,\mu|_E)$, $h_E \geq 0$, such that

$$\nu_E(A) = \int_A h_E \, d\mu \text{, for all measurable } A \subseteq E$$

and then,

$$\varphi_E(f) = \int_E f(x) h_E(x) \, d\mu(x) = \int_E f \, d\nu_E$$

If φ_E is a state, then in addition

$$\varphi_E(1) = 1 \Leftrightarrow \nu_E(E) = 1$$

By definition 4.4.2 in the approximate framework, an approximate state on $\mathcal{A}^{approx}(L^\infty(\mu))$ is an element

$$\varphi = [\varphi_E]_E \in (\mathcal{A}^{approx}(L^\infty(\mu)))_*$$

such that:
i. Each φ_E is a normal positive functional on $\mathcal{A}_E = L^\infty(E,\mu|_E)$;
ii. The family is compatible with respect to the projective system: if $F \subseteq E$, then

$$\varphi_E|_{\mathcal{A}_F} = \varphi_F$$

where $\mathcal{A}_F = L^\infty(F,\mu|_F)$ is canonically identified with the subalgebra of $L^\infty(E,\mu|_E)$ of functions supported in F;

iii. The global normalization holds:

$$\sup_E \varphi_E(1_E) = 1$$

where 1_E denotes the constant function 1 on E.

Let $\varphi = [\varphi_E]$ be an approximate state. For each E, there exists a finite positive measure ν_E on E, absolutely continuous with respect to $\mu|_E$, such that

$$\varphi_E(f) = \int_E f \, d\nu_E \,, \forall f \in L^\infty(E, \mu|_E)$$

Now fix F,E with $F \subseteq E$. Let $f \in L^\infty(F,\mu|_F)$. We may view f as an element of $L^\infty(E,\mu|_E)$ by extending it by zero outside F. Then, by the compatibility condition

$$\varphi_F(f) = \varphi_E(f)$$

Expressed in integral form, this means

$$\int_F f \, d\nu_F = \int_E f \, d\nu_E = \int_F f \, d\nu_E$$

since f vanishes outside F.

As this equality holds for all bounded measurable f on F, we deduce

$$\nu_F = \nu_E|_F$$

that is, ν_F coincides with the restriction of ν_E to the σ-algebra of subsets of F.

Thus, the family of measures $(\nu_E)_E$ is projectivity compatible in the sense that whenever $F \subseteq E$,

$$\nu_F = \nu_E|_F$$

Moreover, the normalization condition $\sup_E \varphi_E(1_E) = 1$ translates to

$$\sup_E \nu_E(E) = 1$$

because

$$\varphi_E(1_E) = \int_E 1_E \, d\nu_E = \nu_E(E)$$

Therefore, any approximate state can be represented as a compatible family of integrals

$$\varphi_E(f) = \int_E f \, d\nu_E$$

with $\nu_E \ll \mu|_E$ and $\sup_E \nu_E(E) = 1$, and such that $\nu_F = \nu_E|_F$ for all $F \subseteq E$. This proves the inclusion

$$S\left(\mathcal{A}^{approx}(L^\infty(\mu))\right) \subseteq \left\{[\varphi_E] : \varphi_E(f) = \int_E f \, d\nu_E, \nu_E \ll \mu|_E, \sup_E \nu_E(E) = 1, \nu_F = \nu_E|_F : F \subseteq E\right\}$$

Conversely, suppose we are given a family $(\nu_E)_E$ of finite positive measures such that:

a. $\nu_E \ll \mu|_E$ for each E;

b. If $F \subseteq E$, then $\nu_F = \nu_E|_F$;

c. $\sup_E \nu_E(E) = 1$.

For each E, define

$$\varphi_E : L^\infty(E, \mu|_E) \to \mathbb{C}, \varphi_E(f) = \int_E f \, d\nu_E$$

Then,

i) Each φ_E is a positive normal functional on $\mathcal{A}_E = L^\infty(E, \mu|_E)$, since it arises from integration against a measure absolutely continuous with respect to $\mu|_E$.

ii) For $F \subseteq E$ and any $f \in L^\infty(F, \mu|_F)$, viewed as a function on Evanishing outside F,

$$\varphi_E(f) = \int_E f \, d\nu_E = \int_F f \, d\nu_E = \int_F f \, d\nu_F = \varphi_F(f)$$

so that $\varphi_E | \mathcal{A}_F = \varphi_F$. Hence, the family $(\varphi_E)_E$ is compatible and defines an element

$$\varphi = [\varphi_E] \in (\mathcal{A}^{approx}(L^\infty(\mu)))_*$$

iii) Finally, the global normalisation condition

$$\sup_E \nu_E(E) = 1$$

implies

$$\sup_E \varphi_E(1_E) = \sup_E \int_E 1_E \, d\nu_E = \sup_E \nu_E(E) = 1$$

which is precisely the normalisation required for an approximate state. Therefore, any such family $(\nu_E)_E$ defines an approximate state φ = $[\varphi_E]$, and we have the reverse inclusion.

Conclusion. Combining the former results, we conclude that

$$S\left(\mathcal{A}^{approx}(L^\infty(\mu))\right) = \{[\varphi_E] : \varphi_E(f) = \int_E f\, d\nu_E\, , \nu_E \ll \mu|_E, \sup \nu_E(E) = 1, \nu_F = \nu_E|_F : F \subseteq E\}$$

which proves corollary 6.2.1. □

6.3. Example 3: The Space $C(\beta\mathbb{N})$

Let $\beta\mathbb{N}$denote the Stone–Čech compactification of the integers. The Banach space $C(\beta\mathbb{N})$ of continuous functions is non-separable, with separable subspaces given by $C(K)$ for compact metrizable subsets $K \subset \beta\mathbb{N}$. Define $\mathcal{A}_K = C(K)$ and the inductive limit

$$\mathcal{A}^{approx}(C(\beta\mathbb{N})) = \lim_{\substack{\rightarrow \\ K\subset\beta\mathbb{N}}} \; C(K)$$

Proposition 6.3.1. The approximate algebra $\mathcal{A}^{approx}(C(\beta\mathbb{N}))$ is abelian and coincides with the algebra of bounded continuous functions on $\beta\mathbb{N}$with locally separable support.

Proof. Each $C(K)$ is abelian, and the inductive limit preserves commutativity. For each $f \in C(\beta\mathbb{N})$, the restriction $f|_K$ defines a compatible system in $C(K)$. The approximate limit consists of functions whose support lies in the directed union of separable compact subsets, which are dense in $\beta\mathbb{N}$. □

Corollary 6.3.1 (*Approximate Gelfand representation*). Every abelian approximate C^* -algebra $\mathcal{A}^{approx}$ is *-isomorphic to a function algebra $C^{approx}(\Omega)$, where Ω is a locally compact Hausdorff space with a locally separable base. More precisely, there exists a locally compact Hausdorff space Ω and an isometric *-isomorphism

$$\Phi: \mathcal{A}^{approx} \longrightarrow C^{approx}(\Omega)$$

where $C^{approx}(\Omega)$ denotes the C^*-algebra of continuous complex-valued functions on Ω whose supports are contained in directed unions of separable compact subsets of Ω.

Proof. We assume throughout that $\mathcal{A}^{approx}$ is an abelian approximate C^*-algebra obtained as an inductive limit of separable abelian C^*-algebras:

$$\mathcal{A}^{approx} = \lim_{\substack{\rightarrow \\ F\in\mathcal{F}}} \; \mathcal{A}_F$$

where $(\mathcal{F},\leq)$ is a directed set, and for each $F \in \mathcal{F}$,

a) $\mathcal{A}_F$ is a separable abelian C^*-algebra,
b) The connecting maps $\iota_{FG}:\mathcal{A}_F \to \mathcal{A}_G$ (for $F \subseteq G$) are unital *-homomorphisms,
c) Elements of $\mathcal{A}^{approx}$ are equivalence classes $[a_F]$of compatible families $(a_F)_{F\in\mathcal{F}}$ with $\iota_{FG}(a_F) = a_G$ whenever $F \subseteq G$,
d) $\|[a_F]\| = \sup_F \|a_F\|$.

Since each $\mathcal{A}_F$ is a separable abelian C^* -algebra, the classical Gelfand theory applies: there exists a locally compact Hausdorff space Ω_F (the spectrum of $\mathcal{A}_F$) and an isometric *-isomorphism

$$\Gamma_F: \mathcal{A}_F \longrightarrow C_0(\Omega_F)$$

where $C_0(\Omega_F)$ denotes the $C*$ -algebra of continuous complex-valued functions on Ω_F vanishing at infinity. Explicitly,

i) Points $\omega \in \Omega_F$ correspond to non-zero characters (multiplicative linear functionals) $\chi_\omega : \mathcal{A}_F \to \mathbb{C}$,
ii) $\Gamma_F(a_F)$ is the function $\Omega_F \to \mathbb{C}$ given by

$$\Gamma_F(a_F)(\omega) = \chi_\omega(a_F)$$

Moreover, for each inclusion $F \subseteq G$, the connecting *-homomorphism $\iota_{FG} : \mathcal{A}_F \to \mathcal{A}_G$ induces a continuous map between spectra

$$\phi_{GF} : \Omega_G \to \Omega_F$$

defined by composition of characters:

$$\phi_{GF}(\omega_G) = \omega_G \circ \iota_{FG}, \omega_G \in \Omega_G$$

These maps satisfy the usual compatibility

$$\phi_{HF} = \phi_{GF} \circ \phi_{HG} : F \subseteq G \subseteq H$$

We now construct a topological space Ω which plays the role of a global spectrum for the inductive system. Consider the disjoint union

$$\coprod_{F \in \mathcal{F}} \Omega_F$$

and introduce an equivalence relation $\sim$ as follows: we declare $\omega_F \in \Omega_F$ and $\omega_G \in \Omega_G$ to be equivalent if there exists some $H \in \mathcal{F}$ with $F \subseteq H$, $G \subseteq H$, and

$$\phi_{HF}(\omega_H) = \omega_F, \phi_{HG}(\omega_H) = \omega_G$$

for some $\omega_H \in \Omega_H$. Intuitively, two characters are equivalent if they are compatible images of a common character on a larger algebra $\mathcal{A}_H$.

Define

$$\Omega := \left(\coprod_{F \in \mathcal{F}} \Omega_F \right) / \sim$$

We endow Ω with the final topology making all natural injections

$$j_F : \Omega_F \longrightarrow \Omega, \omega_F \mapsto [\omega_F]$$

continuous. The resulting space $\Omega$is locally compact and Hausdorff, and it has a locally separable base because:
1. Each ΩF is locally compact, separable, and Hausdorff (as spectrum of a separable C^*-algebra),
2. The topology on Ω is locally generated by the images $j_F(U)$ of open subsets $U \subseteq \Omega_F$,
3. Any point in Ω admits a neighbourhood homoeomorphic to an open subset of some Ω_F, hence with separable topology.

Thus Ω is a locally compact Hausdorff space with a base consisting of sets which are images of separable open subsets.

We define $C^{approx}(\Omega)$ as the C^*-algebra of continuous functions on Ωwhich arise compatibly from the local algebras $C_0(\Omega_F)$.

More precisely, consider families $(f_F)_{F \in \mathcal{F}}$ with

$$f_F \in C_0(\Omega_F)$$

such that:

A) Compatibility: for all $F \subseteq G$, $f_F = f_G \circ \phi_{GF}$ on Ω_F, i.e. $f_F(\omega_F) = f_G(\omega_G) : \omega_F = \phi_{GF}(\omega_G)$.
B) Uniform boundedness: $sup_F \|f_F\|_\infty < \infty$. Such a family defines a well-defined continuous function $f : \Omega \to \mathbb{C}$ by $f([\omega_F]) := f_F(\omega_F)$, which does not depend on the chosen representative of the equivalence class.

We then define

$$C^{approx}(\Omega) = \{f : \Omega \to \mathbb{C} \text{ continuous} : f \text{ arises from a compatible family } (f_F)\}$$

with point wise operations and norm

$$\| f \| = \sup_F \| f_F \|_\infty$$

Equivalently, $C^{approx}(\Omega)$ is (up to canonical identification) the inductive limit

$$C^{approx}(\Omega) \cong \lim_{F} \rightarrow \ C_0(\Omega_F)$$

with respect to the pullback maps induced by the continuous maps ϕ_{GF}.

We now define a mapping

$$\Phi : \mathcal{A}^{approx} \longrightarrow C^{approx}(\Omega)$$

by means of the local Gelfand transforms. An element $a \in \mathcal{A}^{approx}$ is represented by a compatible family $(a_F)_{F \in \mathcal{F}}$ with $a_F \in \mathcal{A}_F$ and

$$\iota_{FG}(a_F) = : F \subseteq G$$

For each F, define

$$f_F := \Gamma_F(a_F) \in C_0(\Omega_F)$$

We claim that $(f_F)_F$ is a compatible family in the sense we have seen. Indeed, if $F \subseteq G$, then the naturality of the Gelfand transform yields the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_F & \overset{\iota_{FG}}{\rightarrow} & \mathcal{A}_G \\ \Gamma_F \downarrow & & \downarrow \Gamma_G \\ C_0(\Omega_F) & \overset{\phi^*_{GF}}{\leftarrow} & C_0(\Omega_G), \end{array}$$

where $\phi_{GF}{}^*$ is the pullback map $(\phi^*_{GF} g)(\omega_F) = g(\phi^{-1}_{GF}(\omega_F))$ when properly interpreted.

In particular, for $a_F \in \mathcal{A}_F$,

$$\Gamma_F(a_F) = \Gamma_G(\iota_{FG}(a_F)) \circ \phi_{GF} = f_G \circ \phi_{GF}$$

Thus, the family $(f_F)_F$ satisfies $f_F = f_G \circ \phi_{GF}$, i.e. it is compatible. Therefore $(f_F)_F$ defines a unique function $f \in C^{approx}(\Omega)$, and we set $\Phi(a) := f$.

Linearity and multiplicativity of Φ follow directly from the fact that each Γ_F is a *-homomorphism and the inductive system is compatible:
i. For $a,b \in \mathcal{A}^{approx}$ represented by $(a_F)_F$ and $(b_F)_F$,

$$\Phi(a + b) = [\Gamma_F(a_F + b_F)] = [\Gamma_F(a_F) + \Gamma_F(b_F)] = \Phi(a) + \Phi(b)$$

ii. Similarly,

$$\Phi(ab) = [\Gamma_F(a_F b_F)] = [\Gamma_F(a_F)\Gamma_F(b_F)] = \Phi(a)\Phi(b)$$

iii. For the involution,

$$\Phi(a^*) = [\Gamma_F(a_F^*)] = [\Gamma_F(a_F)^*] = \Phi(a)^*$$

Thus Φ is a *-homomorphism. For the norm, we have

$$\| \Phi(a) \| = \sup_F \| \Gamma_F(a_F) \|_\infty = \sup_F \| a_F \| = \| a \|$$

because each Γ_F is an isometric *-isomorphism and the norm in $\mathcal{A}^{approx}$ is defined as

$$\| a \| = \sup_F \| a_F \|$$

Hence, Φ is isometric.

Let $f \in C^{approx}(\Omega)$. By definition, f is represented by a compatible family $(f_F)_F$ with $f_F \in C_0(\Omega_F)$ and

$$f_F = f_G \circ \phi_{GF} \text{for } F \subseteq G$$

For each F , use the inverse Gelfand transform $a_F := \Gamma_F^{-1}(f_F) \in \mathcal{A}_F$ .The compatibility of the f_F's implies compatibility of the a_F's. Indeed, for $F \subseteq G$,

$$f_F = f_G \circ \phi_{GF} \Longrightarrow \Gamma_F(a_F) = \Gamma_G(a_G) \circ \phi_{GF}$$

By the naturality of the Gelfand transform, $\Gamma_F(a_F) = \Gamma_G(\iota_{FG}(a_F)) \circ \phi_{GF}$. Comparing with $\Gamma_G(a_G) \circ \phi_{GF}$ and using injectivity of Γ_G , we obtain $\iota_{FG}(a_F) = a_G$. Hence, $(a_F)_F$ is a compatible family, and therefore defines an element $a = [a_F]\ F \in \mathcal{A}^{approx}$. By construction, $\Phi(a) = [\Gamma_F(a_F)] = [f_F] = f$. Thus $\Phi$is surjective.

Conclusion. We have constructed an isometric *-isomorphism

$$\Phi: \mathcal{A}^{approx} \longrightarrow C^{approx}(\Omega)$$

where Ω is a locally compact Hausdorff space with a locally separable base, obtained as an inductive limit of the local spectra Ω_F . This establishes the approximate Gelfand representation and proves corollary 6.3.1. □

6.4. Structural Comparison

The preceding examples exhibit the following unifying phenomena:

Property	Classical (separable)	Approximate (non-separable)
Topology	Sequentially *weak**-compact	Net-based *weak**-compact
Duality	Unique pre-dual	Projective limit of pre-duals
Spectral theory	Sequential spectral mapping	Directed spectral mapping
Reflexivity	Guaranteed for W*-algebras	Approximate reflexivity holds
States	Countably additive	Locally additive, globally consistent
Representation	Hilbert-space GNS	Directed GNS on inductive Hilbert limits

Theorem 6.2 (*Structural coherence*). Let $\mathcal{A}^{approx}(X)$ be an approximate W^* -algebra associated with a non-separable Banach space X . Then the following are equivalent:

a. Each local component $\mathcal{A}_F$ is reflexive and weak*-closed.
b. The approximate pre-dual $(\mathcal{A}^{approx}(X))_*$ is complete.
c. The set of approximate states is compact in the approximate weak*-topology.

Proof. (1) ⇒ (2): The projective limit of reflexive dual pairs is complete.
(2) ⇒ (3): Compactness follows from Tychonoff's theorem applied to the product of compact local state spaces.
(3) ⇒ (1): Weak*-compactness ensures the closure of each local component and thus its reflexivity. □

6.5. Summary

The examples above confirm that the approximate framework successfully extends classical operator algebra theory to non-separable Banach spaces. In each case, local separable structures provide a consistent approximation that preserves spectral, dual, and algebraic properties globally. Approximate reflexivity, *weak**-compactness, and Gelfand-type representation remain valid, even though global separability is lost.

The next section discusses the broader implications of these results, compares them with the classical theory, and outlines potential directions for future research, particularly in noncommutative geometry and quantum analysis.

7. Discussion and Perspectives

The development of approximate C*- and W*-algebras on non-separable Banach spaces presented in this work provides a consistent and general framework that bridges the gap between classical operator algebra theory and the analytic structures encountered in non-separable or large-dimensional functional settings. The use of directed systems of separable components allows for the retention of essential topological and spectral properties while avoiding the constraints imposed by countable bases or sequential compactness.

7.1. Theoretical Implications

From a purely functional-analytic standpoint, the theory confirms that the fundamental algebraic and topological principles governing operator algebras remain valid in non-separable contexts when reformulated in terms of nets and directed limits. The approximate □□- and W^*- framework preserves:

a. The C^*-identity, ensuring norm stability under involution.
b. Positivity and spectral calculus, through the coherence of local spectra.
c. *Weak- duality**, via the projective limits of pre-duals.
d. Reflexivity, interpreted as approximate bi-dual closure.
e. Faithful representation, generalized by the *approximate GNS theorem*.

These results indicate that non-separability, though analytically challenging, does not destroy the internal consistency of the operator algebraic structure. Instead, it requires a reformulation of convergence and compactness, replacing sequences by directed nets and local–global compatibility conditions.

7.2. Relation to the Classical Theory

The approximate framework retains all the key results of the separable theory—such as the Gelfand–Naimark and Bicommutant theorems—but in a *net-theoretic* rather than *sequential* setting. When X is separable, the directed system (X_F) becomes countable, and all approximate notions reduce to their classical counterparts:

Concept	Classical setting	Approximate setting
C^*-identity	$\parallel T^*T \parallel = \parallel T \parallel^2$	Preserved exactly
*Weak**-duality	Unique pre-dual	Projective limit of pre-duals
Spectrum	Sequential closure	Directed closure of local spectra
Reflexivity	Automatic in W^*-algebras	Approximate reflexivity via bidual limit
Representation	GNS over Hilbert space	Approximate GNS over inductive Hilbert system

Thus, the approximate construction can be regarded as a strict generalization of the standard operator algebra theory, smoothly extending it to non-separable environments without altering its fundamental logical structure.

7.3. Conceptual Consequences for Operator Theory

The notion of approximate *weak*- closure* introduced here generalizes the standard concept of topological closure and allows a refined understanding of continuity, duality, and boundedness in spaces lacking separability. In particular:

A) Approximate weak- continuity* replaces sequential weak-* continuity in the definition of normal states.
B) Approximate bicommutant closure yields an operational criterion for identifying approximate W^*-algebras.
C) Approximate representations provide a natural extension of cyclic and factorial decompositions to directed inductive systems.

This reformulation reveals that the essential content of operator theory—its algebraic–topological duality—does not depend on separability per se but on the existence of coherent local structures capable of reproducing the same functional relationships.

7.4. Potential Applications

The approximate operator algebra framework has potential applications in several mathematical and physical domains:

i. Noncommutative Geometry. The concept of approximate W^*-algebra suggests a pathway to define approximate spectral triples, where the Dirac operator acts locally on separable substructures and coherence is preserved globally. This can yield a generalization of Connes' noncommutative manifolds to non-separable analytic spaces [11].

ii. Quantum Theory and Mathematical Physics. In quantum field theory and statistical mechanics, infinite systems often require non-separable Hilbert spaces [10]. Approximate algebras provide a rigorous analytical tool for studying observables, states, and representations in such contexts.

iii. Banach Space Theory. The approximate C*-framework interacts naturally with the geometry of Banach spaces, offering new perspectives on reflexivity, weak compactness, and dual embeddings [6].

iv. Functional Analysis and Logic. The formal structure of approximate limits and weak*-closure parallels logical and categorical formulations of model theory and topological algebra, suggesting potential interdisciplinary links.

7.5. Open Problems and Research Directions

Several directions arise naturally from the results presented:

I. Approximate K-Theory. Defining K -groups for approximate C^* -algebras and studying their invariants remains an open question.

II. Approximate Spectral Triples. A rigorous formulation of approximate Dirac operators could lead to a consistent extension of noncommutative geometry beyond separable algebras.

III. Approximate Tomita–Takesaki Theory. Investigating modular structures and automorphism groups for approximate W*-algebras may illuminate the dynamics of nonseparable quantum systems.

IV. Approximate Tensor Products: Extending the framework to tensor product constructions would allow the study of bipartite systems and entanglement phenomena in non-separable contexts.

V. Connections with Category Theory. Formalizing approximate algebras as inductive–projective objects in a suitable categorical setting could unify the algebraic and topological aspects of this theory.

7.6. Final Remarks

The approximate operator algebra theory developed here demonstrates that nonseparable Banach spaces are not an obstacle but an opportunity to generalise and deepen the classical understanding of operator algebras. By systematically replacing sequential arguments with directed topological constructions, we have achieved a coherent extension of C^* - and W^* -theory to the non-separable domain. This framework opens new possibilities for analysis, geometry, and physics—where infinity and non-separability, rather than being pathologies, become natural and structurally consistent features of the mathematical landscape [12].

References

1. Neumann, J. V. (1955). Mathematical foundations of quantum mechanics.
2. Sakai, S., & Sakai, S. (1971). *C*-algebras and W*-algebras* (Vol. 60). Berlin: Springer.
3. Kadison, R. V., & Ringrose, J. R. (1983). *Fundamentals of the theory of operator algebras. Vol. I: Elementary theory*. Boston, MA: Birkhäuser.
4. Kadison, R. V., & Ringrose, J. R. (1986). *Fundamentals of the theory of operator algebras. Volume II: Advanced theory* (Vol. 25). New York: Academic press.
5. Takesaki, M. (2003). *Theory of operator algebras II* (Vol. 125, p. 518). Berlin: Springer.
6. Lindenstrauss, J., & Tzafriri, L. (2013). *Classical Banach spaces II: function spaces* (Vol. 97). Springer Science & Business Media.
7. Akhiezer, N. I., & Glazman, I. M. (1993). Theory of linear operators in Hilbert space. New York: Dover Publications
8. Gel'fand, I. M., Raikov, D. A., & Shilov, G. E. E. (1946). Commutative normed rings. *Uspekhi Matematicheskikh Nauk, 1*(2), 48-146.
9. Reed, M., Simon, B., Simon, B., & Simon, B. (1972). *Methods of modern mathematical physics* (Vol. 1). New York: Academic press.
10. Bratteli, O., & Robinson, D. (1982). Operator algebras and quantum statistical mechanics. Bull. Amer. Math. Soc, 7, 425.
11. Connes, A. (1994). Noncommutative Geometry (San Diego: Acad. Press).
12. Reed, M., Simon, B., Simon, B., & Simon, B. (1978). Methods of modern mathematical physics (Vol. IV. New York: Academic press.