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Research Article

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Analysis and Control of Measles Dynamic Models

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Abstract
Measles is a communicable and deadly viral disease that can be contracted on contact with an infected individual or via airborne propagules Effective and efficient strategies must be implemented to minimize the damage caused by measles , and to do this, we must understand the dynamics of the measles transmission and implement control methods that are beneficial and cost-effective. In this work, bifurcation analysis and multiobjective nonlinear model predictive control is performed on two dynamic models involving measles transmission. Bifurcation analysis is a powerful mathematical tool used to deal with the nonlinear dynamics of any process. Several factors must be considered, and multiple objectives must be met simultaneously. The MATLAB program MATCONT was used to perform the bifurcation analysis. The MNLMPC calculations were performed using the optimization language PYOMO in conjunction with the state-of-the-art global optimization solvers IPOPT and BARON.The bifurcation analysis revealed the existence of branch and limit points and the MNLMPC calculations converged to the Utopia solution in both the models. The branch and limit points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in both models.

Keywords: Bifurcation, Optimization, Control, Measles

1. Introduction

Investigated the chaos and complexity in measles models using a comparative numerical study [1]. researched the space, persistence, and dynamics of measles epidemics [2]. Determined the factors that affect the prevalence of maternal antibody to measles virus throughout infancy [3]. Roberts and applied a mathematical model to predict and prevent measles epidemics in NewZealand [4]. Studied the dynamics of measles epidemics estimating scaling of transmission rates using a time series SIR model [5]. Discussed the average cost of measles cases and adverse events following vaccination in industrialised countrie [6]. Modelled measles outbreaks considering branching processes, variation, growth, and extinction of populations [7]. Provided an estimation of transmission intensity for a measles epidemic in Niamey, Niger [8]. Developed a mathematical model for control of measles by vaccination [9] . Reported the unacceptably high mortality related to measles epidemics in Niger, investigated the effectiveness of measles vaccination and vitamin A treatment [10,11]. Performed a mathematical analysis of the effect of Immunization on the dynamical spread of measles [12]. Performed a mathematical analysis of the effect of area on the dynamical spread of measles[13]. Studied the transmission dynamics and performed optimal control of measles epidemics [14]. Performed an optimal control analysis of the dynamic spread of measles [15]. Performed computational modelling and optimal control studies of measles epidemic in human population [16]. This work aims to perform bifurcation analysis and multiobjective nonlinear control (MNLMPC) studies in two measles transmission models, which are discussed in (model 1) and model 2). The paper is organized as follows [15,16]. First, the model equations are presented, followed by a discussion of the numerical techniques involving bifurcation analysis and multiobjective nonlinear model predictive control (MNLMPC). The results and discussion are then presented, followed by the conclusions.

2. Model Equations (Model 1) [16]

The variables sv, ev, iv, rv) represent susceptible individuals, individuals exposed to measles (i.e., infected but show no clinical symptoms of measles), individuals infected with measles, and individuals who recovered from measles.

The model equations are

$$\begin{aligned} av &= \frac{(1-u3)\beta\ ivsv}{1+k1iv} \\ bv &= \frac{u2iv}{1+k2iv} \\ \frac{d(sv)}{dt} &= \Lambda - av - (\mu + u1)sv \\ \frac{d(ev)}{dt} &= av - (\mu + \varepsilon)ev \\ \frac{d(iv)}{dt} &= \varepsilon ev - bv - (\gamma + \mu + d)iv \\ \frac{d(rv)}{dt} &= bv + u1sv + \gamma iv - \mu rv \end{aligned} \tag{1}$$

The model base parameters are

$\Lambda = 10; \mu = 0.2;\ \gamma = 0.4;\ \varepsilon = 1.2;\ \theta = 0.01; \beta = 0.8; k1 = 0.1; k2 = 2; d = 0.5; u1 = 0.1; u2 = 0.2; u3 = 0.3;$

The control variables are u1,u2,u3.

2.1. Model Equations (model 2) [15] (model 2)

The variables (sv, ev, iv, jv, rv) represent susceptible individuals, individuals exposed to measles, individuals infected with measles, individuals infected with measles and isolated, and individuals who have recovered from measles.

The model equations are

$$\begin{aligned} \frac{d(sv)}{dt} &= \pi + \sigma rv - (1-u1)\lambda sv - \mu sv \\ \frac{d(ev)}{dt} &= (1-u1)\lambda sv - (k+\mu)ev \\ \frac{d(iv)}{dt} &= kev - (\sigma + \mu + \delta)iv - (2-u2-u3)iv \\ \frac{d(jv)}{dt} &= \xi iv - (\mu + \delta) jv - (1-u3) jv \\ \frac{d(rv)}{dt} &= (1-u3)iv + (1-u3) jv - (\sigma + \mu) rv \\ \lambda &= \frac{\beta\eta(iv+ij)}{sv+iv+ev+jv+rv} \end{aligned} \tag{2}$$

The base parameter values are

$\sigma = 0.2; k = 0.09; \mu = 0.1; \pi = 2000; \delta = 0.1; \xi = 0.1; \eta = 0.01; \beta = 0.2; u1 = 0.5; u2 = 0.88; u3 = 0.3;$

2.2. Bifurcation Analysis

The MATLAB software MATCONT is used to perform the bifurcation calculations. Bifurcation analysis deals with multiple steady-states and limit cycles. Multiple steady states occur because of the existence of branch and limit points. Hopf bifurcation points cause limit cycles. A commonly used MATLAB program that locates limit points, branch points, and Hopf bifurcation points is MATCONT [17,18]. This program detects Limit points(LP), branch points(BP), and Hopf bifurcation points(H) for an ODE system

$$\frac{dx}{dt} = f(x, \alpha) \tag{3}$$

$x \in R^n$ Let the bifurcation parameter be α. Since the gradient is orthogonal to the tangent vector,
The tangent plane at any point $w = [w_1, w_2, w_3, w_4, ... w_{n+1}]$ must satisfy

$$Aw = 0 \tag{4}$$

Where A is

$$A = [\partial f / \partial x \quad | \partial f / \partial \alpha] \tag{5}$$

where $\partial f / \partial x$ is the Jacobian matrix? For both limit and branch points, the Jacobian matrix $J = [\partial f / \partial x]$ must be singular.
For a limit point, there is only one tangent at the point of singularity. At this singular point, there is a single non-zero vector, y, where Jy = 0. This vector is of dimension n. Since there is only one tangent the vector $y = (y_1, y_2, y_3, y_4, ... y_n)$ must align with $\hat{w} = (w_1, w_2, w_3, w_4, ... w_n)$. Since

$$J\hat{w} = Aw = 0 \tag{6}$$

the n+1 th component of the tangent vector $w_{n+1} = 0$ at a limit point (LP).
For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w. This implies that

$$\begin{aligned} Az &= 0 \\ Aw &= 0 \end{aligned} \tag{7}$$

Consider a vector v that is orthogonal to one of the tangents (say w). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since w and v are orthogonal, $w^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ w^T \end{bmatrix} v = 0$ which implies that B is singular.

Hence, for a branch point (BP) the matrix B = $\begin{bmatrix} A \\ w^T \end{bmatrix}$ must be singular. At a Hopf bifurcation point,

$$\det(2f_x(x, \alpha) @ I_n) = 0 \tag{8}$$

@ indicates the bialternate product while I_n is the n-square identity matrix. Hopf bifurcations cause limit cycles and should be eliminated because limit cycles make optimization and control tasks very difficult. More details can be found in [19-21].

2.3. Multiobjective Nonlinear Model Predictive Control(MNLMPC)

The rigorous multiobjective nonlinear model predictive control (MNLMPC) method developed by was used [22].

Consider a problem where the variables $\sum_{t_{i=0}}^{t_i = t_f} q_j(t_i)$(j = 1, 2..n) have to be optimized simultaneously for a dynamic problem

$$\frac{dx}{dt} = F(x, u) \tag{9}$$

t_f being the final time value, and n the total number of objective variables and u the control parameter. The single objective optimal control problem is solved individually optimizing each of the variables $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ The optimization of $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ will lead to the values q_j^*. Then, the multiobjective optimal control (MOOC) problem that will be solved is

$$\min(\sum_{j=1}^{n}(\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^*))^2 \qquad (10)$$
$$subject\ to\ \frac{dx}{dt} = F(x,u);$$

This will provide the values of u at various times. The first obtained control value of u is implemented and the rest are discarded. This procedure is repeated until the implemented and the first obtained control values are the same or if the Utopia point where ($\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) = q_j^*$ for all j) is obtained.

Pyomo is used for these calculations [23]. Here, the differential equations are converted to a Nonlinear Program (NLP) using the orthogonal collocation method The NLP is solved using IPOPT and confirmed as a global solution with BARON [24, 25].

The steps of the algorithm are as follows

1. Optimize $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ and obtain q_j^*.

2. Minimize $(\sum_{j=1}^{n}(\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^*))^2$ and get the control values at various times.

3. Implement the first obtained control values

4. Repeat steps 1 to 3 until there is an insignificant difference between the implemented and the first obtained value of the control variables or if the Utopia point is achieved. The Utopia point is when $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) = q_j^*$ for all j.

Demonstrated that when the bifurcation analysis revealed the presence of limit and branch points tthe MNLMPC calculations to converge to the Utopia solution [26]. For this, the singularity condition, caused by the presence of the limit or branch points was imposed on the co-state equation [27]. If the minimization of q_1 lead to the value q_1^* and the minimization of q_2 lead to the value q_2^* The MNLPMC calculations will minimize the function $(q_1 - q_1^*)^2 + (q_2 - q_2^*)^2$. The multiobjective optimal control problem is

$$\min \quad (q_1 - q_1^*)^2 + (q_2 - q_2^*)^2 \quad subject\ to\ \frac{dx}{dt} = F(x,u) \qquad (11)$$

Differentiating the objective function results in

$$\frac{d}{dx_i}((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) = 2(q_1 - q_1^*)\frac{d}{dx_i}(q_1 - q_1^*) + 2(q_2 - q_2^*)\frac{d}{dx_i}(q_2 - q_2^*) \qquad (12)$$

The Utopia point requires that both and $(q_1 - q_1^*)$ are $(q_2 - q_2^*)$ zero. Hence

$$\frac{d}{dx_i}((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) = 0 \qquad (13)$$

The optimal control co-state equation (Upreti; 2013)[27] is

$$\frac{d}{dt}(\lambda_i) = -\frac{d}{dx_i}((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) - f_x\lambda_i; \quad \lambda_i(t_f) = 0 \qquad (14)$$

λ_i is the Lagrangian multiplier. t_f is the final time. The first term in this equation is 0 and hence

$$\frac{d}{dt}(\lambda_i) = -f_x \lambda_i ; \lambda_i(t_f) = 0 \tag{15}$$

At a limit or a branch point, for the set of ODE $\frac{dx}{dt} = f(u, x)$ f_x is singular. Hence there are two different vectors-values for $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i) > 0$ and $\frac{d}{dt}(\lambda_i) < 0$. In between there is a vector $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i) = 0$. This coupled with the boundary condition $\lambda_i(t_f) = 0$ will lead to $[\lambda_i] = 0$ This makes the problem an unconstrained optimization problem, and the optimal solution is the Utopia solution.

3. Results and Discussion

In model 1, the bifurcation diagram revealed a limit and a branch point at (s, ev, iv,rv,u3) values of (33.024004, 0.066285, 0.062246, 16.691849, 0.943219) and (33.333333, 0, 0, 16.666667, 0.943125). u3 is the bifurcation parameter (Figure. 1a).

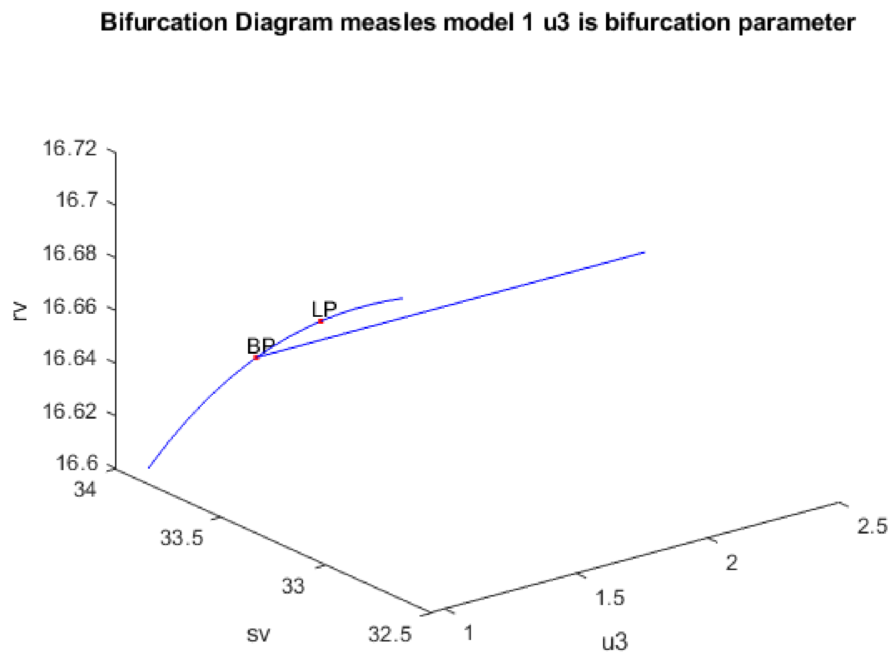

Figure: 1a Bifurcation Diagram for Measles model 1

For theMNLMPC calculations, sv(0)is set to 1000; ev(0) is set to 80; and iv(0) is set to 80 $\sum_{t_i=0}^{t_i=t_f} rv(t_i)$ were maximized and resulted in a value of 2045.88 and $\sum_{t_i=0}^{t_i=t_f} iv(t_i)$ was minimized and resulted in a value of 100. The overall optimal control problem will involve the minimization of $(\sum_{t_i=0}^{t_i=t_f} iv(t_i) - 100)^2 + (\sum_{t_i=0}^{t_i=t_f} rv(t_i) - 2045.88)^2$ was minimized subject to the equations governing the model. This led to a value of zero (the Utopia point).

The MNLMPC values of the control variables, u1, u2, and u3 were 0.18479, 0.5016, and 0.9847. The various MNMPC figures are shown in figures 1b-1f. The control profiles u1, u2, and u3. (Figure. 1e) exhibited noise, and this was remedied using the Savitzky-Golay filter to produce the smooth control profiles u1sg, u2sg, and u3sg (Figure. 1f). It is seen that the presence of the limit and branch points is beneficial because it allows the MNLMPC calculations to attain the Utopia solution, validating the analysis of [26].

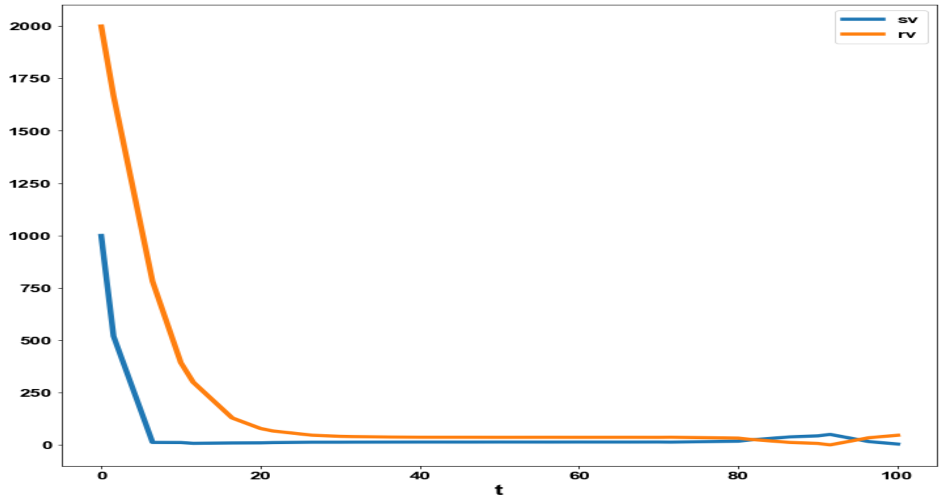

Figure: 1b Bifurcation Diagram For Measles Model 1

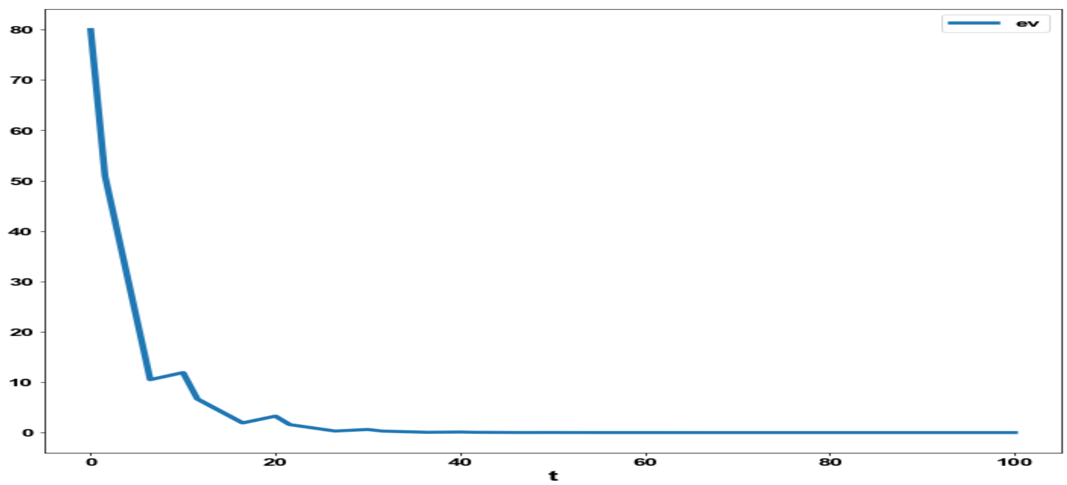

Figure :1c Mnlmpc For Measles Model 1(Sv, Rv Profiles)

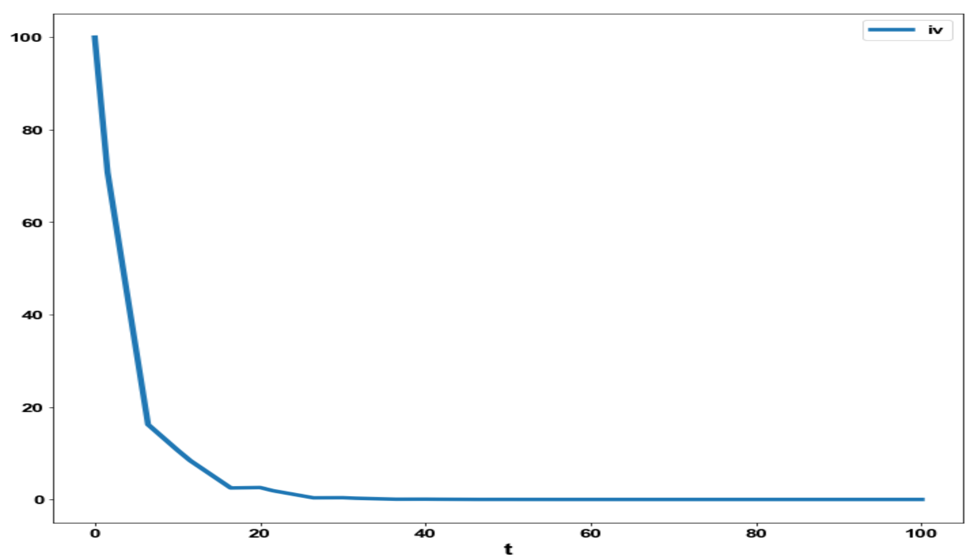

Figure: 1d Mnlmpc For Measles Model 1(Ev Profile)

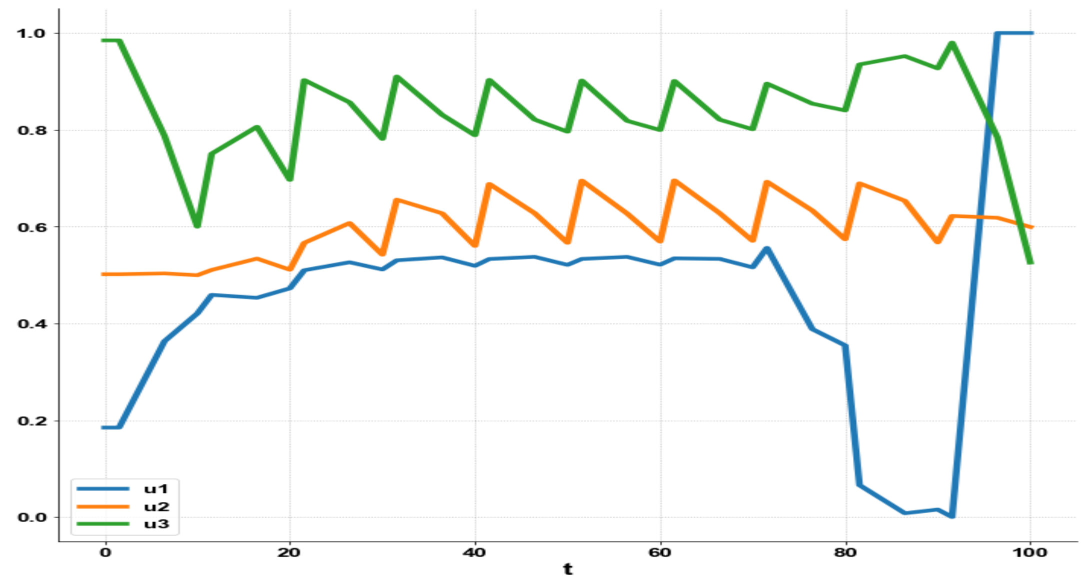

Figure: 1e Mnlmpc For Measles Model 1(Iv Profile)

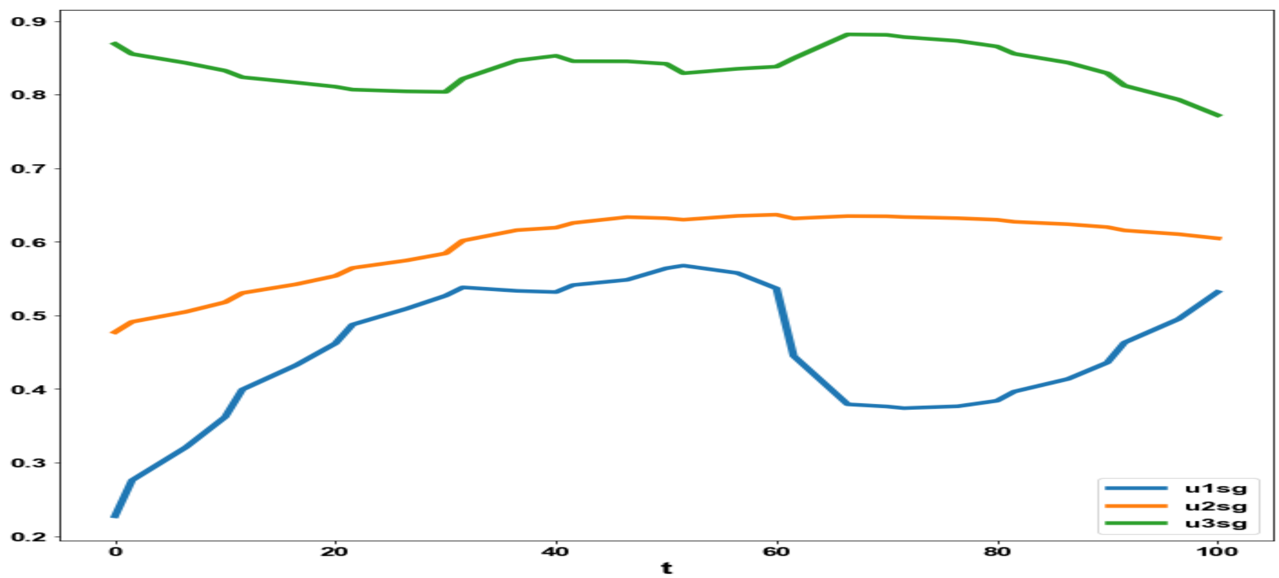

Figure:1f Mnlmpc For Measles Model 1(Control Profiles Noise Exhibited)

In model 2, the bifurcation diagram revealed 2 branch points and 2 limit points at (sv, ev, iv,jv, rv,u3) values of (20000 0 0 0 0 1.199852) ; (20000 0 0 0 0 1.519674); (20000.007454 -0.000029 -0.000008 -0.005545 0.003699 1.199852); (20000.008330 -0.000013 -0.003489 0.001091 0.004153 1.519674). u3 is the bifurcation parameter (Figure. 2a).

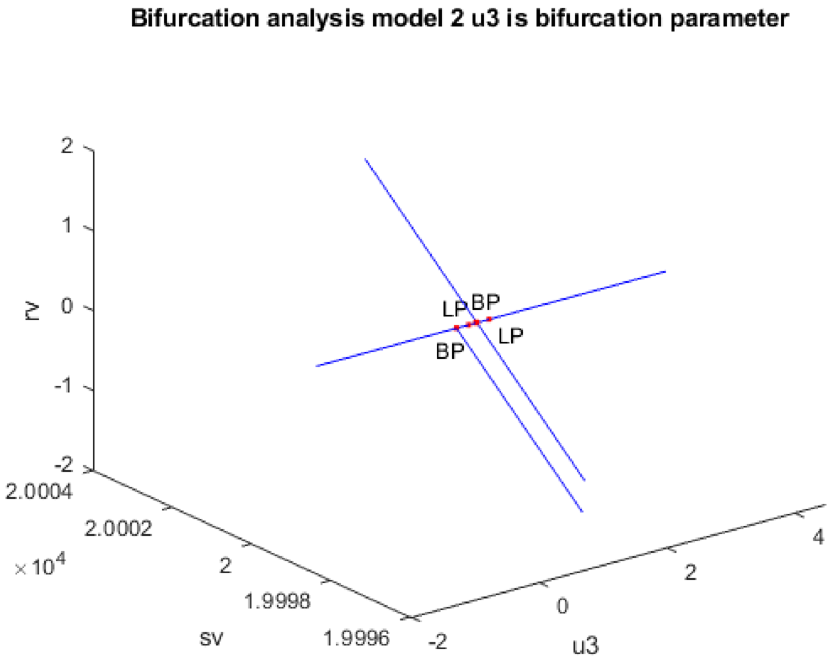

Figure: 2a Bifurcation Analysis For Measles Model 2

For theMNLMPC calculations, sv(0)is set to 1000 ; $\sum_{t_i=0}^{t_i=t_f} rv(t_i)$ were maximized and resulted in a value of 2000 and $\sum_{t_i=0}^{t_i=t_f} iv(t_i)$ was minimized and resulted in a value of 0. The overall optimal control problem will involve the minimization of $(\sum_{t_i=0}^{t_i=t_f} iv(t_i)-0)^2+(\sum_{t_i=0}^{t_i=t_f} rv(t_i)-2000)^2$ was minimized subject to the equations governing the model. This led to a value of zero (the Utopia point).

The MNLMPC values of the control variables, u1, u2, and u3 were 0.9884, 0.5194, and 0.9705. The various MNMPC figures are shown in Figures 2b-2f. The control profiles u1, u2, and u3. (Figure. 2e) exhibited noise, and this was remedied using the Savitzky-Golay filter to produce the smooth control profiles u1sg, u2sg, and u3sg (Figure. 2f). It is seen that the presence of the limit and branch points is beneficial because it allows the MNLMPC calculations to attain the Utopia solution, validating the analysis of [26]. In both cases, the presence of branch and limit points caused the MNLMPC calculations to attain the Utopia solution, validating the analysis of [26].

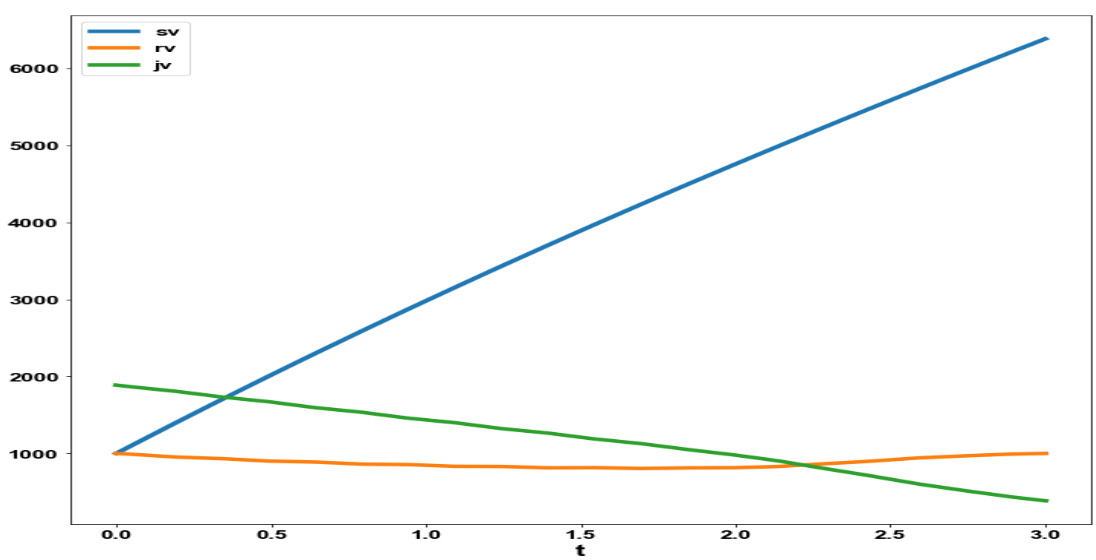

Figure: 2b Mnlmpc For Measles Model 2 (Sv, Jv, Rv Profiles)

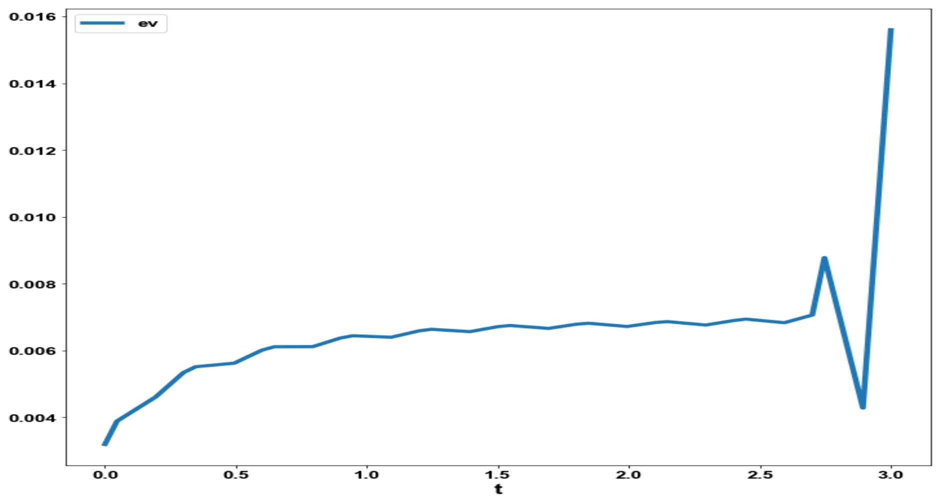

Figure: 2c Mnlmpc For Measles Model 2 (Ev Profile)

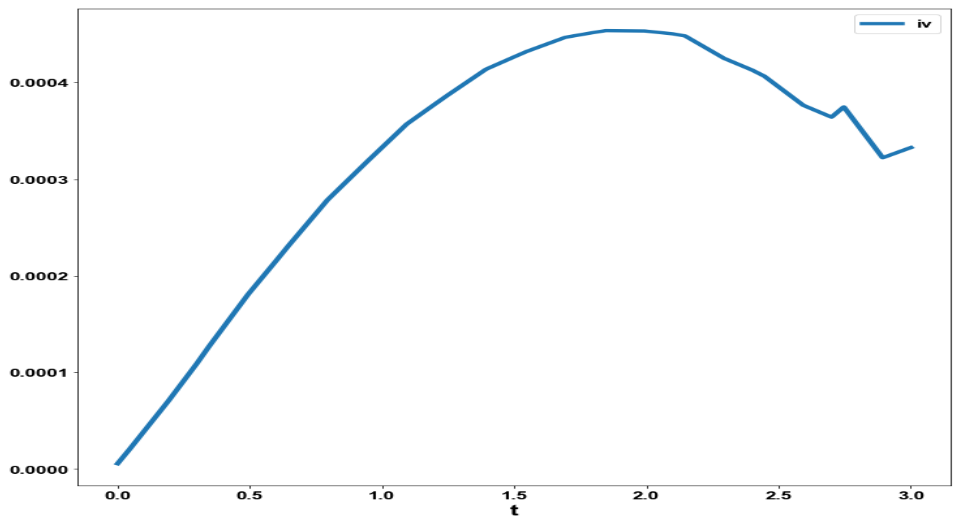

Figure: 2d Mnlmpc For Measles Model 2 (Iv Profile)

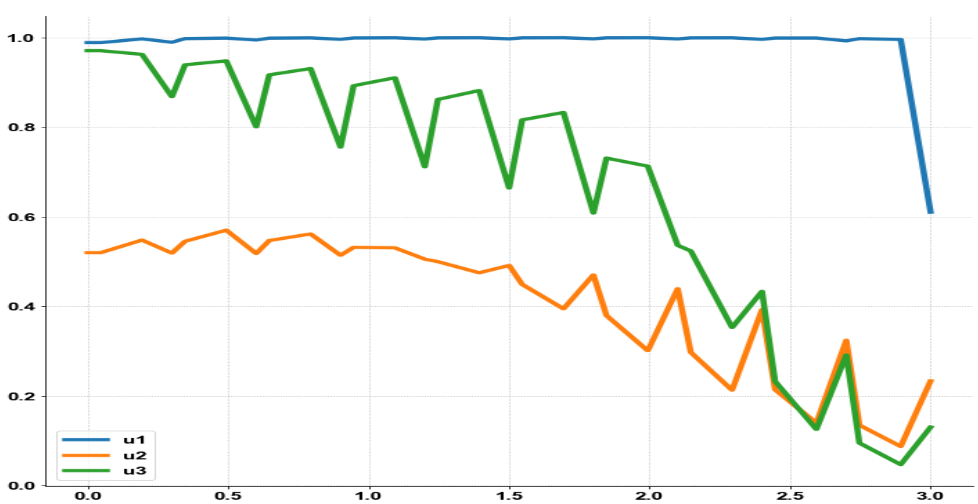

Figure: 2e Mnlmpc For Measles Model 2(Control Profiles Noise Exhibited)

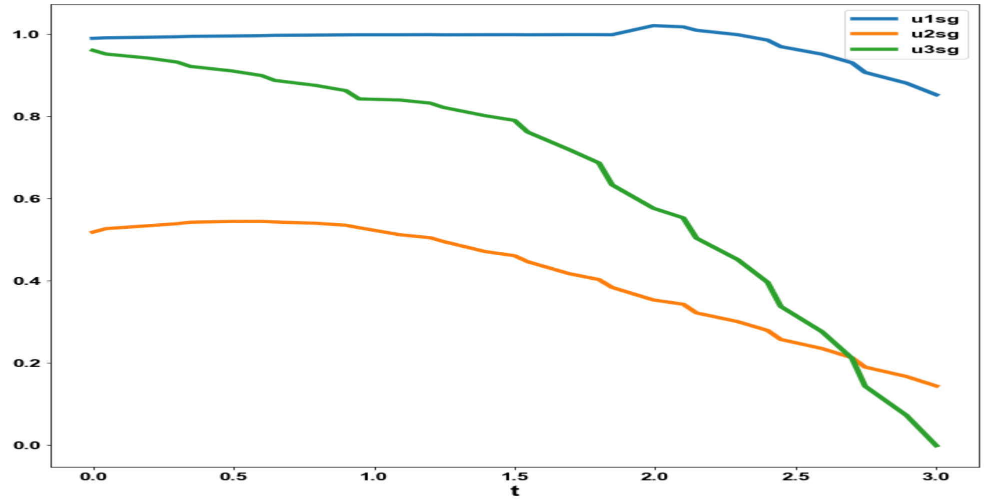

Figure: 2 f Mnlmpc For Measles Model 2(Control Profiles Noise Eliminated)

4. Conclusions

Bifurcation analysis and multiobjective nonlinear control (MNLMPC) studies in two measles transmission models. The bifurcation analysis revealed the existence of branch and limit points. The branch and limit points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in the models. A combination of bifurcation analysis and Multiobjective Nonlinear Model Predictive Control(MNLMPC) for measles transmission models is the main contribution of this paper.

Data Availability Statement

All data used is presented in the paper

Conflict of interest

The author, Dr. Lakshmi N Sridhar has no conflict of interest.

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