Review Article

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An Elegant Formula Derivation for The Circumference of An Ellipse

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Abstract

It is a widely accepted fact that the circumference of ellipse has no discrete formula. This happens, in part, because the orthodox methods of calculating circumference of an ellipse run into a system of infinite series and, in part, since there have been little concerted efforts to explore other ways to calculate an explicit formula for ellipse. However, this article claims that some minor revisions in a concerted manner, if proceeded in a different direction, can result in a full-fledged, distinct and explicit formula for the circumference of an ellipse, that is elegant yet simple. The article progresses in an orthodox manner, at first, and then displays, how a solution becomes possible in relatively simple calculus. The formula is a generalized form for all ellipses and as a special case, derives the regular formula for a circle. The formula has been tested on various mathematical calculation platforms available on internet and the results are very close to what the website claims. However, this article claims that the actual calculations result from the formula derived in the article and not from those available on the internet.

Keywords: Elliptic Measurements, Double Integration, Formula Derivation, Approximations, Infinite Series

1. Introduction

An ellipse is a geometrical figure that has a significant importance in mathematics. An ellipse is a two-dimensional shape in geometry that may be defined from its axes, foci or from different other paradigms [1]. While studying conic sections, if a plane intersects a cone at an angle to the base of the cone, the intersection results in an ellipse. It has two main foci. For every point on the elliptic curve, the sum of the distances from the two focal points is constant. A circle is also an ellipse since its centre is the point where both the foci are located [2].

Definition 1: An ellipse is defined as the locus of all the points on a curve in a 2-dimensional plane such that the sum of distances of all the points under consideration from two fixed points (known as foci) remains constant.

An elliptic figure is associated with many other terms in mathematics that may be named as foci, major axis, minor axis, directrix, latus rectum, eccentricity, etc. All these parameters are linked to any ellipse that may be framed in a 2-dimensional plane. Only a curve that has all these parameters defined well may only be termed as an ellipse [3]. There may exist other oval shapes that should not be confused with ellipse. An elliptic intersection defined from a conic section is described in the Figure 1.

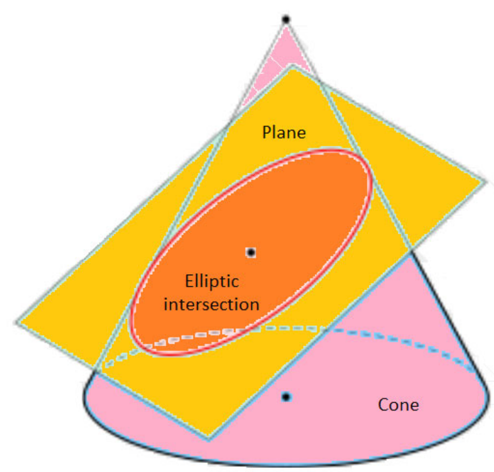

Figure 1: A geometrical formative description of a mathematical ellipse.

It is a matter of concern that even after thousands of years of development of maths, the exact, discrete and explicit formula for circumference of an ellipse still remains a mystery [4-6]. This article is inclined in a manner that it is more of a claim rather than inductive research in the field of mathematics. It is claimed that a formula for circumference of an ellipse is derivable and elegant yet so.

The article includes a progressive outlook and all the concepts and equations discussed have been tried to be properly framed. It should not be surprising that the final formula derived is a very eye-catching formula and it is claimed that the actual formula for calculating the circumference of an ellipse is the one provided in the article and not the approximate formulae that are in use currently.

The article is divided into 4 sections. Section 1 has provided a brief description of what follows in the article. Section 2 provides the literature survey and some detailed definitions of concepts to be used. Section 3 discusses all theoretical framework including the derivation of the circumference of the ellipse and Section 4 concludes the article. It should be mentioned that the developed theory is simply a claim currently that needs to be verified.

2. Literature Survey

Currently, it is agreed upon that there is no exact (accurate) formula to calculate the perimeter or circumference of an ellipse. There have been futile efforts, but some formulae exist today that provide a good approximation to the circumference of the ellipse. The formulae exist on the basis of the major axis length and minor axis length of the ellipse. The definitions of major and minor axis of ellipse are provided.

Definition 2: A major axis of the ellipse is defined as the longest diameter in the ellipse that passes through the center and it is usually denoted as 'a'.

Definition 3: A minor axis of the ellipse is defined as the shortest diameter in the ellipse that passes through the center and it is usually denoted as 'b'.

Based on the lengths on major and minor axes a and b, many minor and major approximations for the Perimeter P have been given. Some of these formulae include:

$$P = \pi(a+b) \quad \ldots(1)$$

$$P = \pi\sqrt{2(a^2+b^2)} \quad \ldots(2)$$

$$P = \pi\left[\frac{3}{2}(a+b) - \sqrt{ab}\right] \quad \ldots(3)$$

An Indian mathematician, Ramanujan provided even better approximations using his deep insights of infinite series [7, 8]. The formulae provided by him currently approximate to:

$$P = \pi\left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right] \quad \ldots(4)$$

$$P = \pi(a+b)\left(1 + \frac{3h}{10+\sqrt{4-3h}}\right) \quad \ldots(5)$$

Some of the well-established infinite series formulae include:

$$P = 2a\pi\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1}{2}.\frac{3}{4}\right)^2 \frac{e^4}{3} - \left(\frac{1}{2}.\frac{3}{4}.\frac{5}{6}\right)^2 \frac{e^6}{5} + \ldots\right] \quad \ldots(6)$$

$$P = \pi(a+b)\left(1 + \frac{1}{4}h + \frac{1}{64}h^2 + \frac{1}{256}h^3 + \ldots\right) \quad \ldots(7)$$

There exist some formulae that make use of the integration operation that might be summarized as follows:

$$P = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2-x^2)}}\,dx \quad \ldots(8)$$

$$P = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin\theta^2}\,d\theta \quad \ldots(9)$$

The above referenced formulae are used today on mathematical websites that provide the direct answer to the perimeter or circumference of ellipse when the lengths of major and minor axis a and b are plugged in.

However, we claim that a better, elegant and yet a simple formula can be developed that is seen to provide answers very close to the ones that are produced by the mathematical website tools on the internet.

3. Theoretical Framework Development

It is known that the most basic mathematical equation of an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots(10)$$

The parametric form of this equation in terms of the angle θ that the radial line forms with the major axis, can be defined as:

$$x = a\cos\theta \quad \ldots(11)$$

$$y = b\sin\theta \quad \ldots(12)$$

Now, we know that, for any point on ellipse, the equation between the radial line r and a differential part of the circumference dl can be given as:

$$dl = r\,d\theta \quad \ldots(13)$$

Now, r in terms of θ can be described as:

$$x^2 + y^2 = r^2 \quad \ldots(14)$$

$$\Rightarrow a^2\cos^2\theta + b^2\sin^2\theta = r^2 \quad \ldots(15)$$

$$\Rightarrow r = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \quad \ldots(16)$$

Plugging (16) in (13), we get,

$$dl = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta \quad \ldots(17)$$

Squaring both sides, we get:

$$dl^2 = (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta^2 \quad \ldots(18)$$

Now, it can be seen that there are squared differential terms instead of linear ones. Hence, we need to apply a double integration operation.

Hence,

$$\iint dl^2 = \iint_0^{2\pi} a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta^2 \quad \ldots(19)$$

On solving, we get,

$$\int l\, dl = \frac{a^2}{2}\int_0^{2\pi} \frac{-[\sin 2(2\pi) - \sin 2(0)] + 2\pi - 0}{2} d\theta$$

$$+ \frac{b^2}{2}\int_0^{2\pi} \frac{[\sin 2(2\pi) - \sin 2(0)] + 2\pi - 0}{2} d\theta \quad \ldots(20)$$

$$\Rightarrow \frac{l^2}{2} = \frac{a^2}{2}\int_0^{2\pi} \pi\ d\theta + \frac{b^2}{2}\int_0^{2\pi} \pi\ d\theta \quad \ldots(21)$$

$$\Rightarrow l^2 = 2\pi^2(a^2+b^2) \quad \ldots(22)$$

$$\Rightarrow l = \sqrt{2}\pi(a^2+b^2)^{1/2} \quad \ldots(23)$$

$$\text{Or, } P(Perimeter) = 2\pi\sqrt{\frac{a^2+b^2}{2}} \quad \ldots(24)$$

Special Case: For a circle, a and b both can be taken equal to r. Hence, the formula specializes to: P=2πr.

Hence, we have derived an efficient, simple and elegant formula for the circumference of an ellipse.

4. Conclusion

Hence, this article derives the formula for perimeter or the circumference of an ellipse in a simple, yet elegant manner. It is worth mentioning that the formula is applicable to the special case of circle as well. On matching the results with those from the mathematical tools websites, it was found that the formula gives results in close proximity with those are expected. Hence, the derived formula is a good candidate to the existing approximation techniques. Also, it can be claimed that this might actually be the exact, accurate formula for derivation of circumference of an ellipse.

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