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A Mathematical Model for Top Motions

Albina Omorova* and Ryspek Usubamatov

Kyrgyz State Technical University, Bishkek, Kyrgyzstan

***Corresponding author**
Albina Omorova, Kyrgyz State Technical University, Bishkek, Kyrgyzstan.
E-mail: alomorova15@gmail.com

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Abstract

The topic of the top motions is not new but the known publications represent wrong mathematical models for its gyroscopic effects. Recent investigations have demonstrated the physics of gyroscopic effects is more complex. On any spinning objects are acting the system of interrelated internal torques generated by their rotating mass elements and center mass. The inertial torque is produced by the centrifugal, common inertial and Coriolis forces, as well as the change in the angular momentum. These inertial torqueses represent the fundamental principle of gyroscope theory. The new inertial torques enables deriving mathematical models for the motions of any rotating objects that were impossible for a long time. The aim of this work is to represent mathematical models for the motions of the well-balanced top on the flat horizontal surface. This work describes the physics of the top motions and closes the problem of many years of discussion. The new analytical approach for the top's motions definitely responds to the practical results and represents a good example of the educational process.

Keywords: Physics of Gyroscopic Effects, Inertial Torque, Top

Introduction

Since the Industrial Revolution brilliant scientists and ordinary researchers have investigated, developed and added new interpretations of the gyroscopic effects, which analytical models and physics of acting forces were represented by simplified approaches and assumptions. The applied theory of gyroscopes emerged mainly in the twentieth century due to the intensification of the work of mechanisms with rotating components [1, 2]. Gyroscopic effects are used in gyroscopic devices in aerospace engineering, as well as on ships and other industries [3, 4]. The importance of the action of gyroscopic inertial forces on rotating objects is obvious and all textbooks of classical mechanics have chapters on the gyroscope theory [5-8]. Numerous and valuable publications have dedicated to gyroscopic effects and their applications in engineering [9, 10].

However, in known publications, mathematical models for the gyroscopic effects do not seem to match their practical applications in gyroscopic devices. This is the reason the gyroscope theory still attracts many researchers who seek to discover true gyroscope theory.

Mechanically, a gyroscope is a spinning disc, in which the simplest design represents a top. The motions of a top have been described analytically in numerous publications, and with complex numerical modeling based on Lagrangian dynamics on Euler coordinates that are solved with computer software [11, 12]. However, all publications contain approximations, assumptions, and simplifications, and explain gyroscope effects by the physical principle of the angular momentum [13, 14]. Gyroscope effects still represent a problem that remains to be solved. The physics of gyroscopic effects are more complex than represented in the theories known to date. Recent investigations into the physical principles of gyroscope motions demonstrate the system of interrelated inertial forces acting upon a spinning disc. Gyroscope effects generated by the action of the centrifugal, common inertial, and Coriolis forces, and the change in the angular momentum of the spinning disc [15-17]. The action of these forces is manifested by the resistance and precession torques which formulate the motions of the gyroscopic devices [18, 19].

The equations of the inertial torques demonstrate the proportional dependence on a spinning disc mass moment of inertia and its angular velocity, as well as on the angular velocity of its precession [13]. The mathematical models for this inertial torques clearly describe the physics of gyroscopic effects and explain the unusual motions of gyroscopic devices. The aim of this work is to represent mathematical models for the motions of the well-balanced top on the flat horizontal surface. The analytical approach is based on the action of the system interrelated inertial forces generated by the mass elements and center mass of a spinning top. The practical tests and observations confirm the mathematical model for the op motion.

Methodology

The new mathematical models for the top motions are formulated by the action of the system of interrelated internal torques. Observation of an inclined fast-spinning top demonstrates the spiral motion on the flat surface with following asymptotically motion to the vertical position of its axis. The spinning top stabilizes itself and its spin axis preserves a vertical position until to the minimal angular velocity that leads to the wobbling and then to side fall. This phenomenon of the top's motion is the result of the action of inertial internal torques generated by the spinning mass elements and the centre-mass. The circular or spiral motion of the top is the result of the action of the frictional force on the spherical surface of the tip of the top's leg.

The spiral motion of the top is being considered in terms of machine dynamics. The acting forces on the top are its weight, frictional force of the leg's tip at the point of contact the leg with the horizontal surface and the system of inertial forces mentioned above. The analysis of top motions is conducted using the example of a top that is tilted and spinning in a counter clockwise direction. The motion of the top spinning in a counter clockwise direction is considered around the point of support O that demonstrated in Figure 1. If the axis of a top is adjusted on the angle γ to the horizontal and released, then, under the action of the weight of the top, frictional force, and inertial torques the top's axis will begin to presses at about the vertical and horizontal. The frictional force acting on the tip of the leg starts to move the top around its gravity centre and the top's axis describes a conical surface. The action of external and inertial torques leads to the decrease of the angular velocity of the top's spin. Hence, the values of inertial torques are decreasing gradually, while the precession velocities correspondingly increase. When the angular velocity of the top becomes smaller, the internal torques becomes weaker. In this case, the tip of the top's leg describes a clearly visible spiral curve with a decrease of its radius of curvature. This situation leads to the vertical approach of the top axis that manifests its stabilization.

The motion of a spinning top that inclined to a horizontal flat surface is complex. The equation of motion is formulated by the action of several forces and torques on the top that is as follows: the external torque generated by action of the weight of an inclined top; the torque generated by the centrifugal force of the rotating top's centre-mass around axis *oy*; the external torque generated by the frictional force acting on the top's leg and around its axis.Other acting inertial torques is generated by the rotating masses mentioned above and represented in several publications [15, 16]. All forces and torques acting on the top are demonstrated in Figure 1.

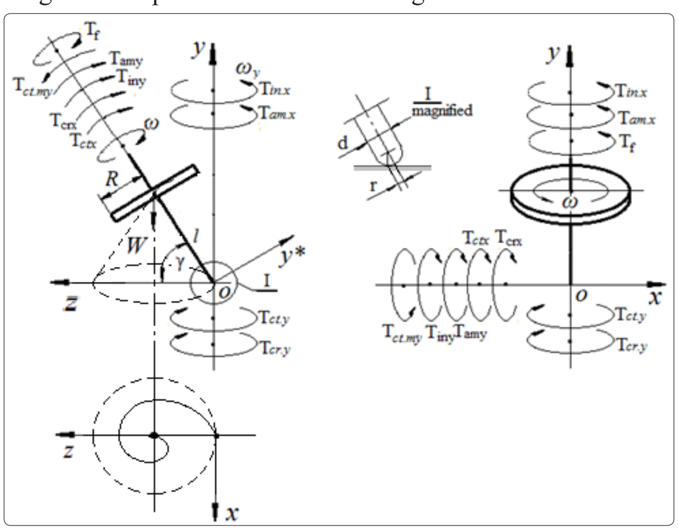

Figure 1: Forces acting on a spinning top

The parameters defined above allow for the formulation of a mathematical model for a top's motion around axes ox and oy in Euler's form. The torques acting on the top are similar to the torques acting on the gyroscope suspended from a flexible cord [16]. Both models consider the motion of the spinning disc with one-side support. As such, the mathematical model for a top's motion is represented by the following system of equations:

$$J_x \frac{d\omega_x}{dt} = T + T_{ct.my} - T_{ctx} - T_{crx} - T_{amy}\cos\gamma\eta \tag{1}$$

$$J_y \frac{d\omega_y}{dt} = T_f + (T_{inx} + T_{amx} - T_{cry})\cos\gamma \tag{2}$$

$$\omega_y = -\left(\frac{2\pi^2 + 8}{\cos\gamma} + 2\pi^2 + 9\right)\omega_x \tag{3}$$

where $J_i = (mR^2/4) + ml^2$ is the top's mass moment of inertia around axis i [5-7]; m is the top's mass; is the external torque generated by action of the top weight; g is the gravity acceleration, l is the length of the leg; $F_{ct.mx} = ml\omega x^2\sin\gamma\cos\gamma$ is the force generated by the centrifugal force of the rotating top's centre mass around axis *ox*; $T_f = mgfl\cos\gamma$ is the torque generated by the frictional force acting on the tip and turns the top around it's the centre mass in the counter clockwise direction, f is the coefficient of the sliding friction between the leg and flat surfaces; $T_{ct.my} = F_{ct.my} l\sin\gamma = ml\cos\gamma\omega_y^2 l\sin\gamma = ml^2\cos\gamma\sin\gamma\omega_y^2$ is the torque generated by the centrifugal force of the rotating top's centre mass around axis *oy*;ω_y is the precession velocity of the top around axis *oy*;η is the coefficient of the change in the value of the inertial torques; other expressions are as specified above.

The action of the frictional torque T_f turns the top around axis *oy* and increases the value of the precession torques ($T_{px} = T_{inx} + T_{amx}$). The value of the precession torque T_{amy} acting around axis ox is also decreases due to the interdependency of the actions of the inertial torques around axes. This action is expressed by the coefficient η of the change in the precession torque T_{amy} that is presented by the following equation:

$$\eta = \frac{T_{px} + T_f}{T_{px}} = 1 + \frac{mgfl}{\left(\frac{2}{9}\pi^2 + 1\right)J\omega\omega_x} \tag{4}$$

Where T_{px} is the precession torques originated on axis *ox* but acting around axis *oy*, other components are as specified above.

The methodology towards a solution for equation (1) - (2) by using the ratio of the precession velocities of a top around axes (equation (3)) is represented in [16]. Substituting equations (3) and (4) and equations of inertial torques [15] into equations (1) and (2) and simplification yield the following equations of the top's motion around axis *ox*:

$$J_x \frac{d\omega_x}{dt} = mgl\cos\gamma - ml^2\cos\gamma\sin\gamma\omega_x^2 + ml^2\cos\gamma\sin\gamma\left[\left(\frac{2\pi^2+8}{\cos\gamma}+2\pi^2+9\right)\omega_x\right]^2 -$$
$$\left(\frac{2\pi^2+8}{9}\right)J\omega\omega_x - [2\pi^2+8+(2\pi^2+9)\cos\gamma]J\omega\omega_x \times \left[1+\frac{mgfl}{\left(\frac{2\pi^2+9}{9}\right)J\omega\omega_x}\right] \tag{5}$$

$$J_x \frac{d\omega_x}{dt} = mgl\left[\cos\gamma - 9\left(\frac{2\pi^2+8}{2\pi^2+9}+\cos\gamma\right)f\right] - \left[\frac{10(2\pi^2+8)}{9}+(2\pi^2+9)\cos\gamma\right]J\omega\omega_x -$$
$$\left[1-\left(\frac{2\pi^2+8}{\cos\gamma}+2\pi^2+9\right)^2\right]ml^2\cos\gamma\sin\gamma\omega_x^2 \tag{6}$$

Where all parameters are as specified above.

Self-Stabilization

The practical observation of a tilted spinning top proves its capacity for self-stabilization. The axis of the tilted spinning top goes to the vertical position by the action of the inertial torques generated by the rotating mass elements, which values are bigger than torques generated by the top's weight and inertial torque generated by the centre mass. The necessary condition for a top's self-stabilization is formulated by separating variables acting in the counter-clockwise and clockwise directions of the right side of equation (6) that expresses by the following equation:

$$mgl\left[\cos\gamma - 9\left(\frac{2\pi^2+8}{2\pi^2+9}+\cos\gamma\right)f\right] - ml^2\cos\gamma\sin\gamma\left[1-\left(\frac{2\pi^2+8}{\cos\gamma}+2\pi^2+9\right)^2\right]\omega_x^2 =$$
$$\left[\frac{10(2\pi^2+8)}{9}+(2\pi^2+9)\cos\gamma\right]J\omega\omega_x \tag{7}$$

Analysis of equation (7) shows that the equilibrium of the acting torques depends on three main variable components, i.e. the angular velocity ω of the top, the velocity of precession ω_x, and the angle γ of its inclination. The stabilization process is intensive when the value of the torque of the right side equation is big and in addition, the length of the top's leg should be short, and i.e. the centre-mass of the top is located towards the tip of the leg. The spinning top with a long leg and a small radius of the disc manifest the less stability. The top of high spinning velocity generates the high value of the torque presented at the right side of equation (7). This value is always bigger than the value of the torques on the left side of equation (7). It means, the property of the top for self-stabilization is permanent and depends only on the speed of the top rotation. The running top starts to wobble when the speed of the top is always close to zero that leads to loose of its stability.

Working Example

The example considers the motion of a disc-type top whose technical data is represented in Table 1. The centre-mass of the top is located on the plane of the thin disc. The spinning top initially possesses an inclined axle and rotating around a vertical axis.

Table 1: Technical data of the top

	Parameter	Data
	Angular velocity, ω	1000 rpm
	Radius of the disc, R	0.025 m
	Length of the leg, l	0.02 m
	Angle of tilt, γ	75.0°
	Mass, m	0.02 kg
	Coefficient of friction, f	0.1
Mass moment of inertia, kgm²	Around axis oz, $J = mR^2/2$	0.625×10^{-5}
	Around axes ox and oy of the centre mass, $J = mR^2/4$	0.3125×10^{-5}
	Around axes ox and oy, $J_x = J_y = mR^2/4 + ml^2$	1.1125×10^{-5}

For the solution, equation (6) is transformed, simplified and variables separated that yield the following differential equation:

$$\frac{J_x d\omega_x}{ml^2\cos\gamma\sin\gamma\left[1-\left(\frac{2\pi^2+8}{\cos\gamma}+2\pi^2+9\right)^2\right]\omega_x^2-\left[\frac{10(2\pi^2+8)}{9}+(2\pi^2+9)\cos\gamma\right]J\omega\omega_x+ mgl\left[\cos\gamma-9\left(\frac{2\pi^2+8}{2\pi^2+9}+\cos\gamma\right)f\right]}=dt \quad (8)$$

Substituting initial data into equation (8) brings the following expression:

$$\frac{1.1125\times10^{-5}d\omega_x}{0.02\times0.02^2\cos75^o\sin75^o\left[1-\left(\frac{2\pi^2+8}{\cos75^o}+2\pi^2+9\right)^2\right]\omega_x^2-\left[\frac{10(2\pi^2+8)}{9}+(2\pi^2+9)\cos75^o\right]0.625\times10^{-5}\times1000\times\frac{2\pi}{60}\omega_x+ 0.02\times9.81\times0.02\times\left[\cos75^o-9\times\left(\frac{2\pi^2+8}{2\pi^2+9}+\cos75^o\right)\times0.1\right]}=dt \quad (9)$$

Simplification of equation (9) and transformation yields the following solution:

$$-\frac{3.011320372\times10^{-4}d\omega_x}{\omega_x^2+0.677806882\omega_x+0.064776789}=dt \quad (10)$$

The denominator on the left side of equation (10) represents the quadratic equation with the following expression:

$$-\frac{3.011320330\times10^{-4}d\omega_x}{(\omega_x+0.562686267)(\omega_x+0.115120615)}=dt \quad (11)$$

Equation (11) is converted into integral forms with definite limits and yields the following equation:

$$\frac{1}{0.447565652}\int_0^{\omega_x}\left(\frac{1}{\omega_x+0.115120615}-\frac{1}{\omega_x+0.562686267}\right)d\omega_x=-3320.802473378\int_0^t dt \quad (12)$$

Integrals of Equation (12) is tabulated and presented the integrals $\int\frac{dx}{x+a}=\ln|a+x|+C$ and $\int\frac{dx}{x-a}=\ln|a-x|+C$

with the following solutions: $\ln(\omega_x+0.115120615)\Big|_0^{\omega_x}-\ln(\omega_x+0.562686267)\Big|_0^{\omega_x}=-1486.277124161t$

that giving the rise to the following: $\left(\frac{\omega_x+0.115120615}{0.115120615}\right)-\ln\left(\frac{\omega_x+0.562686267}{0.562686267}\right)=-1486.277124161t$

The next transformation yields the following result:

$$\omega_x+0.115120615=e^{-1486.277124161t}\times4.887797611\times(\omega_x+0.562686267) \quad (13)$$

Analysis of the right component of Equation (13) demonstrates that the expression has a small value of a high order that can be neglected. The solution ofequation (13) and (3) yields the following values of the precession angular velocities for the top around axes *ox* and *oy*:

$$\omega_x = -0.115120615 \quad rad/s$$

$$\omega_y = \left(\frac{2\pi^2+8}{\cos 75^o} + 2\pi^2 + 9\right) \times 0.115120615 = 15.646650797 \quad rad/s \tag{14}$$

Where the sign (-) for ω_x means the turn of the top axel in the clockwise direction toward the vertical location.

Self-stabilization

The presented data of a tilted spinning top allow the condition for its self–stabilization to be checked. Substituting the defined data above and from Table 1 into equation (7) yield the following result:

$$0.02 \times 9.81 \times 0.02 \times \left[\cos 75^o - 9 \times \left(\frac{2\pi^2+8}{2\pi^2+9} + \cos 75^o\right) \times 0.1\right] -$$
$$0.02 \times 0.02^2 \cos 75^o \sin 75^o \left[1 - \left(\frac{2\pi^2+8}{\cos 75^o} + 2\pi^2 + 9\right)^2\right] \times 0.115120615^2 =$$
$$\left[\frac{10(2\pi^2+8)}{9} + (2\pi^2+9)\cos 75^o\right] 0.625 \times 10^{-5} \times 1000 \times \frac{2\pi}{60} \times 0.115120615 \tag{15}$$

That giving rise to the following

$$-0.\ 007560157 < 0.002882717 \tag{16}$$

The right component of equation (16) is bigger than the left one. It means the inertial torques acting on the top turn up to vertical with the spiral motion of the top on the surface.

Results and Discussion

The mathematical models for the top motions on the horizontal surface are based on the action of the top's weight, frictional forces, and the system of inertial torques generated by the rotating mass elements and centre mass of a spinning top. The action of a top's weight produces inertial torques that is interrelated and acted at one time and expresses precession motions. This inertial torques present the action of centrifugal, common inertial, Coriolis forces as well as the change in the angular momentum. The new physical principles enable formathematical models of an inclined top precession motion and its self-stabilization to be formulated, i.e., rotates vertically.

Conclusion

In the gyroscope theory, the top's motion was one of the most complex and intricate in terms of analytical solutions. Known mathematical models for the top motions are accepted with simplifications and do not adequately express the real physics of acting forces. The new mathematical models for gyroscope torques consider the interdependent action of the system of inertial forces generated by the rotating mass elements and centre mass of the spinning objects. The new physical principles for gyroscope motions were used for modelling of inclined a top's motions and its self-stabilization. The obtained mathematical model for the top complex motionis distinguished from the models in known publications, which does not interpret the physics of gyroscopic effects. The application of the correct mathematical model for the top's motion is a good example of educational processes.

Nomenclature

g – Gravity acceleration
F - Centrifugal force
$F_{ct,my}$ - Centrifugal force of a top's centre-mass rotating around axis oy
i - Index for axis ox or oy
J - Mass moment of inertia of a top
J_i - Mass moment of inertia of a top around axis i
l – Length of a top's leg
m - Mass of a top
T - Load torque
$T_{am.i}$, T_{cti}, $T_{cr.i}$, $T_{in.i}$ - Torque generated by the change in the angular momentum, centrifugal, Coriolis and common inertial forces acting around axis i, respectively
$T_{r.i}$, T_{pi} – Resistance and precession torque acting around axis i, respectively
t -Time
γ - Tilt angle of a top's axis
ω - Angular velocity of a top
ω_i - Angular velocity of a top around axis i

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