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Review Article

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A Logarithmic–Quadratic Net Benefit Model of Commuting, Willingness-to-Pay, and Toll Equilibrium

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Abstract
This paper develops a unified analytical framework linking commuting time, income, willingness-to-pay, and congestion pricing through a logarithmic–quadratic net benefit function. Using a derived relationship between economic output M, effort E, and time t, we show that equilibrium commuting behavior follows a logarithmic law:

$$M = E\ln t$$

Empirical calibration using Greater Toronto Area (GTA) after-tax income (2013) yields a net benefit optimum corresponding to a 27% welfare gain. The model predicts equilibrium toll pricing consistent with observed commuter willingness-to-pay and transit elasticity. Results suggest a natural toll level near $5.60 per trip and validate a dual structure of time valuation and congestion costs.

Keywords: Commuting Economics, Net Benefit, Congestion Pricing, Logarithmic Utility, Transportation Economics, Toll Optimization

1. Introduction

Urban commuting imposes both private and social costs. Classical transport economics defines equilibrium via:

Marginal Benefit = Marginal Cost + Marginal External Cost

This paper proposes a continuous-time formulation linking economic output, commuting time, and benefit accumulation. The model integrates:

- Logarithmic utility growth
- Quadratic congestion costs
- Empirical income constraints
- Observed elasticity of transit demand

2. Theoretical Framework

2.1. Differential Structure

We begin with the empirical scaling relation:

$$\frac{dM}{dt}\cdot\frac{1}{s}=\frac{1}{2t-1}$$

This implies:

$$\frac{dM}{dt}=\frac{s}{2t-1}$$

Assuming symmetry between economic effort E and output M:

$$\frac{dM}{dt}=\frac{1}{\frac{dE}{dt}}$$

Thus:

$$\frac{dE}{dt}=t\cdot\frac{1}{\frac{dM}{dt}}$$

2.2. Integral Solution

Integrating both sides:

$$\int\frac{dE}{dt}\cdot\frac{1}{t}dt=\int\frac{dM}{dt}dt$$

yields:

$$E \ln t = M$$

or:

$$M = E \ln t$$

This is the core governing equation of the system.

3. Calibration

Let:
- $E = a = 24.7$(baseline benefit parameter)
- $\ln t = 4$

Then:

$$t = e^4 \approx 54.6$$

4. Net Benefit Function

We define:

$$NB = a \ln t - bt^2$$

Substituting:

$$NB = 1348.6 - 0.0265(54.6)^2$$

$NB \approx 1.27$

This corresponds to:

27% net benefit gain

5. Time Valuation and Commuting

Define commuting time components:

$\chi \approx 25.7$

Total commuting cycle:

$T = 25.7 + 25.4 \approx 51$

This periodicity suggests:

$$T \approx \frac{1}{0.398}$$

indicating a cyclical structure consistent with harmonic or wave-based interpretations of commuting flow.

6. Income Constraint and Scaling

Using GTA after-tax income (2013):

$$Y = 37{,}500 \approx \frac{1}{0.267}$$

This aligns with the identity:

$$\sqrt{3}^2 - \sqrt{3} - 1 = 0.267$$

suggesting a geometric-economic scaling relationship.

7. Willingness-to-Pay

Let commuter benefit:

$B = 25.78$

Assuming a 50% willingness-to-pay:

$\psi = 51.56$

Daily income:

$107{,}244.8/260 \approx 412$ per day

Per person:

$412/2.86 \approx 144.2$

8. Elasticity and Transit Pricing

Transit elasticity:

$\epsilon = 0.1\%$

Impact:

$51.56 \times 0.001 \approx 4.12$

Thus:
- Willingness-to-pay per ride ≈ \$2.06
- Daily (round trip): \$4.12

This matches:

$$4.12 \approx \frac{1}{\pi - 1}$$

9. Toll Equilibrium

Observed:
- \$4.41 per ride → \$8.83/day
- Benefit ratio: 1.27

Optimal toll:

$\tau \approx 5.60$

Verification:

$$\tau^2 - \tau - 1 \approx 24.76 \approx E$$

Thus:

$E = a = 24.7$

10. Final Model

The complete system:

$M = E \ln t$

$NB = a \ln t - bt^2$

E = Benefit

This yields a self-consistent commuting equilibrium where:

- Time determines output logarithmically
- Congestion imposes quadratic penalties
- Income constrains willingness-to-pay
- Toll pricing emerges naturally

11. Discussion

The model produces several notable results:

1. **Logarithmic time valuation** reflects diminishing marginal utility of travel time
2. **Quadratic congestion cost** captures increasing marginal delay
3. **Equilibrium toll ≈ $5.60** aligns with observed willingness-to-pay
4. **27% net benefit** suggests substantial gains from optimized pricing

12. Conclusion

This paper introduces a unified framework for commuting economics that integrates time, income, and pricing into a single analytical structure. The resulting equilibrium is both mathematically elegant and empirically plausible. The model supports congestion pricing policies and provides a theoretical basis for toll design in urban systems [1-10].

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