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A Brief Introduction to Splitting Of Primes over Number Fields

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Abstract

The study of Dedekind Zeta Functions over a number field extension uses different aspects of both Algebraic and Analytic Number Theory. In this paper, we shall learn about the structure and different analytic aspects of such functions, namely the domain of its convregence and analyticity at different points of $\mathbb{C}$ when the function is defined over any finite field extension K over $\mathbb{Q}$. Moreover, given any two Number Fields L and K over $\mathbb{Q}$ with L being Normal over K, our intention is to classify and study the primes in K which split completely in L. Also, we shall explore some special cases related to this result.

Keywords and Phrases: Dirichlet Series, Dedekind Zeta Functions, Number Field Extension, Riemann Hypothesis, Holomorphic Functions, Partial Dedekind Zeta Functions, Polar Density, Finite Extension, Class Number Formula, Dirichlet L-Functions.

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Secondary 11R44, 11S15, 32A60, 32A10.

Motivation to Study Dedekind Zeta Functions

Dedekind Zeta Functions over an algebraic number field *K* over $\mathbb{Q}$, usually denoted by $\zeta_K(s)$, where, $s \in \mathbb{C}$, is often considered as the generalization of the *Riemann Zeta Functions* over $\mathbb{Q}$ (It is to be noted that, in the case for the *Riemann zeta functions*, we take the field *K* mentioned above to be $\mathbb{Q}$ itself.)

Dedekind zeta functions can be expressed as a *Dirichlet Series*, as well as it has an *Euler Product Expansion*, which we shall discuss thoroughly in the next section.

Worth mentioning that, Dedekind zeta functions satisfy a *functional equation*. Moreover, they can be analytically extended to a *meromorphic function* on the complex plane $\mathbb{C}$ with only a simple pole at $s = 1$. Indeed, it helps us conclude about various properties of the number field *K* over $\mathbb{Q}$.

Another aspect of Dedekind zeta functions is the *Riemann Hypothesis*, which is till date considered as one of the very few unsolved problems in the field of mathematics, the statement of the conjecture is as follows:

$$If\ \zeta_K(s) = 0\ and,\ 0 < Re(s) < 1,\ then,\ Re(s) = \tfrac{1}{2}.$$

A Brief Overview of Dirichlet Series

1. Introduction

Suppose we consider the Dirichlet's Series of complex numbers defined by,

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}\ ,$$

where, an's are fixed complex numbers ∀ n, and the co-efficients of different terms in the series.

$s = x + iy$; $x, y \in \mathbb{R}$ is a complex variable.

First we observe the analyticity of the *Dirichlet Series* on different points of $\mathbb{C}$.

1.1 Analytic Properties

Definition 2.2.1 (Holomorphic Functions) A *holomorphic function* is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain.

More generally, the term *holomorphic* is used in the same sense as an *analytic function*.

Let us mention the following lemma regarding analyticity of *Dirichlet Functions* on $\mathbb{C}$:

Lemma 2.2.1. *Suppose for the above Dirichlet Series, the partial sum of co-efficients,* $\sum_{n \leq t} a_n = O(t^r)$ *for some real number* $r > 0$. *Then the Dirichlet Series converges for every* $s \in \mathbb{C}$ *with Re(s) > r , and is analytic, as well as holomorphic function on that half-plane.*

Proof. Here, for the proof of lemma (2.2.1), we apply Weierstrass Theorem for analyticity of power series.

Theorem 2.2.2. *(Weierstrass Theorem) Suppose that, for every $n \in \mathbb{N}$, the function $f_n(z)$ is analytic in the region Ω_n, also the sequence $f_n(z)$ converges to a limit function $f(z)$ in a region Ω, and converges uniformly on every compact subset of Ω. Then, $f(z)$ is analytic in Ω, moreover, $f'_n(z) \longrightarrow f'(z)$ uniformly on every compact subset of Ω.*

As evident from the statement of Theorem (2.2.2), it only suffices to establish the absolute convergence of the series on every compact subset of the half-plane. Defining the partial summation of the co-efficients as, $A_k = \sum_{n=1}^{k} a_n$, we obtain, for any m, $M \in \mathbb{N}$ with, $1 \leq m < M$,

$$\sum_{n=m}^{N} \frac{a_n}{n^s} = \sum_{n=m}^{M} \frac{A_n}{n^s} - \sum_{n=m}^{M} \frac{A_n}{n^s} = \frac{A_M}{M^s} - \frac{A_{m-1}}{m^s} + \sum_{n=m}^{M-1} A_n\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right). \qquad (2.1)$$

Since, by statement, the partial sum is $O(t^r)$, therefore, $\exists$ a positive $B \in \mathbb{R}$ such that, $|A_n| \leq Bn^r$, $\forall\ n \in \mathbb{N}$. Applying this condition in equation (2.1), we get,

$$|\sum_{n=m}^{M} \frac{a_n}{n^s}| \leq B\left(\frac{M^r}{|M^s|} + \frac{(m-1)^r}{|m^s|} + \sum_{n=m}^{M-1} n^r \left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right|\right) \qquad (2.2)$$

Therefore,

$$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| \leq |s| \int_{n}^{n+1} \frac{dt}{|t^{s+1}|} = |s| \int_{n}^{n+1} \frac{dt}{t^{x+1}} \leq \frac{|s|}{n^{x+1}}$$

(Since, assuming $s = x + iy$, where, $x = Re(s) \in \mathbb{R}$ and, $y \in \mathbb{R}$, it can be written that, $\frac{1}{n^s} - \frac{1}{(n+1)^s} = s \int_{n}^{n+1} \frac{dt}{t^{s+1}}$, and hence, the above result follows).

Which yields, from equation (2.2),

$$|\sum_{n=m}^{M} \frac{a_n}{n^s}| \leq B\left(M^{r-x} + m^{r-x} + |s| \sum_{n=m}^{M-1} n^{r-x-1}\right). \qquad (2.3)$$

Since, we have,

$$\sum_{n=m}^{M-1} n^{r-x-1} \leq \int_{m-1}^{\infty} t^{r-x-1} dt$$

(This improper integral is valid, by Comparison Test)

$$= \lim_{v \to \infty} \int_{m-1}^{v} t^{r-x-1} dt = \frac{(m-1)^{r-x}}{x-r}\ , \text{ for any } m > 1,$$

Hence, having put $m \to \infty$ and $M \to \infty$, from (2.3), it is evident that,

$$|\sum_{n=m}^{M} \frac{a_n}{n^s}| \longrightarrow 0$$

for any fixed $s = x + iy$ in the half-plane, $x > r$.

This implies that, the partial sum of the *Dirichlet Series* converges, consequently, we conclude that, the series itself converges $\forall\ s = x + iy$ with $x > r$.

Moreover, since each such compact set in the half-plane

$$\{s = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}\ \&\ Re(s) = x > r\}$$

is bounded (we can have, $|s| \leq B_1$ for some positive real $B_1 > 0$) and is away from the line, $Re(s) = x = r$ (Since, $x > r \Longrightarrow x - r \geq \epsilon$ for some $\epsilon > 0$). It implies that, the convergence of the *Dirichlet Series* is also uniform on every compact subset of the half-plane, $x > r$, $x = Re(s)$.

Thus, we have an uniform estimate,

$$|\sum_{n=m}^{M}\frac{a_n}{n^s}| \leq B\left(\frac{1}{m^\epsilon}+B_1\frac{1}{\epsilon(m-1)^\epsilon}\right)$$

and the proof of the lemma (2.2.1) is complete.

Remark 2.2.3. As an appropriate example reflecting the significance of Lemma (2.2.1), we can observe that, the *Riemann Zeta Function* defined by,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

converges. Since the partial sum of the co-efficients of the above series is = $O(n)$, hence, applying Lemma (2.2.1), it can be verified that, the function is analytic on the half-plane, $x > 1$.

2 Dedekind Zeta Function

2.1 Formal Definition

Definition 3.1.1. (Dedekind Zeta Function) As a more generalized case of *Riemann Zeta Function* for a number field K over $\mathbb{Q}$, we have the following definition of *Dedekind Zeta Function* for the half-plane, $Re(s) = x > 1$,

$$\zeta_k(s)=\sum_{n=1}^{\infty}\frac{j_n}{n^s},$$

where, $s \in \mathbb{C}$, such that, $Re(s) = x > 1$.
And we also define for every $n \in \mathbb{N}$,

j_n = Number of ideals I of the Ring of Integers of K over $\mathbb{Q}$ (denoted by, $\mathcal{O}_K$) with, $\mathcal{N}(\mathcal{I})$ $= n$,

2.2 Analyticity of $\zeta_K(s)$

For determining the domain of $\zeta_K(s)$, where the function is analytic, we introduce an important result,
Theorem 3.2.1. *Let $K/\mathbb{Q}$ be a finite extension of number fields with $[K : \mathbb{Q}] = n$ (say), and $\mathcal{O}_k$ be the ring of integers of K. We define, for each positive $t \geq 0$, $i(t)$ = Number of ideals I of $\mathcal{O}_K$ with $\mathcal{N}(\mathcal{I})$ $\leq t$, and also, for each ideal class C (say), define, $i_C(t)$ = Number of ideals I in C with $\mathcal{N}(\mathcal{I})$ $\leq t$, i.e.,*

$$i(t)=\sum_{C} i_C(t)$$

It is important to note that, the above sum is finite.
Then ∃ a number k_ς, depending on $\mathcal{O}_K$ but independent of the ideal class C, such that,

$$i_C(t)=k_\varsigma t+\epsilon_C(t)$$

Where the error term is denoted by $\epsilon_C(t)$, which is $O(t^{1-\frac{1}{n}})$. *In other terms, the ratio,* $\frac{\epsilon_C(t)}{t^{1-\frac{1}{n}}}$ *is bounded by a finite real number as $t \to \infty$.*

Remark 3.2.2. From the above theorem, we can conclude that, the partial sum of co-efficients in the *Dedekind Zeta Function*, $\sum_{n\leq t} j_n$ is $O(t)$. Therefore, by lemma (2.2.1), ζ_K converges and is analytic on the half-plane, $Re(s) > 1$.
Lemma 3.2.3. *Dedekind Zeta Function ζ_K can be extended to a meromorphic function on the half-plane,*

$$Re(s)>1-\frac{1}{n},\ where,\ n=[K:\mathbb{Q}],$$

where the function is analytic everywhere except at the point s = 1, where it has a simple pole of order 1.

***Remark* 3.2.4.** An analogue of the above statement is that, $(s - 1)\zeta_K(s)$ is analytic on the entire half-plane, $Re(s) > 1 - \frac{1}{n}$.

Proof. Considering the series,

$$f(s)=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+......,$$

we observe that, $f(s)$ converges to an analytic funtion on the entire half-plane, $Re(s) > 0$ (Applying Lemma (2.2.1)).
Let, $s = x + iy \in \mathbb{C}$. Now, we can write,

$f(s) = (1 - 2^{1-s})\zeta(s)$ for, $x > 1$ (From Definition of $f(s)$).

where, $\zeta(s)$ is the *Riemann Zeta Function* over $\mathbb{Q}$.

Therefore, the function $\frac{f(s)}{1-2^{1-s}}$ is an extension of ζ to a *meromorphic function* on the half-plane, $x > 0$. Hence the function above can have poles, if,

$$1 - 2^{1-s} = 0, \qquad \text{i.e.,} \qquad s = 1.$$

Now, we can also observe that, $f(s) = 0$ at the points,

$$s_k = 1 + \frac{2k\pi i}{log2}, \qquad k = \pm 1, \pm 2, ...$$

Again, considering another series,

$$g(s) = 1 + \frac{1}{2^s} - \frac{2}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s}........$$

similarly, we can deduce that,

$$g(s) = (1 - 3^{1-s})\zeta(s) \qquad \text{for, } x > 1 \text{ (From Definition of } g(s)).$$

and consequently, the function, $\frac{f(s)}{1-3^{1-s}}$ is an extension of ζ to a meromorphic function on the half-plane, $x > 0$ with poles at every such s satisfying, $1 - 3^{1-s} = 0$, i.e.,

$$s'_k = 1 + \frac{2k\pi i}{log3}, \qquad k = \pm 1, \pm 2, ...$$

Clearly, the set of points s_k and s'_k are disjoint, $k \neq 0$.

Since, $\lim_{s \to s_k^+} \zeta(s)$ is finite, therefore, we have $\zeta(s) = \frac{f(s)}{1-2^{1-s}}$ does not really have poles at the points s_k, $k \neq 0$, and hence the function is analytic everywhere except at $s = 1$.

Assuming this as the definition of the extension of ζ (also denoting the extension as ζ), we can express ζ_K as,

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{j_n - h\mathrm{k}_{\text{¸}}}{n^s} + h\mathrm{k}_{\text{¸}}\zeta(s), \text{ for, } Re(s) = x > 1$$

where,

h = Number of ideal classes in $\mathcal{O}_K$

k¸ = The constant term (as mentioned in Theorem (3.2.1)).

Applying Theorem (3.2.1) and Lemma (2.2.1), we conclude that, $\zeta_K(s)$ can be described as a *Dirichlet Series* with co-efficients, $j_n - h_k$, $\forall n$. Moreover, it converges to an analytic function

$$x > 1 - \frac{1}{n}, \text{ where, } n = [K : \mathbb{Q}]$$

(It can be deduced that, the partial sum of the co-efficients, $\sum_{n \leq t} (j_n - h\mathrm{k}_{\text{¸}})$ is $O(t^{1-\frac{1}{n}})$.)

Combining above with the result obtained for the extension of ζ, the proof is complete.

2.3 Other Definitions of ζ_K

A priori using the notion of absolute convergence of ζ_K, we can redefine ζ_K in the following manner:

Definition 3.3.1. (Dedekind Zeta Function) We define the Dedekind Zeta function as,

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{\mathcal{N}(\mathcal{I})^s}, \qquad \text{for, } Re(s) > 1.$$

where, we define,

$$\mathcal{N}(\mathcal{I}) = \text{Norm of an ideal } I \subset \mathcal{O}_K = |\mathcal{O}_K / I|,$$

and, the sum is performed over every non-zero ideal I of $\mathcal{O}_K$.

***Remark* 3.3.1.** Also it is to be noted that the order of the summation is unspecified since it is not required.

We can further provide a third definition of the *Dedekind Zeta Functions* as,

Definition 3.3.2. (Dedekind Zeta Function) We define ζ_K as the following product,

$$\zeta_K(s) = \prod_{\mathcal{P}\subset\mathcal{O}_K} (1 - \frac{1}{\mathcal{N}(\mathcal{P})^s})^{-1}, \qquad \text{for, } Re(s) > 1.$$

where,

$$\mathcal{P} := \text{Prime ideal of } \mathcal{O}_K.$$

***Remark* 3.3.2.** In definition (3.3.2), we are taking the product over every prime ideal P of $\mathcal{O}_K$.
***Remark* 3.3.3.** The product defined above yields a finite value, since, for every prime ideal $P \subset \mathcal{O}_K$, we have, $\mathcal{N}(\mathcal{P}) > 1$, hence each term in the product is finite.
At this juncture, one may be curious to ask,

Are these definitions of ζ_K equivalent to each other ?

The answer of course, is YES. We state the following proposition in order to establish our claim.

Proposition 3.3.4. Definitions (3.1.1),(3.3.1) and (3.3.2) of ζ_K are equivalent.
Proof. First, we shall establish that, definition (3.3.1) implies definition (3.1.1) :
We choose $n \in \mathbb{N}$ and define the ideal class of n as:

$$[n] := \{\mathcal{I} \subset \mathcal{O}_K \mid \mathcal{N}(\mathcal{I}) = n\}$$

Then, we can observe that, by our previous definition of j_n,

$$j_n = |[n]|.$$

Now, taking the sum over every such ideal class corresponding to eevery $n \in \mathbb{N}$, we obatin the definition (3.1.1) of ζ_K.

Next, we shall prove that, definition (3.3.2) implies definition (3.3.1) :

For every prime ideal, $P \subset O_K$, we have previously mentioned that,

$$\mathcal{N}(\mathcal{P}) > 1 \implies \frac{1}{\mathcal{N}(\mathcal{P})} < 1.$$

Expanding each term, $\left(1 - \frac{1}{\mathcal{N}(\mathcal{P})}\right)^{-1}$ as an infinite series in terms of every such $\mathcal{N}(\mathcal{P})$,

$$\zeta_K(s) = \prod_{\mathcal{P}} \left(1 + \frac{1}{\mathcal{N}(\mathcal{P})} + \frac{1}{\mathcal{N}(\mathcal{P})^2} + \frac{1}{\mathcal{N}(\mathcal{P})^3} + ...\right)$$

Since, the norm map is multiplicative and every non-zero ideal is either a prime, or it can be expressed as a product of finite powers of prime ideals, therefore, expanding the product over all the prime ideals P of $\mathcal{O}_K$, we obtain the definition (3.3.1) of ζ_K.

4. Density of Primes over Number Field Extensions

Now, as we have already defined *Dedekind Zeta Function* over a number field $K/\mathbb{Q}$, we proceed towards defining and observing different results regarding density of primes which split completely over a number field extension.

4.1 Important Definitions

Definition 4.1.1. (Partial Dedekind Zeta Functions) Let $K/\mathbb{Q}$ be a number field and, we denote,

$$\mathcal{A} := \text{Any set of primes of } \mathcal{O}_K.$$

Then, the *Dedekind Partial Zeta Functions* over K is defined as:

$$\zeta_{K,\mathcal{A}}(s) = \sum_{\mathcal{I}\in[\mathcal{A}]} \frac{1}{\mathcal{N}(\mathcal{I})^s} = \prod_{\mathcal{P}\in\mathcal{A}} \left(1 - \frac{1}{\mathcal{N}(\mathcal{P})^s}\right)^{-1}$$

where $[\mathcal{A}]$ denotes the semigroup of ideals generated by $\mathcal{A}$; in other words, $\mathcal{I} \in \mathcal{A}$ iff, $\forall$ prime ideals, $\mathcal{P}|\mathcal{I} = \Rightarrow \mathcal{P} \in \mathcal{A}$.

Definition 4.1.2. (Polar Density) We have, by definition of *Partial Dedekind Zeta Functions,*

$$\zeta_{K,\mathcal{A}}(s) = \prod_{\mathcal{P}\in\mathcal{A}} \left(1 - \frac{1}{\mathcal{N}(\mathcal{P})^s}\right)^{-1}, \qquad \text{for } Re(s) > 1.$$

If, for some integral power $n \geq 1$, $\zeta^n_{K,\mathcal{A}}$ can be extended to a meromorphic function in a neighbourhood of $s = 1$, such that the function shall have a pole of order m at $s = 1$, then we define the *Polar Density* of $\mathcal{A}$ to be, $\mathcal{D} = \frac{m}{n}$.

Remark 4.1.1. For example, it can be observed that, a finite set has polar density $\mathcal{D} = 0$, and a set containing all but a finitely many elements has polar density $\mathcal{D} = 1$.

Remark 4.1.2. If two sets of primes , $\mathcal{A}$ and $\mathcal{B}$ of the number field K differ only by the primes $\mathcal{P}$ such that, for each such $\mathcal{P}$, $\mathcal{N}(\mathcal{P})$ is not a prime, then the polar density of the set $\mathcal{A}$ exists $\Leftrightarrow$ the polar density of $\mathcal{B}$ exists.

Furthermore, if $\mathcal{D}_{\mathcal{A}}$ and, $\mathcal{D}_{\mathcal{B}}$ be the *polar densities* of the sets $\mathcal{A}$ and $\mathcal{B}$ respectively, then we shall have,

$$\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{B}}.$$

(Since the corresponding zeta functions differ by a factor which is analytic and non-zero in the neighbourhood of $s = 1$).

Having all the neccessary tools we need, we can now study about the corresponding densities of sets of primes that split completely over a finite field extension.

4.2 The Main Theorem

Theorem 4.2.1. *Suppose we consider L and K to be two number fields over $\mathbb{Q}$ such that, the extension, L/K is normal. Then, the set of primes in K which splits completely in L has polar density, $\mathcal{D} = \frac{1}{n}$, where, $n = [L : K]$.*

Proof. Define,

$$\mathcal{A} = \{\mathfrak{p} \subset \mathcal{O}_K \quad | \ \mathfrak{p} \text{ is prime, \& } \mathfrak{p} \text{ splits completely in } L\},$$
$$\mathcal{B} = \{\wp \subset \mathcal{O}_L \quad | \ \mathfrak{p} \text{ is prime, \& } \wp|\mathfrak{p} \text{ for some } \mathfrak{p} \in \mathcal{A}\}.$$

Clearly, we can say that, the set $\mathcal{A}$ is infinite (Follows from the fact that, there exists infinitely many primes in a number field K which splits completely over the extension L). Hence, the density of $\mathcal{A}$ is non-zero.
Let, $\mathfrak{p} \in \mathcal{A}$ such that, $\exists$ primes, $\wp_1, \wp_2, ..., \wp_g \in \mathcal{B}$ (say) and,

$$\mathfrak{p} = \wp_1\wp_2........\wp_g$$

Since, $\mathfrak{p} \in \mathcal{A} \implies \mathfrak{p}$ splits completely in L, hence, we have, for each, $1 \leq i \leq g$,
The inertial degree,

$$f_i = [\mathcal{O}_L/\wp_i : \mathcal{O}_K/\mathfrak{p}] = 1, \qquad \forall\ 1 \leq i \leq g.$$
$$\implies |\mathcal{O}_L/\wp_i| = |\mathcal{O}_K/\mathfrak{p}|, \text{ for every } 1 \leq i \leq g.$$
$$\implies \mathcal{N}(\wp_i) = \mathcal{N}(\mathfrak{p}), \text{ for every } 1 \leq i \leq g.$$

Again, by a well-known result, we know that, if e_i be the *ramification degree* and f_i be the corresponding *inertial degree* of each $\wp_i$ for every $1 \leq i \leq g$, then,

$$\sum_{i=1}^{g} e_i f_i = n = [L : K].$$

But, Since $\mathfrak{p}$ splits completely in L, hence, $e_i = f_i = 1$ for every $1 \leq i \leq g$.

$$\implies g = n.$$

Similarly, we can obtain similar sort of prime decomposition in L.
Therefore, we obtain,

$$\zeta^n_{K,\mathcal{A}} = \zeta_{L,\mathcal{B}}.$$

Thus, it only suffices to prove that, $\zeta_{L,\mathcal{B}}$ has a pole of order 1 at $s = 1$.

Here, we clearly observe that, $\wp \in \mathcal{B} \implies \mathcal{N}(\wp)$ is prime, and this statement holds for all primes $\wp$ except finitely many, which are ramified over K.

Therefore,

$$\zeta_L = \zeta_{L,\mathcal{B}}.H(s)$$

where, $H(s)$ is a finite factor which is an analytic function and non-zero in a neighbourhood of $s = 1$. Therefore, our claim is established and, hence, $\zeta^n_{K,\mathcal{A}}$ can be extended to a meromorphic function in a neighbourhood of $s = 1$, having pole of order 1 at $s = 1$, hence the polar density of $\mathcal{A}$ is ,

$$\mathcal{D}_{\mathcal{A}} = \mathcal{D} = \frac{1}{n} = \frac{1}{[L:K]}.$$

And we have proved our desired result. □

4.3 Some Special Cases

As a corollary to the above theorem, we state this result with the intention of obtaining the density of primes in a number field K that split completely in an extension field, say L, which is not normal over K.

Corollary 4.3.1. *Let L/K be any finite extension of number fields with $[L : K] = n$ (say) and, we define,*

$$M = \textit{Normal Closure of } L/K \\ = \overline{L_N} \textit{ (say)}$$

Then, the set of primes in K that splits completely over M has polar density,

$$\mathcal{D} = \frac{1}{[M:K]} \ .$$

Proof. Clearly, here, we can observe that, M/K is a normal extension. Let us define the set , $\mathcal{A} = \{\mathfrak{p} \subset \mathcal{O}_K \ \mid \mathfrak{p}$ is prime, & $\mathfrak{p}$ splits completely in $M\}$.
Then, applying the result proved in Theorem (4.2.1), we can deduce that, the polar density of $\mathcal{A}$ is , $\mathcal{D}_{\mathcal{A}} = \frac{1}{[M:K]}$.

Thus, our only objective is to prove that, the primes in K which split completely over M will also split completely over L.

$\Leftrightarrow$ A prime will split completely in L iff, it splits completely in M.

Here, we prove an important result:

Lemma 4.3.2. *Let, L/K be a finite extension ($[L:K]=n(say)$) of number fields over $\mathbb{Q}$, and $\mathfrak{p}$ be a prime in K. If $\mathfrak{p}$ is unramified or splits completely in L, then the same holds in the normal closure M of L/K.*

Proof. We know that, the normal closure is the composite field of subfields σL, where, $\sigma \in Gal(M/L)$, i.e.,

$$M = \prod_{\sigma \in Gal(M/L)} \sigma L$$

Let, $\mathfrak{p} \subset \mathcal{O}_K$ be a prime in K which splits completely over L.

Then, $\exists$ primes, $\mathfrak{P}_1, \mathfrak{P}_2, ..., \mathfrak{P}_g$ in $\mathcal{O}_L$ such that,

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1\mathfrak{P}_2........\mathfrak{P}_g$$

Under any automorphism map $\sigma \in Gal(M/L)$, we get, from above,

$$\mathfrak{p}\mathcal{O}_{\sigma L} = (\sigma\mathfrak{P}_1)(\sigma\mathfrak{P}_2)........(\sigma\mathfrak{P}_g)$$

(Since $\mathfrak{p}$ is invariant under σ.)

Since, $\mathfrak{p}$ splits completely over L, therefore, we obtain that, $\mathfrak{p}$ splits completely in M.

Conversely, applying similar process, we can conclude that, if a prime splits completely in M/K, then it splits completely in L/K also.

Hence our Lemma (4.3.2) is proved. □

Applying Lemma (4.3.2), we have proved the Corollary (4.3.1) . □

Corollary 4.3.3. *$K/\mathbb{Q}$ is a number field, and, f is a monic and irreducible polynomial over $\mathcal{O}_K$. Define,*

$$\mathcal{A} = \{\mathfrak{p} \in \mathcal{O}_K \mid f \textit{ splits into linear factors over } \mathcal{O}_K/\mathfrak{p}\}$$

If L/K is the splitting field of f, then, the set $\mathcal{A}$ has polar density,

$$\mathcal{D}_{\mathcal{A}} = \frac{1}{[L:K]}$$

Proof. Fixing one of the roots of the polynomial f, and denoting it by α (say), we intend to observe the primes $\mathfrak{p} \in K$ which splits in the extension $K[\alpha]/K$.

Applying a well-known theorem, we can say that, for infinitely many primes $\mathfrak{p} \in K$, $\mathfrak{p}$ splits completely in $K[\alpha] \Longleftrightarrow f$ splits into linear factors over $\mathcal{O}_K/\mathfrak{p}$.

Since, $L/K[\alpha]/K$ is the normal closure of $K[\alpha]$, hence, applying corollary (4.3.1), we obtain, by definition of the set $\mathcal{A}$,

$$\mathcal{D}_{\mathcal{A}} = \frac{1}{[L:K]}$$

And the statement of the corollary (4.3.3) is established. □

Corollary 4.3.4. *Let, $H < \mathbb{Z}_m^\times$, then, the set,*

$$\mathcal{A} = \{\mathfrak{p} \in \mathbb{Z} \mid \mathfrak{p} \textit{ is a prime, \&} \quad \mathfrak{p}(mod\ m) = \overline{\mathfrak{p}} \in H\}$$

has polar density,

$$\mathcal{D}_{\mathcal{A}} = \frac{|H|}{\varphi(m)}.$$

Note: *$\overline{\mathfrak{p}}$ denotes the congruence class of $\mathfrak{p}(mod\ m)$.*

Proof. Consider the cyclotomic extension, $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, where, ζ_m is the primitive m^{th} root of unity, which is a normal extension, as well as separable. Thus it has Galois Group, $Gal(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = \mathbb{Z}_m^\times$. Suppose , we define L to be the fixed field of H, then we have, for any prime, $\mathfrak{p} \nmid m$, $\mathfrak{p}$ splits completely in $L \Leftrightarrow \overline{\mathfrak{p}} \in H$. Hence, the result follows. □

Corollary 4.3.5. *A normal extension of a number field K can be uniquely characterized by the set, say $\mathcal{A}$ of primes $\mathfrak{p} \in K$, which split completely in it.*
An equivalent statement of the above corollary is, $\exists$ a one-one inclusion-reversing correspondence between the normal extension L/K and the set of primes defined above as $\mathcal{A}$.

Proof. Suppose, L/K and L'/K be two normal extensions, which corresponds to the same set $\mathcal{A}$ of primes in K that split completely in respectable extension fields. Therefore, we can say that,

$$\begin{aligned} \mathcal{A} &= \{\mathfrak{p} \in K \mid \mathfrak{p} \text{ is a prime, \&} \quad \mathfrak{p} \text{ splits completely in } L\} \\ &= \{\mathcal{P} \in K \mid \mathcal{P} \text{ is a prime, \&} \quad \mathcal{P} \text{ splits completely in } L'\}. \end{aligned}$$

Suppose that, $M = LL'$. Then, by theorem, we can conclude that,

$$\mathcal{A} = \{\wp \in K \mid \wp \text{ is a prime, \&} \quad \wp \text{ splits completely in } M\}.$$

Consequently, having the same polar density, we have, by theorem (4.2.1),

$$\begin{aligned} [M:K] &= [L:K] = [L':K] \\ &\Rightarrow L = L'. \end{aligned}$$

And hence, the corollary (4.3.5) holds true. □

Remark 4.3.6. In this paper, we have only observed about how a set of primes over a number field looks like in any finite field extension, and also applied the analytic concepts of *Dedekind Zeta Functions* over number fields to obtain an expression for the density of such sets. This article also reflects densities of such sets of primes for a special case of the cyclotomic extensions over $\mathbb{Q}$.

Remark 4.3.7. It is also worth mentioning that, there are various applications of *Dedekind Zeta Functions*, one of which is to obtain the well-known ***Class Number Formula***.

Remark 4.3.8. Another important property of these functions, which can often be said to be the relation between *Dedekind Zeta Functions* and *Dirichlet's L-functions* is that, Dedekind Zeta Functions can be factored into L-functions having a simpler functional equation.This is also an important fact linked to one of the most famous conjectures [2, p. 18] in number theory:

Artin's Conjecture on Analytic continuation of L-Series.

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