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Review Article

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Upper and Lower Bounds of the State Variable of $M/G/1$ PSFFA Model of the Non-Stationary $M/E_k/1$ Queueing System

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Abstract
The current work reports the upper bound of the state variable of the of the $M/G/1$ Pointwise Steady Fluid Flow Approximation (PSFFA) model of non-stationary $M/E_k/1$ queueing system. Concluding remarks combined with future work are provided.

Keywords: $M/G/1$ PSFFA Model, $M/E_k/1$ Queue, Threshold, Upper and Lower Bounds, k Set of Phases.

1. Introduction

In the given context, a mathematical model called PSFFA is normally used to simulatively, to solve the state variable's a non-linear differential equation in a queue. The PSFFA method utilizes steady state queueing relationships to determine the structure of the fluid flow differential equation. This approach is accurate and offers advantages such as versatility, simplicity in simulating queueing systems, and computational efficiency. Additionally, these techniques have the potential to serve as a fundamental mathematical framework for developing the dynamic network's controlability [1-6].

Let $\mu(t)$ reads as the time-dependent mean service rate and $\lambda(t)$ denotes the time-dependent mean arrival rate.

As the system's ensemble average number at time t, we define $x(t)$ as the state variable, with temporal change denoted by, $x^{\cdot}(t) = \frac{dx(t)}{dt}$. Let $f_{in}(t)$ and $f_{out}(t)$ stand for, respectively, the time-dependent the system's flow in and out. $f_{in}(t)$, $f_{out}(t)$ and $x(t)$ are equationally related as:

$$x^{\cdot}(t) = -f_{out}(t) + f_{in}(t) \quad (1.1)$$

$f_{out}(t)$ relates to server utilization, $\rho(t)$ by

$$f_{out}(t) = \mu\rho(t) \quad (1.2)$$

For an infinite queue waiting space, we have

$$f_{in}(t) = \lambda(t) \quad (1.3)$$

Thus, (1.1) rewrites to:

$$x^{\cdot}(t) = -\mu\rho(t) + \lambda(t),\ 1 > \rho(t) = \frac{\lambda(t)}{\mu(t)} > 0 \quad (1.4)$$

In (1.4), approaching a steady state zone, implies $x.(t) = 0$, that is

$$x = G_1(\rho) \quad (1.5)$$

Additonally, supposing the numerical invertibility of $G_1(\rho)$,

$$\rho = G_1^{-1}(x) \quad (1.6)$$

Consequently,

$$x^{\cdot}(t) = -\mu(G_1^{-1}(x(t))) + \lambda(t) \quad (1.7)$$

The associated G_1 functional for $M/E_k/1$ queue (c.f., [1]) reads:

$$G_1(x) = \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) \quad (1.8)$$

Therefore, the PSFFA model of the time varying $M/E_k/1$ system with k set of phases is determined by:

$$x^{\cdot}(t) = -\mu\left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) + \lambda(t) \quad (1.9)$$

with $\lambda(t) = A + Bsin(wt + C)$

The $M/E_k/1$ queue based on a model where patients arrive at a service facility following a Poisson process and are served in phases with 'k' services available at a rate of $k\alpha$ each. The system operates on a first-come-first-served basis, with a server that can handle a specific range of patients at a time, following an exponential service time distribution with rate μ.

Additionally, an ambulance in the system has a random idle policy based on the number of patients in the queue, and limited seats are available for transporting patients to the hospital in case of accidents, as in Fig.1(c.f., [7]).

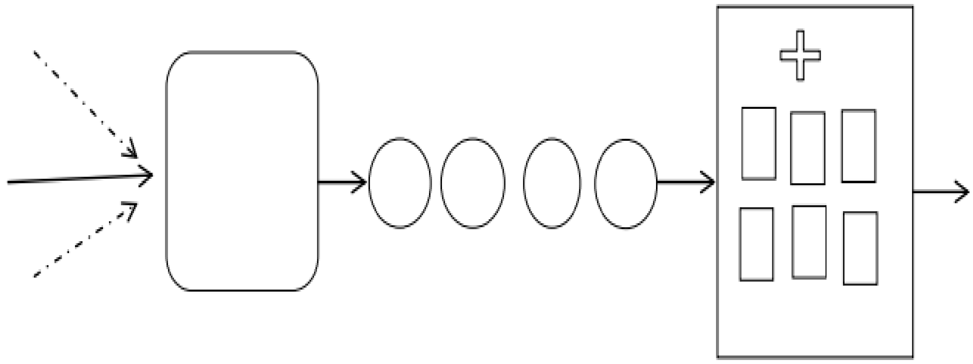

Fig.1

2. Preliminary Theorem (PT) [3]

Let f be a function that is defined and differentiable on an open interval (c,d).
If $f'(x) > 0$ (< 0) for all $x \in (c, d)$,then f is increasing (decreasing)on (c, d) (1.10)

3. Upper and lower bounds of $x(t)$ (c.f., (1.9))

Theorem 1 $x(t)$ of the non-stationary $M/E_k/1$ queueing system satisfies the following inequality:

$$\left(1+\frac{1}{k}\right)\frac{\rho(t)}{2} < x(t) < \frac{k\rho(t)}{1-k\rho(t)},\ \rho(t) \in (0,1) \tag{2.1}$$

Proof
We have

$$x^{\cdot}(t) = -\mu\left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) + \lambda(t) \qquad \text{(c.f., (1.9))}$$

By PT, $x(t)$ is increasing $\Leftrightarrow x.(t) > 0$. Consequently

$$-\mu\left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) + \lambda(t) > 0 \tag{2.2}$$

Hence, it follows that:

$$\rho(t) = \frac{\lambda(t)}{\mu} > \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) = \frac{2(k-1)x}{(k-1)(k(x+1)+\sqrt{(k^2+2kx+k^2x^2)})} = \frac{2x}{(k(x+1)+\sqrt{(k^2+2kx+k^2x^2)})} > \frac{x}{k(x+1)} \tag{2.3}$$

Therefore,

$$k\rho(t)[1+x(t)] > x(t) \tag{2.4}$$

$$k\rho(t) > (1-k\rho(t))\, x(t) \tag{2.5}$$

(2.5) implies:

$$x(t) < \frac{k\rho(t)}{1-k\rho(t)} \qquad \text{(c.f., (2.1))}$$

By the Preliminary Theorem (PT), $x(t)$ is decreasing if and only if $x.(t) > 0$. Consequently

$$-\mu\left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) + \lambda(t) < 0 \tag{2.6}$$

Hence, it follows that:

$$\rho(t) = \frac{\lambda(t)}{\mu} < \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) = \frac{2(k-1)x}{(k-1)(k(x+1)+\sqrt{(k^2+2kx+k^2x^2)}} < \frac{2kx}{k+1} \tag{2.7}$$

Therefore,

$$\left(1+\frac{1}{k}\right)\frac{\rho(t)}{2} < x(t) \qquad \text{(c.f., (2.1))}$$

Hence, the proof follows.
It is noted by (2.1), that at the instability phase ($\rho(t) \to 1$)

$$lim_{\rho(t)\to 1} x(t) < \frac{k}{1-k} \tag{2.8}$$

(2.8) interprets as the limit as ($\rho(t) \to 1$) has an upper bound which is k dependent. As ($k \to 1$),

$$lim_{\rho(t),k\to 1} x(t) < \infty \tag{2.9}$$

Moreover, assuming $k \to \infty$, (2.1) reduces to

$$\frac{\rho(t)}{2} < x(t) < \frac{\rho(t)}{1-\rho(t)},\ \rho(t) \in (0,1) \tag{2.10}$$

Clearly, (2.10) shows that the derived bounds inequality of the underlying $x(t)$ model generalizes of the non-stationary $M/D/1$ queue.

4. Conclusion and Future Work

For the first time ever, the current study reports the upper bound of the state variable of the *M/G*/1 PSFFA model of the non-stationary $M/E_k/1$ queueing system. Furthermore, it is discovered that the state variable of the derived bounds inequality of the state variable of the *M/G*/1 PSFFA model of the non-stationary *M/D*/1 queueing system is a special case of the state variable of the *M/G*/1 PSFFA model of the non-stationary $M/E_\infty/1$ queueing system. Future research will define the state variable bounds for other non-stationary *M/G*/1 PSFFA models.

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