ISSN: 2834-7706

Research Article

Journal of Mathematical Techniques and Computational Mathematics

# Multi-Attribute Group Decision-Making Problem Based On Q-Rung Orthopair Fuzzy Set Under Confidence Environment

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**Submitted:** 2023, Oct 17; **Accepted:** 2023, Dec 20; **Published:** 2024, Feb 20

**Citation:** Muhammad, J., Rahman, K., Hussain, I. (2024). Multi-Attribute Group Decision-Making Problem Based On Q-Rung Orthopair Fuzzy Set Under Confidence Environment. *J Math Techniques Comput Math, 3*(2), 01-23.

**Abstract**

*Emergency decision-making for communities and countries is an important and critical tool. It improves the effectiveness and reliability to response emergencies which minimizes the rate of casualties, environmental damages and economic losses. In the event of emergency decision-making, the main issue is extreme imprecision, ambiguity, and fuzziness. This paper is devoted to the study of q-rung orthopair fuzzy aggregation operators under the confidence levels and their applications to multiple-attribute group decision making (MAGDM) problems. The concept of q-rung orthopair fuzzy set (q-ROFS) is used as tool to describe undetermined information and is superior to the intuitionistic set (IFS) and Pythagorean fuzzy set (PFS). The distinguishing feature of the q-ROFS is that the sum of the q-th power of the membership degree and the q-th power of non-membership degree is bounded by 1. As a result, the range of uncertain information that it may describe is expanded. In this work, we focus on MAGDM problems under the fuzzy environment. First, based on aggregation operators some drawbacks of the already existing MAGDM methods are analyzed.*

*Moreover, we present some modified operational laws and some of their properties to overcome these drawbacks. Next, related to q-ROFS fuzzy-weighted averaging (q-ICROFWA) and fuzzy-weighted power averaging (q-ICROFWPA) aggregation operator under confidence levels along with their properties are presented. By using these operators' q-ICROFWA and q-ICROFWPA an advanced method is proposed to deal with MAGDM problems in fuzzy environment. At last, the validity and feasibility of this method is illustrated with some numerical examples.*

**Keywords:** q-ICROFWA, q-ICROFWPA, q-ROFS, MAGDM Problem

## 1. Introduction

Decision-making plays an important role in our daily life. Most of the time, we face emergencies and need to make quick decisions based on available information. Depending on the nature of the problem and situation, each decision requires a different theory and context. To handle these situation Zadeh, started this journey by introducing the concept of fuzzy set (FS), having just one element known as membership function describing only the satisfaction degree of an object while no information about the dissatisfaction. With the passage of time this theory failed to deal with data acquiring the information of interest of dissatisfaction and satisfaction. Due to this drawback, Atanassov, introduced the notion of IFS, by giving each set element in the form of an ordered pair , where and stands for the degree of membership and non-membership respectively [1,2].

After some time, this field was applied in many real life problems when the IFS cannot be used to situations with data in the form of intervals. So, that mathematician needs some more powerful methods to overcome the mentioned drawbacks. Later, Atanassov and Gargov generalized the idea of intuitionistic fuzzy set and presented the idea of interval-valued intuitionistic fuzzy set, where the grade of membership , and non-membership , are intervals rather than real numbers with condition . As per the requirement for problem solving every method need some operators which play the role of back bone, as seen in the above mentioned literature, therefore, Aggregation operators are very important gears for aggregating the given information in different fields about different alternatives [3]. The importance of operators related to intuitionistic fuzzy environment and interval valued IFS is studied in by many scholars and developed several operators with these different areas and applied them on decision-making problems. Keeping in view the importance of these operators, Wei presented the notion of Hamacher and picture fuzzy aggregation operators and also developed multi-attribute group decision making problem based on the proposed operators [4-9]. Wang et al., presented the idea of new operators named as Muirhead mean operators using picture fuzzy numbers [10,11]. Rahman developed the notion of several averaging and geometric logarithmic aggregation operators based on intuitionistic fuzzy numbers. Tian et al. used Shapley fuzzy measure and presented the notion of power operator and weighted geometric operator. Zhang et al., He et al., Li et al., Yang et al., Meng et al., Ding et al. developed many aggregation operators based on intuitionistic fuzzy environments [12-18].

With increasing the sophistication of human knowledge modeling related to decision making problems and the development of theories, needed more improvement to deal with the highest powers of IFS, Yager introduced a new concept of q-rung orthopair fuzzy set (q-ROFS). In q-ROFS q-th power of membership with the sum of q-th power of non-membership is bounded by one. Moreover, it is proved that IFS and PFS are the special cases of q-ROFS [19]. It shows that q-ROFS is the generalization of the above mentioned methods which covers a wide range of its information and can handle more suitable uncertain environment. Furthermore, Yager in, studied some basic properties of q-ROFS and used it in further studies related to this class. Next, Liu et al., studied q-rung method for operators like orthopair fuzzy weighted averaging (q-ROFWA) and orthopair fuzzy weighted geometric (q-ROFWG) and based on these solved more decision problems. Currently, Peng et al., introduced the idea of q-ROFS information measure such as: entropy, similarity measure, inclusion measure and distance measure [20-22].

In addition to these amazing achievements in the study of q-ROFS, all the information related to multi-decision making (MCDM) has not been incorporated into the information fusion step in existing efforts. In the above mentioned methods of MCDM the decision was completely dependent on the basis of criteria only, where no importance were given to the familiarity (known as confidence level) of the experts involved in the process of decision making. Therefore, the familiarity of the observer must be factored into q-ROFS environment to achieve more reliable results in these situations. We focused in this study, to overcome these type of disadvantages by combining the expert's confidence level with q-ROFS evaluated alternatives. The following confidence q-ROFS aggregation operators were used to combine these two sources of information: confidence q-rung orthopair fuzzy weighted geometric (q-CFWG), confidence q-rung orthopair fuzzy weighted average (q-COFWA), confidence q-rung orthopair fuzzy ordered weighted geometric (q-CFOWG), confidence q- rung orthopair fuzzy ordered weighted average (q-COFWOA), are the operators proposed by many scholars and most of their important properties are established. With the use of their parameterizations property (confidence levels), these specified operators may more clearly describe the real-world scenario in a q-rung orthopair fuzzy environment.

In the aforementioned literature, we came to know that in the different times different operators are introduced to overcome the drawbacks of the already existing methods, but still there are some drawbacks. This article is an attempt in that series to overcome the drawback of IFN where the NMED of IFN is zero, this will give its aggregated value zero without regarding the matter that how much is value of others NMED [23-25]. Similarly, for the value 1 of MED, will give its aggregated value 1 without regarding the matter that how much is value of others MED. Keeping in mind these drawbacks, we explained it with examples and some improved q-ROFS operational laws such as q-CROFWOA, q-CRFOWG and q-ICROFWA are introduced.

This article contributed as: (i) the drawbacks of some already existing aggregation operators are studied and analyzed (ii) some new operators such as q-ICROFWA and q-ICROFWPA are introduced (iii) a new method is modified for MAGDM problems on the base of q-ICROFWA and q-ICROFWPA operators (iv) the practicability and feasibility of this modified method is verified with some numerical examples.

## 2. Preliminaries

This section is devoted to the review of some basic concepts, definitions and operators that will be needed later on.

**• Definition 1**

Let a non-empty set X, FS is defined by $T=\{\langle x,\eta_\zeta(x)\rangle : x\in X\}$ with $\eta_\zeta(x)$ is mapped from X to be called the membership degree (MED) of *x*.

Moreover, the notion of FS is generalized to AIFS, consists of membership and non-membership (NOMED) functions, defined as follows [1].

**• Definition 2**

Let X be a fixed set; an AIFS *T* in X is given by $T=\{\langle x,\eta_\zeta(x),\vartheta_\zeta(x)\rangle : x\in X\}$ with $\eta_\zeta(x),\vartheta_\zeta(x)$ are mapped from X to [0, 1] satisfying the conditions $0\le \eta_\zeta(x),\vartheta_\zeta(x)\le 1,$ $0\le \eta_\zeta(x)+\vartheta_\zeta(x)\le 1,\ \forall x\in X,$ and $\eta_\zeta(x),\vartheta_\zeta(x)$ represents MED and NOMED respectively [2].

### 2.1. Some Operational Laws of q-ROF Number

**• Definition 3**

[26] Let X be a fixed set and the q-ROFS G are defined as $G=\{\langle H_G(x),M_G(x)\rangle : x\in X\}$ such that $H_G(x),M_G(x)$ are respectively denote MED and NOMED, satisfying the basic conditions $0\le H_G(x),M_G(x)\le 1,$ and $0\le H_G(x)^q+M_G(x)^q\le 1,$ where $q\ge 1.$ Furthermore, the hesitancy or indeterminacy of the given set is given by: $\pi_G(x)=(H_G(x)^q+M_G(x)^q-H_G(x)^qM_G(x)^q)^{\frac{1}{q}}.$

**• Definition 4**

Let two q-ROFNs be such that $b_1=(\eta_1,\vartheta_1)$ and $b_2=(\eta_1,\vartheta_1),$ then

i) $$b_1\oplus b_2=\left[\left(\eta_1^q+\eta_2^q-\eta_1^q\eta_2^q\right)^{\frac{1}{q}},\vartheta_1\vartheta_2\right]$$

ii) $$b_1\otimes b_2=\left[\vartheta_1\vartheta_2,\left(\eta_1^q+\eta_2^q-\eta_1^q\eta_2^q\right)^{\frac{1}{q}}\right]$$

iii) $$\alpha b_1=\left[\left(1-\left(1-\eta_1^q\right)^{\alpha}\right)^{\frac{1}{q}},\vartheta_1^{\alpha}\right]$$

iv) $$b_1^{\alpha}=\left[\eta_1^{\alpha},\left(1-\left(1-\vartheta_1^q\right)^{\alpha}\right)^{\frac{1}{q}}\right]$$

where $\alpha > 0$, is any real number. Next, let $b = (\eta, \vartheta)$ be a q-ROFN, then the score function is given by $S(b) = \eta^q - \vartheta^q$ and its accuracy function is defined by $H(b) = \eta^q + \vartheta^q$ under the basic condition that $S(b) \in [-1,1]$ and $H(b) \in [0,1]$ [26].

- **Definition 5**

Let two q-ROFNs be $b_1 = (\eta_1, \vartheta_1)$ and $b_2 = (\eta_2, \vartheta_2)$, then its score and accuracy functions will satisfy the following condition [27]s.

i) Let $S(b_1)$ and $S(b_2)$ be the score functions such that $S(b_1) > S(b_2)$ then $b_1 > b_2$.

**ii)** Let $S(b_1) = S(b_2)$ such that the accuracy functions satisfy the condition $H(b_1) > H(b_2)$ then $b_1 > b_2$. Moreover, if $H(b_1) = H(b_2)$, then $b_1 = b_2$.

- **Definition 6**

[27] Power average operator for $X = (x_1, x_2, x_3, ...., x_n)$ can be defined as follows

$$PA(x_1, x_2, x_3, ...., x_n) = \frac{\sum_{j=1}^{n}(1+E(x_j))x_j}{\sum_{j=1}^{n}(1+E(x_j))},$$

where $E(x_j) = \sum_{i=1, i \neq j}^{n} \sup(x_i, x_j)$. If, $\sup(x_i, x_j)$ denotes the degree of support $x_i$ from $x_j$, then

i) $0 \leq \sup(x_i, x_j) \leq 1,$

ii) $\sup(x_i, x_j)$ is commutative

iii) $\sup(x, y) \leq \sup(a, b)$, when $|x-y| > |a-b|$.

## 3. q-CROFN Operators

In this section, we introduce some operators under q-rung orthopair fuzzy numbers.

**• Definition 7**

Let a collection of q-CROFN be $b_j = (\eta_j, \vartheta_j)(1 \leq j \leq n)$, with a weighted vector $\omega = (\omega_1, \omega_2, \omega_3, ..., \omega_j)^T$ and let $\tau_j$ be the confidence level of $b_j$ such that $0 \leq \tau_j \leq 1$, $0 \leq \omega_j \leq 1$, and $\sum_{j=1}^{n} \omega_j = 1$.

$$q\text{-}CROFWA(b_1,b_2,\ldots,b_k)=\left\langle\left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q},\prod_{j=1}^{k}\vartheta_j^{\tau_j\omega_j}\right\rangle \quad (1)$$

$$q\text{-}CROFWG(b_1,b_2,\ldots,b_j)=\left\langle\prod_{j=1}^{n}\eta_j^{\tau_j\omega_j},\left(1-\prod_{j=1}^{n}\left(1-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right\rangle \quad (2)$$

- **Example 1**

Let $b_1=\langle(0.9,0.1),0.3\rangle, b_2=\langle(0.7,0.2),0.2\rangle$, $b_3=\langle(0.8,0.4),0.3\rangle$ and $b_4=\langle(0.7,0.5),0.6\rangle$ be the four q-ROFNs with their respective confidence level. Moreover, the weighted vector is $\omega=(0.4,0.2,0.1,0.3)$ and let $q=2$, then by (1), we have

$$q\text{-}CROFWA(b_1,b_2,b_3,b_4)=\left\langle\left(1-\prod_{j=1}^{4}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/2},\prod_{j=1}^{4}\vartheta_j^{\tau_j\omega_j}\right\rangle$$

$$=\left\langle\begin{array}{l}\left(1-\left(1-0.9^2\right)^{0.3\times0.4}\left(1-0.7^2\right)^{0.2\times0.2}\left(1-0.8^2\right)^{0.3\times0.1}\left(1-0.7^2\right)^{0.3\times0.1}\right)^{1/2}\\,0.1^{0.3\times0.4}\times0.2^{0.2\times0.2}\times0.4^{0.3\times0.1}\times0.5^{0.6\times0.3}\end{array}\right\rangle$$

$$=(0.49,0.61)$$

- **Example 2**

Let $b_1=\langle(0.8,0.2),0.3\rangle$, $b_2=\langle(0.6,0.2),0.2\rangle$, $b_3=\langle(0.7,0.2),0.3\rangle$ and $b_4=\langle(0.5,0.3),0.6\rangle$ be the four q-ROFNs with their respective confidence level. Moreover, the weighted vector is $\omega=(0.4,0.2,0.1,0.3)$ and let $q=2$, then by (1), we have

$$q\text{-}CROFWA(b_1,b_2,b_3,b_4)=\left\langle\left(1-\prod_{j=1}^{4}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/2},\prod_{j=1}^{4}\vartheta_j^{\tau_j\omega_j}\right\rangle$$

$$=\left\langle\begin{array}{l}\left(1-\left(1-0.8^2\right)^{0.3\times0.4}\left(1-0.6^2\right)^{0.2\times0.2}\left(1-0.7^2\right)^{0.3\times0.1}\left(1-0.7^2\right)^{0.3\times0.1}\right)^{1/2}\\,0.2^{0.3\times0.4}\times0.2^{0.2\times0.2}\times0^{0.3\times0.1}\times0.5^{0.6\times0.3}\end{array}\right\rangle$$

$$=(0.40,0)$$

By changing the value of $b_1$ and $b_2$ such that $b_1=\langle(0.5,0.2),0.3\rangle$ and $b_2=\langle(0.4,0.2),0.2\rangle$, then by (1), we have

$$q\text{-}CROFWA(b_1,b_2,b_3,b_4)=\left\langle\left(1-\prod_{j=1}^{4}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/2},\prod_{j=1}^{4}\vartheta_j^{\tau_j\omega_j}\right\rangle$$

$$=\left\langle\begin{array}{l}\left(1-\left(1-0.5^2\right)^{0.3\times0.4}\left(1-0.4^2\right)^{0.2\times0.2}\left(1-0.7^2\right)^{0.3\times0.1}\left(1-0.7^2\right)^{0.3\times0.1}\right)^{1/2}\\ ,0.2^{0.3\times0.4}\times0.2^{0.2\times0.2}\times0^{0.3\times0.1}\times0.5^{0.6\times0.3}\end{array}\right\rangle$$

$$=\left(0.2804,0\right).$$

The above proposed operators have some drawbacks such as the NOMED in the equation (1) and MED of equation (2) are their respective entries products which will give a zero by taking only one zero entry in the data as shown in example 1. Similarly, in example 2, it is shown that by changing the values of MED does not affect the calculated value of NOMED. Therefore, it does not provide the actual information to the decision makers. These operators can be improved by using the idea of to overcome the mentioned drawbacks as follows [28].

Let $b=(\eta,\vartheta)$, $b_1=(\eta_1,\vartheta_1)$ and $b_2=(\eta_2,\vartheta_2)$ the q-ROFN, then

i) $$b_1\oplus b_2=(\eta_1,\vartheta_1)\oplus(\eta_2,\vartheta_2)=\left\langle\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)^{\frac{1}{q}},\left(\prod_{j=1}^{2}\left(1-\eta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)\right)^{\frac{1}{q}}\right\rangle$$

ii) $$b_1\otimes b_2=(\eta_1,\vartheta_1)\otimes(\eta_2,\vartheta_2)=\left\langle\left(\prod_{j=1}^{2}\left(1-\vartheta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)\right)^{\frac{1}{q}},\left(1-\prod_{j=1}^{2}\left(1-\vartheta_j^q\right)\right)^{\frac{1}{q}}\right\rangle$$

iii) $$\alpha b=\left\langle\left(1-\left(1-\eta^q\right)^{\alpha}\right)^{\frac{1}{q}},\left(\left(1-\eta^q\right)^{\alpha}-\left(1-\eta^q-\vartheta^q\right)^{\alpha}\right)^{\frac{1}{q}}\right\rangle,$$

iv) $$b_1^{\alpha}=\left\langle\left(\left(1-\vartheta^q\right)^{\alpha}-\left(1-\eta^q-\vartheta^q\right)^{\alpha}\right)^{\frac{1}{q}},\left(1-\left(1-\vartheta^q\right)^{\alpha}\right)^{\frac{1}{q}}\right\rangle.$$

- **Theorem 1**

Let $b_1=(\eta_1,\vartheta_1)$, $b_2=(\eta_2,\vartheta_2)$ and $b_3=(\eta_3,\vartheta_3)$ the q-ROFNs, satisfying the condition that $q\geq1$, $\alpha$, $\alpha_1$, $\alpha_2>0$, then

i) $b_1 \oplus b_2 = b_2 \oplus b_1$

ii) $b_1 \otimes b_2 = b_2 \otimes b_1$

iii) $(b_1 \oplus b)_2 \oplus b_3 = b_1 \oplus (b_2 \oplus b_3)$

iv) $(b_1 \otimes b)_2 \otimes b_3 = b_1 \otimes (b_2 \otimes b_3)$

v) $\alpha(b_1 \oplus b_2) = \alpha b_1 \oplus \alpha b_2$

vi) $(b_1 \otimes b_2)^{\alpha} = b_1^{\alpha} \otimes b_2^{\alpha}$

vii) $(\alpha_1 \oplus \alpha_2) b_1 = \alpha_1 b_1 \oplus \alpha_2 b_1$

viii) $b_1^{(\alpha_1 \oplus \alpha_2)} = b_1^{\alpha_1} \otimes b_1^{\alpha_2}$

Proof: By using definition (i) and (ii) with simple straight forward calculation of Theorem (i) and (ii) can easily be proved. Next, we prove here (iii), as follows:
Taking the left-hand side of (iii), implies that

$$(b_1 \oplus b_2) \oplus b_3 = \left\langle \left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)^{\frac{1}{q}}, \left(\prod_{j=1}^{2}\left(1-\eta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)\right)^{\frac{1}{q}} \right\rangle \oplus (\eta_3, \vartheta_3)$$

$$= \left\langle \begin{array}{l} \left(1-\left(1-\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)\right)\left(1-\eta_3^q\right)\right)^{1/q}, \left(\left(1-\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)\right)\left(1-\eta_3^q\right)-\left(1-\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)\right.\right. \\ \left.\left.-\left(\prod_{j=1}^{2}\left(1-\eta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)\right)\right)\left(1-\eta_3^q-\vartheta_3^q\right)\right)^{1/q} \end{array} \right\rangle$$

$$= \left\langle \left(1-\prod_{j=1}^{3}\left(1-\eta_j^q\right)\right)^{1/q}, \left(\prod_{j=1}^{3}\left(1-\eta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)\right)^{1/q} \right\rangle$$

(3)

where the right-hand side of (iii), yields

$$b_1 \oplus (b_2 \oplus b_3) = (\eta_1, \vartheta_2) \oplus \left(1-\prod_{j=2}^{3}\left(1-\eta_j^q\right)\right)^{1/q}, \left(\prod_{j=2}^{3}\left(1-\eta_j^q\right)-\prod_{j=2}^{3}\left(1-\eta_j^q-\vartheta_j^q\right)\right)^{1/q}$$

$$= \left\langle \begin{array}{l} \left(1-\left(1-\eta_1^q\right)\left(1-\left(1-\prod_{j=2}^{3}\left(1-\eta_j^q\right)\right)\right)\right)^{1/q}, \left(\left(1-\eta_1^q\right)\left(1-\left(1-\prod_{j=2}^{3}\left(1-\eta_j^q\right)\right)\right)\right. \\ \left.-\left(1-\eta_1^q-\vartheta_1^q\right)\left(1-\left(1-\prod_{j=2}^{3}\left(1-\eta_j^q\right)\right)-\left(\prod_{j=2}^{3}\left(1-\eta_j^q\right)-\prod_{j=2}^{3}\left(1-\eta_j^q-\vartheta_j^q\right)\right)\right)\right)^{1/q} \end{array} \right\rangle \quad (4)$$

$$=\left\langle\left(1-\prod_{j=1}^{3}\left(1-\eta_j^q\right)\right)^{\frac{1}{q}},\left(\prod_{j=1}^{3}\left(1-\eta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)\right)^{\frac{1}{q}}\right\rangle.$$

Hence (iii) is proved. Similarly, equation (iv) will be proved on the same lines as proved in (iii). Moreover, to prove (v), let the left-hand side

$$\alpha\left(b_1\oplus b_2\right)=\alpha\left\langle\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)^{1/q},\left(\prod_{j=1}^{2}\left(1-\eta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)\right)^{1/q}\right\rangle$$

$$=\left\langle\begin{array}{c}\left(1-\left(\left(1-\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)^{1/q}\right)^{q}\right)^{\lambda}\right)^{1/q}\left(\left(\left(1-\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)^{1/q}\right)^{q}\right)^{\lambda}\right.\\ \left.-\left(1-\left(\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)\right)^{1/q}\right)^{q}-\left(\left(\prod_{j=1}^{2}\left(1-\eta_j^q\right)-\prod_{j=1}^{2}\left(1-\eta_j^q-v_i^q\right)\right)^{1/q}\right)^{q}\right)^{\lambda}\right)^{1/q}\end{array}\right\rangle$$

$$=\left\langle\left(1-\prod_{j=1}^{2}\left(1-\eta_j^q\right)^{\lambda}\right)^{1/q},\left(\prod_{j=1}^{2}\left(1-\eta_j^q\right)^{\lambda}-\prod_{j=1}^{2}\left(1-\eta_j^q-\vartheta_j^q\right)^{\lambda}\right)^{1/q}\right\rangle. \tag{5}$$

Similarly, for right-hand side, we have

$$\alpha_1b_1\oplus\alpha_2b_2=\left\langle\left(1-\left(1-\eta_1^q\right)^{\alpha_1}\right)^{1/q},\left(\left(1-\eta_1^q\right)^{\alpha_1}-\left(1-\eta_1^q-\vartheta_1^q\right)^{\alpha_1}\right)^{1/q}\right\rangle\oplus$$
$$\left\langle\left(1-\left(1-\eta_2^q\right)^{\alpha_2}\right)^{1/q},\left(\left(1-\eta_2^q\right)^{\alpha_2}-\left(1-\eta_2^q-\vartheta_2^q\right)^{\alpha_2}\right)^{1/q}\right\rangle$$

$$=\left\langle\begin{array}{c}\left(1-\left(1-\left(\left(1-\left(1-\eta_1^q\right)^{\alpha_1}\right)^{1/q}\right)^{q}\right)\left(1-\left(\left(1-\left(1-\eta_2^q\right)^{\alpha_2}\right)^{1/q}\right)^{q}\right)\right)^{1/q}\\ ,\left(\left(1-\left(\left(1-\left(1-\eta_1^q\right)^{\alpha_1}\right)^{1/q}\right)^{q}\right)\left(1-\left(\left(1-\left(1-\eta_2^q\right)^{\alpha_2}\right)^{1/q}\right)^{q}\right)\right.\\ -\left(1-\left(\left(1-\left(1-\eta_1^q\right)^{\alpha_1}\right)^{1/q}\right)^{q}-\left(\left(1-\eta_1^q\right)^{\alpha_1}-\left(1-\eta_1^q-\vartheta_1^q\right)^{\alpha_1}\right)^{1/q}\right)\\ \left.\left(1-\left(\left(1-\left(1-\eta_1^q\right)^{\alpha_2}\right)^{1/q}\right)^{q}-\left(\left(1-\eta_1^q\right)^{\alpha_2}-\left(1-\eta_1^q-\vartheta_1^q\right)^{\alpha_2}\right)^{1/q}\right)\right)^{1/q}\end{array}\right\rangle$$

$$=\left\langle \left(1-\left(1-\eta_1^q\right)^{\alpha_1+\alpha_2}\right)^{1/q}, \left(\left(1-\eta_1^q\right)^{\alpha_1+\alpha_2}-\left(1-\eta_1^q-\vartheta_1^q\right)^{\alpha_1+\alpha_2}\right)^{1/q}\right\rangle \qquad (6)$$

Similarly, (viii) can be proved by using the same calculation as in the proof (vii).

**• Example 3**

Let $b_1 = (0.6,0.2)$ and $b_2 = (0.5,0)$, be the two q-ROFNs, let $q = 2$ and $a = 3$ then, we have

$$b_1 \oplus b_2 = \left\langle \left(1-\left(1-0.6^2\right)\times\left(1-0.5^2\right)\right)^{\frac{1}{2}}, \left(\begin{array}{l}\left(1-0.6^2\right)\times\left(1-0.5^2\right) \\ -\left(1-0.6^2-0.2^2\right)\times\left(1-0.5^2-0^2\right)\end{array}\right)^{\frac{1}{2}}\right\rangle$$

$$=\langle 0.7211, 0.1732\rangle$$

$$b_1 \otimes b_2 = \left\langle \begin{array}{l}\left(\left(1-0.2^2\right)\times\left(1-0^2\right)-\left(1-0.6^2-0.2^2\right)\times\left(1-0.5^2-0\right)\right)^{\frac{1}{2}}, \\ \left(1-\left(1-0.2^2\right)\times\left(1-0^2\right)\right)^{\frac{1}{2}}\end{array}\right\rangle$$

$$=\langle 0.6573, 0.2\rangle$$

$$\alpha b_1 = \left\langle \left(1-\left(1-0.6^2\right)^3\right)^{\frac{1}{2}}, \left(\left(1-0.6^2\right)^3-\left(1-0.6^2-0.2^2\right)^3\right)^{\frac{1}{2}}\right\rangle,$$

$$=\langle 0.216, 0.2148\rangle.$$

## 4. Some Revised q-CROFS Operators

This section is devoted to the study of improved confidence level q-rung orthopair fuzzy weighted averaging operator q-ICROFWA.

- **Theorem 2**

Let a collection of q-CROFN be $b_j = \left(\eta_j, \vartheta_j\right)(1 \le j \le n)$, with a weighted vector $\omega = \left(\omega_1, \omega_2, \omega_3, \ldots, \omega_j\right)^T$ and let $\tau_j$ be the confidence level of $b_j$ such that $0 \le \tau_j \le 1$, $0 \le \omega_j \le 1$, and $\sum_{j=1}^{n}\omega_j = 1$. Then, the operator q-CROFN remains a q-CROFN and has the following mathematical form:

$$q-ICROFWA\left(b_1, b_2, \ldots, b_n\right)$$

$$=\left\langle \left(1-\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}, \left(\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j\omega_j}-\prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right\rangle \qquad (7)$$

**Proof:** The required result can be proved by using mathematical induction:

• **Step 1:** For ***n=1,*** from the left-hand side of (7), we have

$$q\text{-ICROFWA}\left(b_1\right)=\tau_1\omega_1 b_1$$
$$=\left\langle\left(1-\left(1-\eta_1^q\right)^{\tau_1\omega_1}\right)^{1/q},\left(\left(1-\eta_1^q\right)^{\tau_1\omega_1}-\left(1-\eta_1^q-\vartheta_1^q\right)^{\tau_1\omega_1}\right)^{1/q}\right\rangle \quad (8)$$

Similarly, the right-hand side of (7), implies that

$$\left\langle\left(1-\prod_{j=1}^{1}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q},\left(\prod_{j=1}^{1}\left(1-\eta_j^q\right)^{\tau_j\omega_j}-\prod_{j=1}^{1}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right\rangle$$
$$=\left\langle\left(1-\left(1-\eta_1^q\right)^{\tau_1\omega_1}\right)^{1/q},\left(\left(1-\eta_1^q\right)^{\tau_1\omega_1}-\left(1-\eta_1^q-\vartheta_1^q\right)^{\tau_1\omega_1}\right)^{1/q}\right\rangle \quad (9)$$

Hence, for ***n=1,*** (7) is true. Moreover, it can also be proved for ***n=2,*** proceeding on the same lines as above.

• **Step 2:** Suppose that the (7) is true for ***n=k.***

$$q-ICROFWA\left(b_1,b_2,\ldots,b_n\right)$$
$$=\left\langle\left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q},\left(\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}-\prod_{j=1}^{k}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right\rangle \quad (10)$$

Furthermore, for ***n=k+1***, we have

$$q\text{-ICROFWA}\left(b_1,b_2,\ldots,b_k,b_{k+1}\right)=q\text{-ICROFWA}\left(b_1,b_2,\ldots,b_k\right)\oplus\tau_{k+1}\omega_{k+1}b_{m+1}$$
$$=\left\langle\left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q},\left(\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}-\prod_{j=1}^{k}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right\rangle\oplus$$
$$\left\langle\left(1-\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q},\left(\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}-\left(1-\eta_{k+1}^q-\vartheta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q}\right\rangle \quad (11)$$

$$=\left\langle \begin{array}{l} \left(1-\left(1-\left(\left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right)^q\right)\left(1-\left(\left(1-\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q}\right)^q\right)\right)^{1/q}, \\ \left(\left(1-\left(\left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right)^q\right)\right. \\ \left(1-\left(\left(1-\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q}\right)^q\right)-\left(\begin{array}{l}1-\left(\left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right)^q \\ -\left(\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}-\prod_{j=1}^{k}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\end{array}\right) \\ \left.\left(\begin{array}{l}1-\left(\left(1-\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q}\right)^q \\ -\left(\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}-\left(1-\eta_{k+1}^q-\vartheta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q}\end{array}\right)\right) \end{array}\right\rangle^{1/q}$$

$$=\left\langle \begin{array}{l} \left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q} \\ ,\left(\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\left(1-\eta_{k+1}^q\right)^{\tau_{k+1}\omega_{k+1}}-\prod_{j=1}^{k}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\left(1-\eta_{k+1}^q-\vartheta_{j+1}^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q} \end{array}\right\rangle$$

$$=\left\langle\left(1-\prod_{j=1}^{k+1}\left(1-\eta_j^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q},\left(\prod_{j=1}^{k+1}\left(1-\eta_j^q\right)^{\tau_{k+1}\omega_{k+1}}-\prod_{j=1}^{k+1}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_{k+1}\omega_{k+1}}\right)^{1/q}\right\rangle.$$

Hence, it is proved by mathematical induction that the required result is true for any value of k. Next, we have to prove that $q-ICROFWA(b_1,b_2,\ldots,b_n)$ is still a *q-ROFN*. For this, let $q-ICROFWA(b_1,b_2,\ldots,b_n)=\langle c,d\rangle$ then, if $q-ICROFWA$ is a *q-ROFN*, it will satisfy, the following two conditions:

i) $0\le c,d\le 1,$

ii) $0\le c^q,d^q\le 1,$

For i), it is given that $0\le \eta_j,\vartheta_j\le 1,$ which implies that

$$0 \le 1-\eta_j^q \le 1 \Leftrightarrow 0 \le \left(1-\eta_j^q\right)^{\tau_j \omega_j} \le 1 \Leftrightarrow 0 \le \prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j \omega_j} \le 1$$

$$\Leftrightarrow 0 \le \left(1-\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j \omega_j}\right)^{1/q} \le 1$$

This shows that $0 \le c \le 1$. Moreover, as $0 \le c^q, d^q \le 1$, can be written as

$$0 \le 1-\left(\eta_j^q+\vartheta_j^q\right) \le 1 \Rightarrow 0 \le \left(1-\eta_j^q-\vartheta_j^q\right)^{\omega_j \tau_j} \le 1 \Rightarrow 0 \le \prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\omega_j \tau_j} \le 1.$$

Next, we have

$$1-\eta_j^q \ge 1-\eta_j^q-\vartheta_j^q \Rightarrow \prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j \omega_j} \ge \prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j \omega_j}$$

$$\Rightarrow 0 \le \left(\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j \omega_j}-\prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j \omega_j}\right)^{\frac{1}{q}} \le 1,$$

yields that, $0 \le d \le 1$ and condition i) is proved. Furthermore, for the second condition, we have

$$0 \le c^q+d^q=1-\left(\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j \omega_j}\right)+\left(\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j \omega_j}-\prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j \omega_j}\right.$$

$$\Leftrightarrow 0 \le c^q+d^q=1-\prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j \omega_j} \Leftrightarrow 0 \le \prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j \omega_j} \le 1$$

$$\Leftrightarrow 0 \le 1-\prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j \omega_j} \le 1$$

Thus, condition ii), is proved which shows that $q-ICROFWA\left(b_1, b_2, \ldots, b_n\right)$ is still a *q-ROFN.*

The following theorem shows the idempotency of the *q-ICROFWA*.

- **Theorem 3**

Let a collection $b_j=\left\langle\eta_j, \vartheta_j\right\rangle$ $(1 \le \mathrm{j} \le \mathrm{n})$ of *q-CROFNs* such that $b_j=b=\langle\eta, \vartheta\rangle$, then $q-ICROFWA\ \ (b_1, b_2, \ldots, b_j)=b$.

**Proof:** By applying definition (1), we have

$$q-ICROFWA\ (b_1, b_2, ..., b_j)$$

$$=\left\langle \left(1-\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}, \left(\prod_{j=1}^{n}\left(1-\eta_j^q\right)^{\tau_j\omega_j}-\prod_{j=1}^{n}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\right\rangle$$

$$=\left\langle \left(1-\left(1-\eta^q\right)^{\sum_{j=1}^{n}\tau_j\omega_j}\right)^{1/q}, \left(\left(1-\eta^q\right)^{\sum_{j=1}^{n}\tau_j\omega_j}-\left(1-\eta^q-\vartheta^q\right)^{\sum_{j=1}^{n}\tau_j\omega_j}\right)^{1/q}\right\rangle$$

$$=\langle \eta, \vartheta \rangle.$$

The boundedness of q-CROFNs is shown by the following theorem.

- **Theorem 4**

Let $b_j=\langle \eta_j, \vartheta_j \rangle$ be a collection of q-CROFN with $j=1,2,...,n$ such that $b^-=\left\langle \min_j(\eta_j^q), \max_j(\eta_j^q+\vartheta_j^q-\min_j(\eta_j^q)^{\frac{1}{q}}\right\rangle$ and $b^+=\langle \max_j(\eta_j^q), H\rangle$, then it gives that

$b^- \leq q-ICROFWA\ (b_1, b_2, \ldots, b_n) \leq b^+$, where

$$H=\begin{cases} 0, & \text{if } \min_j(\eta_j^q+\vartheta_j^q) \leq \max_j(\eta_j^q), \\ \min_j(\eta_j^q+\vartheta_j^q)-\ \max_j(\eta_j^q)^{\frac{1}{q}}, & \text{if } \min_j(\eta_j^q+\vartheta_j^q) \geq \max_j(\eta_j^q). \end{cases}$$

**Proof:** (1) For the MED of $q-ICROFWA$, with $(b_1, b_2, \ldots, b_n)=b$, we get

$$\left(1-\prod_{j=1}^{k}\left(1-\min_{1\leq j\leq k}\eta_j^q\right)^{\omega_j\tau_j}\right)^{\frac{1}{q}} \leq \left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\omega_j\tau_j}\right)^{\frac{1}{q}} \leq \left(1-\prod_{j=1}^{k}\left(1-\max_{1\leq j\leq k}\eta_j^q\right)^{\omega_j\tau_j}\right)^{\frac{1}{q}}$$

$$\Leftrightarrow \left(1-\left(1-\min_{1\leq j\leq k}\eta_j^q\right)^{\sum_{j=1}^{k}\omega_j\tau_j}\right)^{\frac{1}{q}} \leq \left(1-\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\omega_j\tau_j}\right)^{\frac{1}{q}} \leq \left(1-\left(1-\max_{1\leq j\leq k}\eta_j^q\right)^{\sum_{j=1}^{k}\omega_j\tau_j}\right)^{\frac{1}{q}}$$

$$\Leftrightarrow \min_{1\leq j\leq k}\eta_j^q \leq \left(1-\left(1-\eta_j^q\right)^{\sum_{j=1}^{k}\omega_j\tau_j}\right)^{\frac{1}{q}} \max_{1\leq j\leq k}\eta_j^q$$

Moreover, for NOMED, we get

$$\left(\prod_{j=1}^{k}\left(1-\max_{1\le j\le k}\eta_j^q\right)^{\omega_j\tau_j}-\prod_{j=1}^{k}\left(1-\max_{1\le j\le k}(\eta_j^q+\vartheta_j^q)\right)^{\omega_j\tau_j}\right)^{\frac{1}{q}}$$

$$\le\left(\prod_{j=1}^{k}\left(1-\eta_j^q\right)^{\omega_j\tau_j}-\prod_{j=1}^{k}\left(1-\eta_j^q-\vartheta_j^q\right)^{\omega_j\tau_j}\right)^{\frac{1}{q}}$$

$$\le\left(\prod_{j=1}^{k}\left(1-\min_{1\le j\le k}\eta_j^q\right)^{\omega_j\tau_j}-\prod_{j=1}^{k}\left(1-\max_{1\le j\le k}(\eta_j^q+\vartheta_j^q)\right)^{\omega_j\tau_j}\right)^{\frac{1}{q}}$$

$$\Rightarrow\left(\min_{1\le j\le k}(\eta_j^q+\vartheta_j^q)-\max_{1\le j\le k}\eta_j^q\right)^{\frac{1}{q}}\le\left(\prod_{j=1}^{k}(1-\eta_j^q)^{\omega_j\tau_j}-\prod_{j=1}^{k}(1-\eta_j^q-\vartheta_j^q)^{\omega_j\tau_j}\right)^{\frac{1}{q}}$$

$$\le\left(\max_{1\le j\le k}(\eta_j^q+\vartheta_j^q)-\min_{1\le j\le k}\eta_j^q\right)^{\frac{1}{q}}$$

In view of the fact that $\left(\prod_{j=1}^{k}(1-\eta_j^q)^{\omega_j\tau_j}-\prod_{j=1}^{k}(1-\eta_j^q-\vartheta_j^q)^{\omega_j\tau_j}\right)^{\frac{1}{q}}\ge 0,$

Then

$$H\le\left(\prod_{j=1}^{k}(1-\eta_j^q)^{\omega_j\tau_j}-\prod_{j=1}^{k}(1-\eta_j^q-\vartheta_j^q)^{\omega_j\tau_j}\right)^{\frac{1}{q}}$$

$$\le\left(\max_{1\le j\le k}(\eta_j^q+\vartheta_j^q)-\min_{1\le j\le k}\eta_j^q\right)^{\frac{1}{q}}.$$

- **Example 4**

By using the data of example 2,

$$q-ICROFWA\ (b_1,b_2,b_3,b_4)=\left\langle\begin{array}{l}\left(1-\prod_{j=1}^{4}\left(1-\eta_j^q\right)^{\tau_j\omega_j}\right)^{1/q},\\ \left(\prod_{j=1}^{4}\left(1-\eta_j^q\right)^{\tau_j\omega_j}-\prod_{j=1}^{4}\left(1-\eta_j^q-\vartheta_j^q\right)^{\tau_j\omega_j}\right)^{1/q}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left(1-\left(1-0.8^2\right)^{0.3\times0.4}\left(1-0.6^2\right)^{0.2\times0.2}\left(1-0.7^2\right)^{0.3\times0.1}\left(1-0.5^2\right)^{0.6\times0.3}\right)^{\frac{1}{2}}\\ ,\left(\begin{array}{l}\left(1-0.8^2\right)^{0.3\times0.4}\left(1-0.6^2\right)^{0.2\times0.2}\left(1-0.7^2\right)^{0.3\times0.1}\left(1-0.5^2\right)^{0.6\times0.3}\\ -\left(1-0.8^2-0.2^2\right)^{0.3\times0.4}\left(1-0.6^2-0.2^2\right)^{0.2\times0.2}\left(1-0.7^2-0.2^2\right)^{0.3\times0.1}\left(1-0.5^2-0.3^2\right)^{0.6\times0.3}\end{array}\right)^{\frac{1}{2}}\end{array}\right\rangle$$

$$=\langle 0.44,0.18\rangle$$

By changing the value of $b_1$ and $b_2$ such that $b_1 = \langle (0.5, 0.2), 0.3 \rangle$ and $b_2 = \langle (0.4, 0.2), 0.2 \rangle$, we have the new result as: $\langle 0.17, 0.33 \rangle$. Hence, by changing the value of MED parameters also change the MED and NOMED of the calculated result, wherein example 2, it was shown that by changing the MED parameters has no effect on the result.

### 4.1. Some Modified Q-Rung Orthopair Averaging Power Fuzzy Operator

• **Definition 8**

Consider $b_j = (\eta_j + \vartheta_j)$, (j=1,2,3,...,n) be the set of *q-CROFN*, such that q-ICROFPA is defined by:

$$q - ICROFPA(b_1, b_2, b_3, ..., b_n) = \otimes_{k=1}^{n} (1 + S(b_k)) b_k \tag{12}$$

With
$$S(b_k) = \sum_{k=1, k \neq l}^{n} \sup(b_k, b_l)$$

Here, $S(b_k)$ denotes the degree of support of the $k^{th}$ *CROFN* with others *CROFN's*, as much their value are closer they will support more each other.

- **Definition 9**

Consider $b_1 = (\eta_1 + \vartheta_1)$ and $b_2 = (\eta_2 + \vartheta_2)$ be the two *q- CROFNS*. Then, the hesitance degree of $b_1$ and $b_2$ is given by the following formula:

$$d(b_1, b_2) = \frac{h\left(\left|\eta_1^q - \eta_2^q\right| + \left|\vartheta_1^q - \vartheta_2^q\right|\right) + (1-h)\left|\varphi_1^q - \varphi_2^q\right|}{2}, \tag{13}$$

with $d \in [0,1]$, $h \in [0,1]$ and $\varphi_j^q = (1 - \eta_j^q - \vartheta_j^q)$, j=1,2. Moreover, from equation (13), different distance measure can be obtained as the parameter $h$ varies as shown in the following cases.

- For $h = 1$, hesitance degree preference maybe reduced to the hamming such that

$$d(b_1, b_2) = \frac{\left|\eta_1^q - \eta_2^q\right| + \left|\vartheta_1^q - \vartheta_2^q\right|}{2}.$$

- For $h = 0.5$, hesitance degree preference maybe reduced to the hamming indeterminacy degree preference distance given by the following relation

$$d(b_1, b_2) = \frac{0.5\left(\left|\eta_1^q - \eta_2^q\right| + \left|\vartheta_1^q - \vartheta_2^q\right| + \left|\varphi_1^q - \varphi_2^q\right|\right)}{2}.$$

Consider the set of $q-CROFN,$ $b_j=(\eta_j,\vartheta_j)$, with $j=1,2,...,m$, then its aggregation still remains a $q-CROFN,$ such that

$$q-ICROFWA(b_1,b_2,...,b_m)=$$

$$\left\langle \begin{array}{l} \left(1-\prod_{j=1}^{m}\left(1-\eta_j^q\right)^{\left(\frac{(1+S(b_j))b_j}{\sum_{j=1}^{m}(1+S(b_j))^{\frac{1}{q}}}\right)}\right)^{\frac{1}{q}}, \\ \left(\left(1-\prod_{j=1}^{m}\left(1-\eta_j^q\right)\right)^{\left(\frac{(1+S(b_j))b_j}{\sum_{j=1}^{m}(1+S(b_j))}\right)}-\prod_{j=1}^{m}\left(1-\eta_j^q-\vartheta_j^q\right)^{\left(\frac{(1+S(b_j))b_j}{\sum_{j=1}^{m}(1+S(b_j))}\right)}\right)^{\frac{1}{q}} \end{array} \right\rangle$$

**Proof:** By taking $\omega_j=\left(\frac{(1+S(b_j))b_j}{\sum_{j=1}^{m}(1+S(b_j))}\right)$ and then proceeding on the same lines as in Theorem 4, the required result can be proved. Hence, to avoid repetition the proof is omitted here. In Definition 9, the weighted vector is not under the consideration which may affect the decision making in many actual situations to overcome these situations; we define the *q-ICROFWA* operator as given below.

- **Definition 10**

Consider $b_j=(\eta_j+\vartheta_j),$ (j=1,2,3,...,n) be the set of *q-CROFN*, such that *q-ICROFPA* is defined by:

$$q-ICRFOWPA\ (b_1,b_2,b_3,...,b_n)=\left(\frac{\oplus_{j=1}^{k}\omega_j\tau_j\left(1+S(b_j)\right)b_j}{\sum_{j=1}^{m}\omega_j\tau_j\left(1+S(b_j)\right)}\right) \qquad (14)$$

For the sake of simplicity, we take $\gamma_j = \left( \frac{\oplus_{j=1}^{k} \omega_j \tau_j \left(1+S(b_j)\right) b_j}{\sum_{j=1}^{m} \omega_j \tau_j \left(1+S(b_j)\right)} \right)$ with $j = 1,2,3,...,k$,

$\omega_j = \omega_1, \omega_2, \omega_3, ..., \omega_k$ and $\tau_j = \tau_1, \tau_2, \tau_3, ..., \tau_k$ with $0 \le \omega_j, \tau_j \le 1$, $\sum_{j=1}^{k} \omega_j = 1$ and $\sum_{j=1}^{k} \tau_j = 1.$

Moreover, eq. (14), can be written as

$$q-ICRFOWPA\ (b_1, b_2, b_3, ..., b_n) = \oplus_{j=1}^{k} b_j \omega_j \tau_j$$

- **Theorem 6**

Consider the set of $q-CROFN$, $b_j = (\eta_j, \vartheta_j)$, with $j = 1,2,...,m$, then according Definition 9, its aggregation still remains a $q-CROFN$, such that

$$q-ICROFWPA(b_1, b_2, b_3, ..., b_n) = \left\langle \left( 1 - \prod_{j=1}^{k} \left(1-\eta_j^q\right)^{\gamma_j} \right)^{\frac{1}{q}}, \left( \prod_{j=1}^{k} \left(1-\eta_j^q\right)^{\gamma_j} - \prod_{j=1}^{k} \left(1-\eta_j^q - \vartheta_j^q\right)^{\gamma_j} \right)^{\frac{1}{q}} \right\rangle.$$

**Proof:** The required proof can be obtained on the same lines as Theorem 2. Hence, to avoid repetition the proof is omitted here.

- **Theorem 7**

Consider the set of $q-CROFN$, $b_j = (\eta_j, \vartheta_j)$, with $j = 1,2,...,k$, with $b_j = b = (\eta, \vartheta)$ then

$$q-ICROFWPA(b_1, b_2, b_3, ..., b_n) = b$$

**Proof:** The required proof can be obtained on the same lines as Theorem 3. Hence, to avoid repetition the proof is omitted here.

- **Theorem 8**

Let $b_j = \langle \eta_j, \vartheta_j \rangle$ be a collection of q-CROFN with $j = 1, 2, ..., n$ such that

$b^- = \left\langle \min_j(\eta_j^q), \max_j(\eta_j^q + \vartheta_j^q - \min_j(\eta_j^q)^{\frac{1}{q}} \right\rangle$ and $b^+ = \left\langle \max_j(\eta_j^q), H \right\rangle$, then it gives that

$b^- \leq q-ICROFWA\left(b_1, b_2, \ldots, b_n\right) \leq \text{b}^+$, where $H$ is the same as defined in Theorem 4.

**Proof:** The required proof can be obtained on the same lines as Theorem 4. Hence, to avoid repetition the proof is omitted here.

## 6. An Approach to MAGDM Method Under Confidence Level

This Section is devoted to the study of multi attribute group decision making (MAGDM) based on the developed operators $q-ICROFWA$ and $q-ICROFWPA$. A real life problem in example 5, is considered to demonstrate the MAGDM method effectively.

Algorithm: Let us consider the set of $k$ decision makers (experts) $D = \langle D_1, D_2, D_3, ..., D_k \rangle$, $C = \langle C_1, C_2, C_3, ..., C_n \rangle$ be the set of $n$ attributes, $A = \langle A_1, A_2, A_3, ..., A_m \rangle$ be m alternatives. Moreover, $b_{ij}^p = \left(\eta_{ij}^p, \vartheta_{ij}^p\right)$ with $p = 1, 2, ..., k$ are the $C_j$ attributes from alternatives of the experts $D_k$. Let $w_p \in [0,1]$ be the weighted vector with $w = (w_1, w_2, w_3, ..., w_k)$ of experts, such that $\sum_{p=1}^{k} w_p = 1$. Similarly, for the confidence levels of the experts, we have $\tau_p \in [0,1]$ with $\sum_{p=1}^{k} \tau_p = 1$ . To effectively assist the developed operators in group decision making problems, the detail of the aforementioned method are described in the following steps:

- In the first step, the decision matrices can be modeled by using the available information. In most cases there are two types of criteria such as cost and benefit, then cost criteria can be normalized by:

$$b_{ij}^{p}=\left(\eta_{ij}^{p},\vartheta_{ij}^{p}\right)=\begin{cases}\left(\eta_{ij}^{p},\vartheta_{ij}^{p}\right), & \text{for the benifit criteria } A_m\\ \left(\vartheta_{ij}^{p},\eta_{ij}^{p}\right), & \text{for the cost criteria } A_m\end{cases}$$

- In the second step, the attribute values can be aggregated by using the following formula

$$S_i^k=q-ICROFWA(b_{i1}^k,b_{i2}^k,\ldots,b_{in}^k)=\left\langle\begin{array}{l}\left(1-\prod_{j=1}^{n}\left(1-\left(\eta_{ij}^{k}\right)^{q}\right)^{\tau_k\omega_k}\right)^{\frac{1}{q}}\\ ,\left(\begin{array}{l}\prod_{j=1}^{n}\left(1-\left(\eta_{ij}^{k}\right)^{q}\right)^{\tau_k\omega_k}\\ -\prod_{j=1}^{n}\left(1-\left(\eta_{ij}^{k}\right)^{q}-\left(\vartheta_{ij}^{k}\right)^{q}\right)^{\tau_k\omega_k}\end{array}\right)^{\frac{1}{q}}\end{array}\right\rangle \tag{15}$$

- Collective preference values $S_i$ can be obtained by using the operator $q-ICROFWPA$, such that

$$S_i=q-ICROFWA(b_i^k,b_i^k,\ldots,b_i^k)$$
$$=\left\langle\begin{array}{l}\left(1-\prod_{k=1}^{p}\left(1-\left(\eta_{i}^{k}\right)^{q}\right)^{\omega_i^k}\right)^{\frac{1}{q}}\\ ,\left(\prod_{k=1}^{p}\left(1-\left(\eta_{i}^{k}\right)^{q}\right)^{\omega_i^k}-\prod_{k=1}^{p}\left(1-\left(\eta_{i}^{k}\right)^{q}-\left(\vartheta_{i}^{k}\right)^{q}\right)^{\omega_i^k}\right)^{\frac{1}{q}}\end{array}\right\rangle \tag{16}$$

- In this step, $S_i$ can be ranked by using the accuracy and score function.

In the last step, the best alternative can be selected by using the best values of $S_i$

## 7. Illustrative Example

This section is devoted to illustrate in detail the application of the proposed method. Furthermore, the superiority and effectiveness of the MAGDM method can be verified with some examples.

### 7.1. Illustrative example on the Proposed Method

**Example 5**

Let a company have a choice to invest the money in the given four alternatives such as $A_1$, $A_2$, $A_3$ and $A_4$, whose weighted vector is $\varpi = (0.4, 0.3, 0.1, 0.2)$. The company must choose the best option keeping in view the following four attributes such as $C_1$, $C_2$, $C_3$ and $C_4$ representing risk analysis, growth analysis, social political impact analysis and environmental analysis respectively. Here, the cost type criteria are represented by $C_1$, $C_3$ and benefit type criteria is given by $C_2$, $C_4$. The decision is made by three experts $D_i (i = 1, 2, 3)$ whose weighted vector is $\omega = (0.3, 0.2, 0.1)$. To choose the best option the detail of the MAGDM method is explained as below with $q$=2.

**Step 1: In this step, we have to construct the decision matrix of q-ROFs**

| | | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|---|
| $D_1$ | $A_1$ | $\langle(0.70, 0.30), 0.30\rangle$ | $\langle(0.60, 0.60), 0.20\rangle$ | $\langle(0.50, 0.70), 0.20\rangle$ | $\langle(0.70, 0.60), 0.10\rangle$ |
| | $A_2$ | $\langle(0.40, 0.80), 0.40\rangle$ | $\langle(0.70, 0.40), 0.30\rangle$ | $\langle(0.70, 0.80), 0.30\rangle$ | $\langle(0.80, 0.30), 0.30\rangle$ |
| | $A_3$ | $\langle(0.70, 0.50), 0.10\rangle$ | $\langle(0.50, 0.40), 0.20\rangle$ | $\langle(0.60, 0.80), 0.30\rangle$ | $\langle(0.50, 0.90), 0.50\rangle$ |
| | $A_4$ | $\langle(0.60, 0.50), 0.30\rangle$ | $\langle(0.50, 0.60), 0.50\rangle$ | $\langle(0.70, 0.60), 0.10\rangle$ | $\langle(0.80, 0.20), 0.40\rangle$ |
| $D_2$ | $A_1$ | $\langle(0.60, 0.70), 0.40\rangle$ | $\langle(0.70, 0.90), 0.30\rangle$ | $\langle(0.60, 0.80), 0.40\rangle$ | $\langle(0.90, 0.20), 0.10\rangle$ |
| | $A_2$ | $\langle(0.50, 0.70), 0.20\rangle$ | $\langle(0.50, 0.70), 0.10\rangle$ | $\langle(0.60, 0.40), 0.20\rangle$ | $\langle(0.80, 0.20), 0.40\rangle$ |
| | $A_3$ | $\langle(0.60, 0.90), 0.40\rangle$ | $\langle(0.90, 0.30), 0.30\rangle$ | $\langle(0.70, 0.80), 0.10\rangle$ | $\langle(0.50, 0.70), 0.40\rangle$ |
| | $A_4$ | $\langle(0.60, 0.80), 0.40\rangle$ | $\langle(0.60, 0.70), 0.40\rangle$ | $\langle(0.90, 0.30), 0.40\rangle$ | $\langle(0.70, 0.30), 0.40\rangle$ |
| $D_3$ | $A_1$ | $\langle(0.90, 0.20), 0.10\rangle$ | $\langle(0.60, 0.80), 0.40\rangle$ | $\langle(0.90, 0.30), 0.20\rangle$ | $\langle(0.90, 0.20), 0.30\rangle$ |
| | $A_2$ | $\langle(0.80, 0.50), 0.40\rangle$ | $\langle(0.90, 0.30), 0.40\rangle$ | $\langle(0.50, 0.80), 0.30\rangle$ | $\langle(0.60, 0.50), 0.40\rangle$ |
| | $A_3$ | $\langle(0.70, 0.40), 0.10\rangle$ | $\langle(0.60, 0.70), 0.30\rangle$ | $\langle(0.60, 0.70), 0.40\rangle$ | $\langle(0.50, 0.80), 0.20\rangle$ |
| | $A_4$ | $\langle(0.60, 0.40), 0.10\rangle$ | $\langle(0.90, 0.20), 0.10\rangle$ | $\langle(0.70, 0.30), 0.10\rangle$ | $\langle(0.60, 0.70), 0.10\rangle$ |

**Table 1: Decision Matrix of q-ROFs**

**Step 2:** The data taken in this example consist of two types cost and benefit. Therefore, Table 1, can be normalized as shown in the Table 2, as below:

| | | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|---|
| $D_1$ | $A_1$ | $\langle(0.30,0.70),0.30\rangle$ | $\langle(0.60,0.60),0.20\rangle$ | $\langle(0.70,0.50),0.20\rangle$ | $\langle(0.70,0.60),0.10\rangle$ |
| | $A_2$ | $\langle(0.80,0.40),0.40\rangle$ | $\langle(0.70,0.40),0.30\rangle$ | $\langle(0.80,0.70),0.30\rangle$ | $\langle(0.80,0.30),0.30\rangle$ |
| | $A_3$ | $\langle(0.50,0.70),0.10\rangle$ | $\langle(0.50,0.40),0.20\rangle$ | $\langle(0.80,0.60),0.30\rangle$ | $\langle(0.50,0.90),0.50\rangle$ |
| | $A_4$ | $\langle(0.50,0.60),0.30\rangle$ | $\langle(0.50,0.60),0.50\rangle$ | $\langle(0.60,0.70),0.10\rangle$ | $\langle(0.80,0.20),0.40\rangle$ |
| $D_2$ | $A_1$ | $\langle(0.70,0.60),0.40\rangle$ | $\langle(0.70,0.90),0.30\rangle$ | $\langle(0.80,0.60),0.40\rangle$ | $\langle(0.90,0.20),0.10\rangle$ |
| | $A_2$ | $\langle(0.70,0.50),0.20\rangle$ | $\langle(0.50,0.70),0.10\rangle$ | $\langle(0.40,0.60),0.20\rangle$ | $\langle(0.80,0.20),0.40\rangle$ |
| | $A_3$ | $\langle(0.90,0.60),0.40\rangle$ | $\langle(0.90,0.30),0.30\rangle$ | $\langle(0.80,0.60),0.10\rangle$ | $\langle(0.50,0.70),0.40\rangle$ |
| | $A_4$ | $\langle(0.80,0.60),0.40\rangle$ | $\langle(0.60,0.70),0.40\rangle$ | $\langle(0.30,0.90),0.40\rangle$ | $\langle(0.70,0.30),0.40\rangle$ |
| $D_3$ | $A_1$ | $\langle(0.20,0.90),0.10\rangle$ | $\langle(0.60,0.80),0.40\rangle$ | $\langle(0.30,0.90),0.20\rangle$ | $\langle(0.90,0.20),0.30\rangle$ |
| | $A_2$ | $\langle(0.50,0.80),0.40\rangle$ | $\langle(0.90,0.30),0.40\rangle$ | $\langle(0.80,0.50),0.30\rangle$ | $\langle(0.60,0.50),0.40\rangle$ |
| | $A_3$ | $\langle(0.40,0.70),0.10\rangle$ | $\langle(0.60,0.70),0.30\rangle$ | $\langle(0.70,0.60),0.40\rangle$ | $\langle(0.50,0.80),0.20\rangle$ |
| | $A_4$ | $\langle(0.40,0.60),0.10\rangle$ | $\langle(0.90,0.20),0.10\rangle$ | $\langle(0.30,0.70),0.10\rangle$ | $\langle(0.60,0.70),0.10\rangle$ |

**Table 2: Normalized Decision Matrix of q-ROFs**

**Step 3: In this step, the attribute values of the matrices $D_{ij}^{k}$ can be aggregated by using the following formula (15), as shown in Table 3:**

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
|---|---|---|---|---|
| $S^1$ | $(0.25,0.80)$ | $(0.36,0.80)$ | $(0.34,0.94)$ | $(0.31,0.21)$ |
| $S^2$ | $(0.38,0.37)$ | $(0.36,0.35)$ | $(0.35,0.29)$ | $(0.25,0.15)$ |
| $S^3$ | $(0.36,0.21)$ | $(0.35,0.28)$ | $(0.36,0.93)$ | $(0.36,0.35)$ |
| $S^4$ | $(0.32,0.95)$ | $(0.31,0.42)$ | $(0.15,0.46)$ | $(0.34,0.86)$ |

**Table 3: Integrated Decision Matrix**

**Table 3: Integrated Decision Matrix**

**Step 4:** This step is devoted to the calculation of preference values $S_i$, by using (16), we have

$S_1 = (0.31, 0.85)$, $S_2 = (0.35, 0.33)$, $S_3 = (0.36, 0.62)$ and $S_4 = (0.31, 0.91)$

- **Step 5:** In this step, $S_i$ can be ranked by using the score function.

$L(S_1) = -0.63$, $L(S_2) = 0.01$, $L(S_3) = -0.25$, $L(S_4) = -0.73$.

- **Step 6:** By the score function, we have: $A_2 \succ A_3 \succ A_1 \succ A_4$. Hence, $A_2$ is the best choice.

Next, by changing the value of $q = 3,5,7,9,11$, we have the following table:

| Values of q | Score function | Ranking |
|---|---|---|
| 3 | $L(S_2) \succ L(S_3) \succ L(S_1) \succ L(S_4)$ | $A_2 \succ A_3 \succ A_1 \succ A_4$ |
| 5 | $L(S_2) \succ L(S_3) \succ L(S_1) \succ L(S_4)$ | $A_2 \succ A_3 \succ A_1 \succ A_4$ |
| 7 | $L(S_2) \succ L(S_1) \succ L(S_3) \succ L(S_4)$ | $A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 9 | $L(S_2) \succ L(S_1) \succ L(S_3) \succ L(S_4)$ | $A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 11 | $L(S_2) \succ L(S_1) \succ L(S_3) \succ L(S_4)$ | $A_2 \succ A_1 \succ A_3 \succ A_4$ |

**Table 4: Results for different q-values**

## 8. Comparative and Sensitive Analysis

The q-ROFS$\left(\eta^q + \vartheta^q \le 1\right)$ with (q ≥ 1 ) is the successful generalization of IFS $(\eta + \vartheta \le 1)$, PFS $\left(\eta^2 + \vartheta^2 \le 1\right)$ and Fermatean fuzzy set , by considering more information to handle the real life problem. Thus, the proposed aggregation operators under q-ROFS environment are more generalized as compared to already existing methods. Moreover, there are some drawbacks such as the NOMED in the equation (1) and MED of equation (2) with their respective entries products which will give zero by taking only one zero entry in the data as shown in example 1. Similarly, in example 2, it is shown that by changing the values of MED does not affect the calculated value of NOMED. Therefore, it does not provide the actual information to the decision makers.

Therefor in this paper, we presented some modified operational laws and some of their properties to overcome these drawbacks. Next, related to q-ROFS fuzzy-weighted averaging (q-ICROFWA) and fuzzy-weighted power averaging (q-ICROFWPA) aggregation operator under confidence levels along with their properties are presented. By using these operators' q-ICROFWA and q-ICROFWPA an advanced method is proposed to deal with MAGDM problems in fuzzy environment. For comparison, if we consider the data with one zero entry then there is no conclusion by using the existing methods as shown in the following table:

| Existing Methods | Score function | Ranking Order | Conclusion |
|---|---|---|---|
| Wang and Liu[30] | $L(S_1) = L(S_2) = L(S_3) = L(S_4)$ | $A_1 = A_2 = A_3 = A_4$ | No |
| Wang and Liu [31] | $L(S_1) = L(S_2) = L(S_3) = L(S_4)$ | $A_1 = A_2 = A_3 = A_4$ | No |
| Proposed method | $L(S_2) \succ L(S_3) \succ L(S_1) \succ L(S_4)$ | $A_2 \succ A_3 \succ A_1 \succ A_4$ | Yes |

**Table: 5 Comparison of existing Methods with one zero entry**

The above comparisons shows that, the operator proposed in this paper is better than the existing other methods for q-ROFS environment under the confidence level. Therefore, it is most reasonable to get conclusion of MAGDM problem of any kind data by using this method.

## 9. Conclusions

The parameter q, in q-ROFS plays an important role to express fuzzy information in wider range than IFS and PFS. This shows that q-ROFS is the generalization of IFS and PFS, where  is not limited to a specific number and can be extend to desirable number needed for available data. In this study, based on aggregation operators some drawbacks of the already existing MAGDM methods are analyzed. Moreover, we presented some modified operational laws and some of their properties to overcome these drawbacks. Next, related to q-ROFS fuzzy-weighted averaging (q-ICROFWA) and fuzzy-weighted power averaging (q-ICROFWPA) aggregation operator under confidence levels along with their properties are presented. By using these operators' q-ICROFWA and q-ICROFWPA an advanced method is proposed to deal with MAGDM problems in fuzzy environment. At last, we solved some numerical examples to show the validity and feasibility of this method. Further, this research can be extended to complex Logarithmic operators, complex Inducing variables, complex Linguistic terms, complex Confidence level, complex Hamacher operators, complex Interval-valued approach, complex Einstein approach, complex Dombi approach, complex Symmetric operators, complex Power operators, complex Hamacher interval approach, complex Dombi interval approach, complex Einstein approach, complex Einstein interval etc.


## Funding

No funding received from external funding agencies.


## Ethical approval

This work does not contain any studies with human participants or animals performed by any of the authors.

## Data availability

All the data is available and within the manuscript, no supplement materials data.

## Declarations

**Conflict of interest**

The authors declare no conflict of interest.

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