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Methods of Solving Partial Differential Equations and Their Application on One Specific Example

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Abstract
This scientific paper investigates the application of the Voltaire-Gurset-Riemann method in solving partial differential equations, using a flickering wire as an example. The method proves to be a powerful tool in the analysis of dynamic systems, providing a deeper understanding of flicker behavior in a wire. The developed numerical solutions enable precise modeling and prediction of the behavior of the flickering structure. This study highlights the key steps in applying the method to a concrete example, providing a useful basis for further research in the field of partial differential equations.

Keywords: Voltaire-Gurset-Riemann Method, Partial Differential Equations, Flickering Wire, Dynamic Systems, Numerical Solution, Behavior Modeling

1. Introduction

In modern engineering and scientific research, solving partial differential equations is essential for understanding dynamic phenomena in various systems. In this context, the Voltaire-Gurset-Riemann method emerges as a powerful tool that provides an efficient solution to these equations. This paper focuses on the application of the mentioned method in the analysis of flickering wire, investigating in detail the numerical aspects and the accompanying implications for the dynamic behavior of the structure.

Our goal is to provide a fundamental understanding of how the Voltaire-Gurset-Riemann method can improve our understanding of flicker in wires and how we can use the obtained solutions in engineering applications. Through precise modeling and analysis of the results, we try to expand the scope of application of this method to different dynamic systems.

This introduction lays the groundwork for further consideration of our research, emphasizing the importance of studying vibrations in strings with the aim of applying the Voltaire-Gurset-Riemann method as a key tool in the analysis of dynamic phenomena.

2. Flickering Wire Equation

The given equation is :

$$\frac{\partial^2 u}{\partial y^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (0<x<l;\ y>0), \qquad (1)$$

where u is the unknown function of the variables x,y, together with the contour conditions:

$$u(0,y)= u(l,y) = 0 \quad (y \geq 0) \qquad (2)$$

where l is a given constant, as well as the initial conditions

$$u(x,0)=g_1(x), u_y(x,0) = g_2(x) \quad (0 \leq x \leq l) \qquad (3)$$

where g_1 and g_2 are given functions. By putting $u(x,y)= X(x)\cdot Y(y)$, we have:

$$\boldsymbol{XY''} - \boldsymbol{a^2X''Y}=0,$$

i.e.

$$\frac{X''}{X} = \frac{1}{a^2}\frac{Y''}{Y} = -\lambda, \qquad (4)$$

where λ is a constant. From (4), we get the following two ordinary differential equations of the second order with constant coefficients:

$$\boldsymbol{X''} + \lambda X = 0, \qquad \boldsymbol{Y''} + a^2 Y = 0. \qquad (5)$$

● Let it be now $\lambda \leq 0$. Let's introduce a shift now $\lambda=-k^2$. The general solutions of the above equations are given by:

$X\ \ (x)= \ C_1 \, cosh\, kx + C_2 \, sinh\, kx\,,\ Y(y) = \ C_3 \, cosh\, aky + C_4 \, sinh\, aky,$ where $C_i, i = \overline{1,4},$ are arbitrary constants.

The contour condition (2) gives

$$C_1(C_3 \cosh aky + C_4 \sinh aky) = 0, (\,C_1 \cosh kl + C_2 \sinh kl)(C_3 \cosh aky + C_4 \sinh aky)=0 \qquad (6)$$

so we get that it is $C_1= C_2= 0$.
If $\lambda = 0$, then it is also k=0 and equations (6) become $C_1\ C_3=0$. On the basis of which we conclude if it is $\lambda \leq 0$, solution to the problem (1)-(2)-(3) is just trivial i.e., u(x,y)=0.
● Let it be $\lambda > 0$. Then it is:

$$X(x) = C_1 \cos\sqrt{\lambda x} + C_2 \sin\sqrt{\lambda x} \qquad (7)$$

Now replacing the contour conditions (2) and (7) we have that:

$$C_1 = 0, C_1 \cos\sqrt{\lambda l} + C_2 \sin\sqrt{\lambda l} = 0,$$

from which it follows that it is:

$$C_2 \sin\sqrt{\lambda l} = 0,\ tj., l\sqrt{\lambda} = n\pi, (n = 1,2, \dots).$$

Therefore, the constant λ can have any values λ_n which are defined by

$$\lambda_n = \frac{n^2\pi^2}{l^2} \qquad \text{(n=1,2,...)}$$

Substituting these values for λ into the second equation (5), we have:

$$Y_n(y) = A_n \cos\frac{na\pi}{l}y + B_n \sin\frac{na\pi}{l}y,$$

where A_n, B_n are arbitrary constants.

So every function u_n defined by:

$$u_n(x,y) = \sin\frac{n\pi}{l}x(A_n \cos\frac{na\pi}{l}y + B_n \sin\frac{na\pi}{l}y),$$

satisfies the equalities (1) and (2), so based on the principle of linear superposition, and the function u, defined by:

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{l}x(A_n \cos\frac{na\pi}{l}y + B_n \sin\frac{na\pi}{l}y)\,, \qquad (8)$$

represents a solution to the problem (1) - (2).
Let's determine the constants now A_n *i* B_n, so that (8) also satisfies conditions (3). Substituting (3) into (8) we have that:

$$g_1(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x, \qquad g_2(x) = \sum_{n=1}^{\infty} B_n \frac{na\pi}{l}\sin\frac{n\pi}{l}x,$$

which makes the problem of determining arbitrary constants A_n, B_n reduced to developing functions g_1,g_2 into Fourier series.

With such determined coefficients A_n, B_n,(8) represents a solution to the problem (1)-(2)-(3).

3. Riemann's Method

Let the hyperbolic equation be given:

$$S(u) = \frac{\partial^2 u}{\partial x \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\delta u}{\delta y} + c(x,y)u = 0,$$

where the functions a,b,c are continuously differentiable in some region G of the plane xOy. Let us now define the operator $\bar{S}$ using:

$$\bar{S}(u) = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial}{\partial x}(au) - \frac{\partial}{\partial y}(bu) + cu.$$

Then we have that is:

$$vS(u) - u\bar{S}(v) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

where is it

$$P = auv + \frac{1}{2}(vu_y - uv_y) = \frac{1}{2}\frac{\delta}{\delta y}(uv) - uM(v)$$

$$Q = buv + \frac{1}{2}(vu_x - uv_x) = \frac{1}{2}\frac{\delta}{\delta x}(uv) - uN(v)$$

$$M(v) = v_y - av, \quad N(v) = v_x - bv.$$

Let C be a closed contour lying inside the region G and let D= intC, and let the conditions under which Green's formula holds are met. Then it is:

$$\bar{\iint_D} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy = \bar{\iint_D} \left(vS(u) - u\bar{S}(v)\right) dxdy - \bar{\int_C} Pdy - Qdx.$$

Let the curve C be composed of lengths YA, AX, which are parallel to the y-axis and the x-axis, respectively, and Γ which passes through points X,Y.

Let's assume it is $\bar{S}(v)=0$ *i* $S(u)=0$. Then it is:

$$\bar{\int_C} Pdy - Qdx = 0$$

so we get that it is:

$$\bar{\int_{AX}} Qdy - \bar{\int_{YA}} Pdx = \bar{\int_{\Gamma}} Mdy - Ndx.$$

On the other hand, we have:

$$\bar{\int_{AX}} Qdy = \frac{1}{2}(u\,v_X - uv_A) - \bar{\int_{AX}} uN(v)dx,$$

$$-\int_{YA}^{-} Pdx = \frac{1}{2}(u\, v_Y - uv_A) - \int_{YA}^{-} uM(v)dx.$$

That's why

$$uv_A = \frac{1}{2}(u\, v_X - uv_Y) + \int_{\Gamma}^{-} Pdy - Qdx + \int_{YA}^{-} uM(v)dx - \int_{AX}^{-} uN(v)dx. \quad (1)$$

Let us now assume that the function v is determined by M(v)=0 along AY and N(v)=0 along AX, and that the values of this function are known on the curve C. From (1) we get:

$$uv_A = \frac{1}{2}(u\, v_X + uv_Y) + \int_{\Gamma}^{-} Pdy - Qdx$$

which can be written in the form:

$$uv_A = \frac{1}{2}(u\, v_X + uv_Y) + \int_{\Gamma}^{-} (lP + mQ)ds \quad (2)$$

where $l = cos\,(n,x)$, $m = cos(n,y)$ are the cosines of the true external vertical to the curve Γ.

In the integral appearing in (2) the expressions figure $u, u_x\, u_y$. However, if we know the values of the function u on the curve Γ and derivative values of the function u in the direction of vertical to Γ, we can determine the derivative values $u_x\, u_y$ on the curve Γ. Indeed, if the values of the function u for are known (x,y)∈ Γ , then the tangential derivative of that function along the curve is also known Γ, so the derivations u_x i u_y can be obtained using tangential and vertical derivatives. Accordingly, we conclude the following:

The problem of solving the equation S(u)=0, where the values of the function u and its vertical derivative $\delta u/\delta n$ on the curve Γ given, it boils down to determining the function v with properties:

$$\bar{S}(v)=0,\ M(v)=0 \text{ za } (x,y)\in AY,\ N(v)=0 \text{ za } (x,y)\in AX.$$

The function v is called the Grin-Riemann function of the problem, and the presented method is the Riemann method. In the following, we will apply the mentioned method to solving the Cauchy problem for the telegraph equation. It should be noted that due to the specificity of the initial conditions, the curve Γ becomes the length, as well as that it is unnecessary to introduce the operator $\bar{S}$.

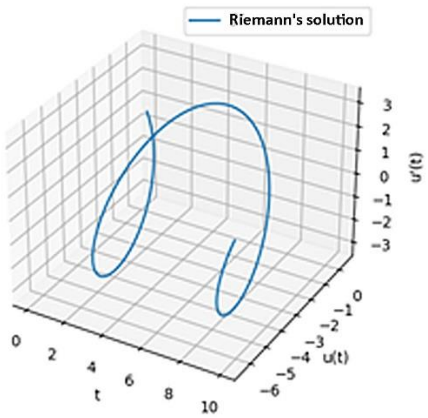

Figure 1: Riemann's Solution Method

This 3D graph shows the solution of the differential equation using the Riemann method. The x axis represents time, while the y and z axes are the results of the function u(t) and its derivative u'(t). The graph shows the evolution of the solution over time, with the initial conditions given by the Riemann function.

4. The Telegraphic Equation

The equation

$$a\frac{\partial^2 u}{\partial \eta^2} + 2b\frac{\partial u}{\partial \eta} - c\frac{\partial^2 u}{\partial \xi^2} = 0, \qquad (1)$$

a,b,c are positive constants, and u is an unknown function of variables η,ξ and is called the telegraph equation.

To simplify equation (1) we will put:

$$\eta = \mu Y,\ \xi = \lambda X, \qquad (2)$$

where λ and μ are constants that we will determine later. Based on (2), we have:

$$\frac{\partial u}{\partial \eta} = \frac{1}{\mu}\frac{\partial u}{\partial Y},\quad \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{\mu^2}\frac{\partial^2 u}{\partial Y^2}, \qquad \frac{\partial u}{\partial \xi} = \frac{1}{\lambda}\frac{\partial u}{\partial X},\quad \frac{\partial^2 u}{\partial \xi^2} = \frac{1}{\lambda^2}\frac{\partial^2 u}{\partial X^2},$$

and equation (1) becomes:

$$\frac{a}{\mu^2}\frac{\partial^2 u}{\partial Y^2} + \frac{2b}{\mu}\frac{\partial u}{\partial Y} - \frac{c}{\lambda^2}\frac{\partial^2 u}{\partial X^2} = 0.$$

Let's choose now λ and μ so it's valid:

$$\frac{\mu^2}{a} = \frac{\mu}{b} = \frac{\lambda^2}{c}, \qquad \text{i.e.,} \qquad \mu = \frac{a}{b},\ \lambda = \frac{\sqrt{ac}}{b}.$$

Equation (3) now becomes:

$$\frac{\partial^2 u}{\partial Y^2} + 2\frac{\partial u}{\partial Y} - \frac{\partial^2 u}{\partial X^2} = 0. \qquad (4)$$

Let us now introduce a new function U that satisfies the following equality:

$$U(X,Y)=e^{-YU(X,Y)}.$$

We have that it is:

$$\frac{\partial u}{\partial Y} = -e^{-YU} + e^{-Y\frac{\partial U}{\partial Y}},\qquad \frac{\partial u}{\partial X} = e^{Y}\frac{\partial U}{\partial X},$$

$$\frac{\partial^2 u}{\partial Y^2} = e^{-YU} - 2e^{-Y\frac{\partial U}{\partial Y}} + e^{-Y\frac{\partial^2 U}{\partial Y^2}},\qquad \frac{\partial^2 u}{\partial X^2} = e^{-Y\frac{\partial^2 U}{\partial X^2}}$$

so equation (4) becomes:

$$\frac{\partial^2 U}{\partial Y^2} - \frac{\partial^2 U}{\partial X^2} - U = 0. \qquad (5)$$

Now let's do a variable shift

$$X = \frac{y+x}{\sqrt{2}},\ Y = \frac{y-x}{\sqrt{2}},\ \text{i.e., } x = \frac{X-y}{\sqrt{2}},\ y = \frac{X+Y}{\sqrt{2}}.$$

We have:

$$\frac{\partial u}{\partial X} = \frac{1}{\sqrt{2}}\frac{\partial u}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial U}{\partial y},\ \frac{\partial u}{\partial Y} = -\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y},$$

$$\frac{\partial^2 U}{\partial X^2} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\frac{\partial^2 U}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 U}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 U}{\partial y^2},$$

$$\frac{\partial^2 U}{\partial Y^2} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2} - \frac{1}{2}\frac{\partial^2 U}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 U}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 U}{\partial y^2},$$

so equation (5) now has the form:

$$-2\frac{\partial^2 U}{\partial x \partial y}\text{-U=0,}$$

i.e.,

$$\frac{\partial^2 U}{\partial x \partial y}+\frac{1}{2}U=0. \qquad (6)$$

Now let's set the following Cauchy problem for equation (1):
Determine the solution of equation (1) that satisfies the conditions:

$$u(\xi,0)=\alpha(\xi), \;\; u\eta(\xi,0)=\beta(\xi). \qquad (7)$$

For equation (4), the initial conditions (7) are::

$$u(X,0)=\alpha_1(X), \;\; u_Y(X,0)=\beta_1(X),$$

and for the equation (5)

$$\text{U(X,0)=}\,\alpha_1(X), -U(X,0)+\;\; U_Y(X,0)=\beta_1(X),$$

i.e.,

$$\text{U(X,0)=}\,\alpha_1(X), \;\;\; U_Y(X,0)=\beta_1(X)+\alpha_1(X)=\beta_2(X).$$

Finally, for equation (6), conditions (7) are:

$$U_{y=x}=g_1(x), \qquad \frac{\partial U}{\partial y}-\frac{\partial U}{\partial x}_{y=x}=g_2(x).$$

Therefore, it is necessary to solve the following problem:
Determine the function U that satisfies the following equation:

$$\frac{\partial^2 U}{\partial x \partial y}+\frac{1}{2}U=0, \qquad (8)$$

and initial conditions $U_{y=x}=g_1(x),\ \frac{\partial U}{\partial y}-\frac{\partial U}{\partial x}_{y=x}=g_2(x),$ where g_1,g_2 are given functions.

Let's introduce the label S(U)=$\frac{\partial^2 U}{\partial x \partial y}+\frac{1}{2}U$. Then it is:

VS(u)-US(V)=V$\frac{\partial^2 U}{\partial x \partial y}-U\frac{\partial^2 V}{\partial x \partial y}=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$,

where P = U$\frac{\partial V}{\partial x}$, Q=V$\frac{\partial U}{\partial y}$.

Let C be a closed contour bounding the region G. Based on Green's formula, we have that:

$$\overline{\iint_G}\big(VS(U)-US(V)\big)dxdy=\overline{\iint_G}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\overline{\int_C}Pdx+Qdy.$$

Assume that U and V are solutions of equation (8). Then S(U)=S(V)=0, and we get:

$$\overline{\int_C}Pdx+Qdy=0. \qquad (9)$$

Let it be $M(x_0,y_0)$ is a point in the plane and let C be the triangle ABM, where A(y_0,y_0), B(x_0,x_0). As on the line MB:dx=0, and on the line AM is dy=0, the integral (9) becomes:

$$\int_{AM}^{-} Pdx + \int_{MB}^{-} Qdy + \int_{BA}^{-} Pdx + Qdy = 0. \qquad (10)$$

However, we have that it is:

$$\int_{MB}^{-} Qdy = \int_{MB}^{-} V\frac{\partial U}{\partial y}dy = \int_{MB}^{-} \frac{\partial}{\partial y}(UV)dy - \int_{MB}^{-} U\frac{\partial V}{\partial y}dy$$

$$= \mathrm{U}(x_0,y_0)\mathrm{V}(x_0,y_0) - \mathrm{U}(x_0,x_0)\ \mathrm{V}(x_0,x_0) - \int_{MB}^{-} U\frac{\partial V}{\partial y}dy.$$

Therefore (10) becomes

$$\int_{AM}^{-} Pdx + UV_M - UV_B - \int_{MB}^{-} V\frac{\partial U}{\partial y}dy + \int_{BA}^{-} Pdx + Qdy = 0,$$

i.e., we have :

$$UV_M = UV_B + \int_{MB}^{-} V\frac{\partial U}{\partial y}dy - \int_{AM}^{-} U\frac{\partial V}{\partial x}dx - \int_{BA}^{-} Pdx + Qdy.$$

Let us now assume that the function V is known to us in the entire plane, and so that $\partial V/\partial y=0$, at each point of length MB, and that it is $\partial V/\partial x=0$ at each point of length AM. Then we get the formula:

$$UV_M = UV_B - \int_{BA}^{-} Pdx + Qdy\,,$$

i.e.,

$$U(x_0,y_0)V(x_0,y_0) = U(x_0,x_0)\ V(x_0,x_0) - \int_{BA}^{-} U\frac{\partial V}{\partial x}dx + V\frac{\partial U}{\partial y}dy. \qquad (11)$$

Since, by assumption, the function V is known, and the condition of the task gives the values for U and $\partial U/\partial y$ on the length BA, we conclude that from (11) we get:

$$U(x_0,y_0) = \frac{1}{V(x_0,y_0)}\left(U(x_0,x_0)\ V(x_0,x_0) - \int_{BA}^{-} U\frac{\partial V}{\partial x}dx + V\frac{\partial U}{\partial y}dy\right),$$

Thus, the function U is determined at an arbitrary point, (x_0,y_0). It remains to construct a function V that has the required properties. Let's assume it is:

$$V(x,y) = W((x-x_0)(y-y_0)) = W(z). \qquad (12)$$

It follows from (12) that:

$$\frac{\partial V}{\partial x} = (y-y_0)W^{`}(z),\ \frac{\partial^2 V}{\partial x\partial y} = W^{`}(z) - zW^{"}(z).$$

Since V must be a solution of equation (6), we have:

$$zW^{"}(z) + W^{`}(z) + \frac{1}{z}W(z) = 0. \qquad (13)$$

By introducing a shift: $\phi = \sqrt{2z},\ i.e.\ \phi^2 = 2z$, we get that it is:

$\frac{dW}{dz}=\frac{1}{\phi}\frac{dW}{d\phi},\frac{d^2W}{dz^2}=\frac{1}{\phi^2}\frac{d^2W}{d\phi^2}$-$\frac{1}{\phi^3}\frac{dW}{d\phi}$,

and equation (13) becomes:

$$\frac{1}{2}\frac{d^2W}{dz^2}+\frac{1}{2\phi}\frac{dW}{d\phi}+\frac{1}{2}W=0,$$

i.e.,

$$\phi\frac{d^2W}{dz^2}+\frac{dW}{d\phi}+\phi W=0. \quad (14)$$

Equation (14) is a special case of the Bessel differential equation $\phi^2 W''+\phi)+W'+(\phi^2-n^2)W=0$, for $n=0$. One of its solutions is given with the Bessel function of the first kind:

$$J_0(\phi)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}(\frac{\phi}{2})^{2k}.$$

That means it is $W= J_0(\sqrt{2z})$ the solution of the differential equation (13), i.e., the function V has the form:

$$V(x,y)=J_0\left(\sqrt{2(x-x_0)(y-y_0)}\right). \quad (15)$$

It follows from (15) that:

$$\frac{\partial V}{\partial x}=\frac{y-y_0}{\sqrt{2(x-x_0)(y-y_0)}}J_0{}',\frac{\partial V}{\partial y}=\frac{x-x_0}{\sqrt{2(x-x_0)(y-y_0)}}J_0{}'.$$

It's on AM length $y=y_0$, and on length Mb is $x=x_0$. That's why it is valid on the length AM $\partial V/\partial x=0$,on the length Mb je $\partial V/\partial y=0$. Therefore:

$$V(x,y)=J_0\left(\sqrt{2(x-x_0)(y-y_0)}\right)$$

is the required function V, by means of which we obtain the solution of the Cauchy problem for the telegraph equation. (O. Todone: Sull integrazione ,Annali di matematica (3) 1 (1898), pag. 1-23)
Solution of the telegraph equation

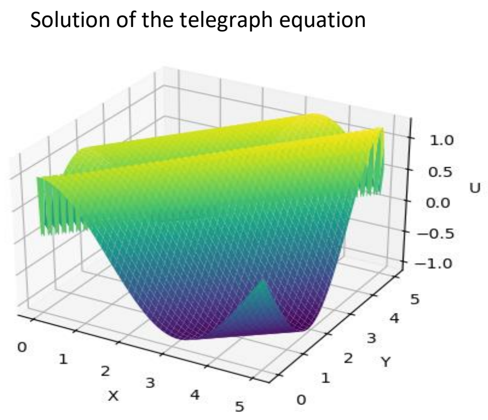

Figure 2: Solution of The Telegraph Equation

This figure numerically solves the telegraph equation using an iterative approach. Parameters a, b and c represent constants in the equation, while the initial conditions are set according to the Cauchy problem. Through iterations, the function U(x, y) is approximated, and the results are displayed in a 3D graph. This process enables the analysis of the evolution of the solution in accordance with the set parameters and conditions. The results are shown in a 3D graph, where the X and Y axes are the spatial coordinates, and the Z axis is the value of the function u. The graph shows the propagation of the waves generated by the initial conditions. It is important to note that this solution is a simple example and provides only an approximate solution. For more accurate results, it may be necessary to use more sophisticated numerical methods or specialized libraries.

5. Voltaire's Method

In this part of the scientific paper, we will solve the equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}=f(x,y,z),$$

where the function u, together with its derivatives of the first order, is given on some surface S. The solution method was given by Voltaire and it represents an extension of the Riemannian method. First, we will introduce an integral formula that will be used for proof later. Let it be:

$$S(u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}.$$

Then it's obvious:

$$vS(u)-uS(v)=\frac{\partial}{\partial x}(vu_x-uv_x)+\frac{\partial}{\partial y}(vu_y-uv_y)-\frac{\partial}{\partial z}(vu_z-uv_z). \qquad (1)$$

If the functions u and v together with their derivatives of the first and second order are continuous, then in the area V bounded by the surface S, it follows directly from formula (1):

$$\bar{\iiint_V}\big(vS(u)-uS(v)\big)dxdydz=\bar{\iint_S}(vu_x-uv_x)dydz+\big(vu_y-uv_y\big)dzdx-(vu_z-uv_z)dxdy. \qquad (2)$$

Let α, β, γ be the angles of the external perpendicular to the surface S and the positive directions of the coordinate axes. Then:

$$\bar{\iint_S}(vu_x-uv_x)dydz+\big(vu_y-uv_y\big)dzdx-(vu_z-uv_z)dxdy$$

$$=\bar{\iint_S} v(u_x\cos\alpha+u_y\cos\beta-u_z\cos\gamma)\,dS$$

$$=\bar{\iint_S} u(v_x\cos\alpha+v_y\cos\beta-v_z\cos\gamma)\,dS.$$

A line n whose direction is determined by a vector ($\cos\alpha,\cos\beta,-\cos\gamma$) we call conormal. Formula (2) then becomes:

$$\bar{\iiint_V}\big(vS(u)-uS(v)\big)dxdydz=\bar{\iint_S}\Big(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\Big)dS, \qquad (3)$$

where $\partial u/\partial n$ is the marked derivative of the function u in the direction n.

Now let's move on to solving the problem. Let's construct a circular cone K with the vertex at a point $P(x_1,y_1,z_1)$, so that the axis of the cone is parallel to the z axis and so that the angle at the vertex P is right. For area V we will take that part of the space that is bounded by the cone K and the surface S.

Let the function v be defined by:

$$v(x,y,z)=\log\frac{z_1-z+\sqrt{(z_1-z)^2-(x_1-x)^2-(y_1-y)^2}}{\sqrt{(x_1-x)^2+(y_1-y)^2}}.$$

Obviously, on the cone K, $S(v)$=0 and v=0. Let's assume that u is the solution to the given problem. Formula (3) cannot be directly applied to the functions u and v and the region V, because the function v is discontinuous along the axis of the cone K, and its derivatives are discontinuous on the cone K.

That is why we will extract the axis of the cone using a circular cylinder C with radius η , whose axis coincides with the axis of the cone K, and we will replace the cone K with a cone K, whose vertex is at point P, its axis coincides with the axis of the cone K and the semi-angle φ at the vertex P is given by $\varphi = \frac{\pi}{4}$ - ε.

Let's form the area V, using the surfaces thus introduced, which consists of that part of the area V which is inside the cone K, and outside the cylinder C.

The area V, it is limited by the part of the surface S located inside K, in the tag S, cylinder C and cone K. Formula (3) can be applied to the functions u and v and the area V, which now reads:

$$\iiint_{V'}^{-} vf(x,y,z)dxdydz = \iint_{K'}^{-}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS + \iint_{K'}^{-}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS + \iint_{C}^{-}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS, \quad (4)$$

because it is S(v) = 0, S(u) = f(x,y,z).
At an arbitrary point of the cone K, at a distance l from the vertex P, the value of the function v and $\partial v/\partial n$ are given with:

$$v = \log\left(ctg\,\varphi + \sqrt{ctg^2\varphi - 1}\right), \frac{\partial v}{\partial n} = -\frac{1}{l}\frac{\sqrt{cos2\varphi}}{sin\varphi}.$$

Accordingly when $\varepsilon \to 0$, i.e., when $\varphi \to \frac{\pi}{4}$, we have that it is:

$$\lim_{\varepsilon\to 0} v = \lim_{\varepsilon\to 0}\frac{\partial v}{\partial n} = 0.$$

Now is:

$$\lim_{\varepsilon\to 0}\overline{\iint_{K'}}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS = 0.$$

We cannot calculate the integral over the cylinder C, because we do not know the values of the functions u and $\partial u/\partial n$ on C. However, the limiting value of that integral can be found when $\eta \to 0$.
Indeed, we can take the surface element of the cylinder C dS=$\omega d\eta dz$, where the angle ω varies from 0 to 2π. On C we have:

$$v = \log(z_1 - z) + \sqrt{(z_1 - z)^2 - \eta^2} - \log\eta,$$

and

$$\frac{\partial v}{\partial n} = \frac{1}{\eta} + \frac{\eta}{\sqrt{(z_1-z)^2-\eta^2}(z_1 - z + \sqrt{(z_1-z)^2-\eta^2})}.$$

Therefore, it is valid

$$\lim_{\eta\to 0}\eta v = 0, \lim_{\eta\to 0}\eta\frac{\partial v}{\partial n} = 1,$$

that's why it is

$$\lim_{\eta\to 0}\overline{\iint_{C}}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS = -2\pi\int_{z_0}^{z_1} u(x_1, y_1, z_1)dz,$$

where z_0 is the point where the axis of the cylinder penetrates the surface S.
Because it is:

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to 0}} \overline{\iiint_{V'}} vf(x,y,z)dxdydz = \overline{\iiint_{V}} vf(x,y,z)dxdydz,$$

taking into account (4) that $\varepsilon \to 0$ and $\eta \to 0$ we get that:

$$\overline{\iiint_{V}} vf(x,y,z)dxdydz = \overline{\iint_{S}} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})dS - 2\pi \int_{z_0}^{z_1} u(x_1,y_1,z_1)dz,$$

from which, after differentiation, it follows:

$$u(x_1,y_1,z_1) = -\frac{1}{2\pi}\frac{\partial}{\partial z}(\overline{\iiint_V} vf(x,y,z)dxdydz - \overline{\iint_S}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})dS). \qquad (5)$$

Since the function v is known, the function f is given, and the value of u and $\partial u/\partial n$ on the surface S is given, formula (5) gives the value of the function u at an arbitrary point (x_1,y_1,z_1), which solves the problem [1].

The method for solving the Cauchy problem for hyperbolic equations with two variables originates from B. Riemann. Although Riemann gave it for some special cases, he in fact directly extends it to the most general hyperbolic linear equations with two variables. V. Voltaire extended the Riemann method, but only for the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = f(x,y,z).$$

Before Voltaire's works G, Kirchhof solved the same problem for an equation with four variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = f(x,y,z,t),$$

and then O. Tedone gave a solution to the Cauchy problem for n variables

$$\frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_{n-1}}^2} - \frac{\partial^2 u}{\partial {x_n}^2} = 0.$$

All the mentioned extensions of the Riemann method refer to special equations of the hyperbolic type with more variables. However, J. Hadamard solved the Cauchy problem for an arbitrary hyperbolic equation with n variables[2].
Voltaire's method for solving a differential equation

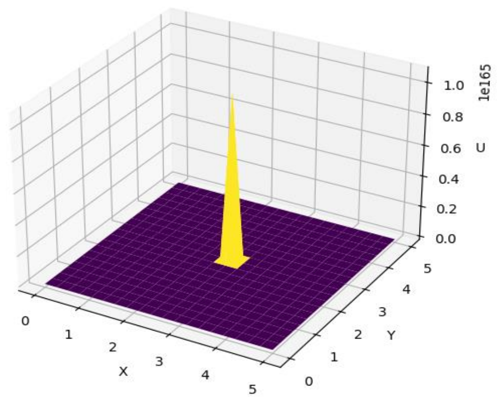

Figure 3: Presentation of The Solution of The Partial Differential Equation By Voltaire's Method

The obtained solution in three-dimensional space shows the spatial distribution of the function u in relation to the parameters *x*, *y* , and *z*. The solution clearly shows the shape and intensity of the function along different values of these parameters. The graph suggests complex changes in the solution of the differential equation with respect to variations in the initial conditions.

6. Goursat's Method

As is well known, a hyperbolic equation can be represented using certain transformations in the form:

$$u_{xy} = a(x,y)u_x + b(x,y)u_y + c(x,y)u + f(x,y). \tag{1}$$

The solution of equation (1) in the domain $\{(x,y)\colon 0 \leq x \leq L, 0 \leq y \leq L\}$ satisfies the conditions:

$$u(x,0)=A(x),\ u(0,y)=B(y), \tag{2}$$

where A and B are given functions such that A(0)=B(0), it is called Gursat's solution, while problem (1)-(2) itself is called Gursat's problem.

In the case that a=b=c=0, the solution to problem (1)-(2) can be determined in the final form.
Indeed, from $u_{xy} = f(x,y)$, after integration by x we get:

$$u_y = u_y(0,y) + \int_0^x f(\xi,y)d\xi, \tag{3}$$

and integrating (3) over y we have

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) + \int_0^y d\eta \int_0^x f(\xi,\eta)d\xi.$$

i.e.,

$$u(x,y) = A(x) + B(y) - A(0) + \int_0^y d\eta \int_0^x f(\xi,\eta)d\xi.$$

Let us now consider the general equation (1). We can replace problem (1)-(2) with the following equivalent problem. Let's solve the integro-differential equation:

$$u(x,y) = \int_0^y \int_0^x (a(\ \xi,\eta)u\xi + b(\xi,\eta)u\eta + c(\xi,\eta)u)d\xi d\eta$$

$$+ A(x)+B(y)-A(0)+ \int_0^y \int_0^x f(\ \xi,\eta)\ d\xi d\eta. \tag{4}$$

We will apply the method of successive approximations to equation (4). We will define for this purpose a series of functions (u_n) using:

$$u_1(x,y) = \mathrm{A(x)} + \mathrm{B(y)} - \mathrm{A(0)} + \int_0^y \int_0^x f(\ \xi,\eta)\ d\xi d\eta,$$

$$u_n(x,y) = u_1(x,y) + \int_0^y \int_0^x (a(\ \xi,\eta)\frac{\partial u_{n-1}}{\partial \xi} + b(\xi,\eta)\frac{\partial u_{n-1}}{\partial \xi} + c(\xi,\eta)u_{n-1})\ d\xi d\eta, \tag{5}$$

where n= 2,3,... Then it is

$$\frac{\partial u_n}{\partial x} = \frac{\partial u_1}{\partial x} + \int_0^x (a(x,\eta)\,\frac{\partial u_{n-1}}{\partial x} + b(x,\eta)\frac{\partial u_{n-1}}{\partial x} + c(x,\eta)u_{n-1})d\eta.$$

$$\frac{\partial u_n}{\partial y} = \frac{\partial u_1}{\partial y} + \int_0^y (a(\xi,y)\,\frac{\partial u_{n-1}}{\partial x} + b(\xi,y)\frac{\partial u_{n-1}}{\partial x} + c(\xi,y)u_{n-1})d\xi. \quad (6)$$

Let's prove that the functional series (u_n) , $(\frac{\partial u_n}{\partial x})$, $(\frac{\partial u_n}{\partial y})$ are uniformly convergent.

As the functions a,b,c are continuous, there exists a constant M such that $|a(x,y)| < M, |b(x,y)| < M, |c(x,y)| < M.$

There is also a constant H such that $|u_1(x,y)| < H$, $\left|\frac{\partial u_1}{\partial x}\right| < H, \left|\frac{\partial u_1}{\partial y}\right| < H.$ Let the above inequalities hold for $0 \le x \le N, 0 \le y \le N.$ Let it be (z_n) defined with:

$$(z_n(x,y) = u_{n+1}(x,y) - u_n(x,y)$$

$$= \int_0^y \int_0^x (a(\xi,\eta)\,\frac{\partial z_{n-1}}{\partial \xi} + b(\xi,\eta)\,\frac{\partial z_{n-1}}{\partial \xi} + c\,(\xi,\eta)z_{n-1}(\xi,\eta))d\,\xi d\eta.$$

We can directly verify that it is:

$$\frac{\partial z_n}{\partial x} = \int_0^y (\mathrm{a}(x,\eta)\,\frac{\partial z_{n-1}}{\partial \xi} + b(x,\eta)\,\frac{\partial z_{n-1}}{\partial \xi} + c\,(x,\eta)z_{n-1}(x,\eta))d\eta,$$

$$\frac{\partial z_n}{\partial y} = \int_0^x (\mathrm{a}\,\xi,y)\,\frac{\partial z_{n-1}}{\partial \xi} + b(\xi,y)\,\frac{\partial z_{n-1}}{\partial \xi} + c\,(\xi,y)z_{n-1}(\xi,y))d\,\xi.$$

It follows from these equalities:

$$|z_1(x,y)| < 3HMxy < 3\mathrm{HM}\frac{(x+y)^2}{2!},$$

$$\left|\frac{\partial z_1}{\partial x}\right| < 3HMy < 3HM(x+y),$$

$$\left|\frac{\partial z_1}{\partial y}\right| < 3HMx < 3HM(x+y),$$

considering that it is $0 \le x \le N, 0 \le y \le N.$ Suppose that the inequalities hold for some n

$$|z_n(x,y)| < 3HM^nK^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}$$

$$\left|\frac{\partial z_n}{\partial x}\right| < 3HM^nK^{n-1}\frac{(x+y)^n}{n!}$$

$$\left|\frac{\partial z_n}{\partial y}\right| < 3HM^nK^{n-1}\frac{(x+y)^n}{n!},$$

where $K=N+2\ge 2$. For n+1 we have:

$$|z_{n+1}| < 3HM^{n+1}K^{n-1}\frac{(x+y)^{n+2}}{(n+2)!}\left(\frac{x+y}{n+3}+2\right) < 3HM^{n+1}K^n\frac{(x+y)^{n+2}}{(n+2)!}$$

$< \frac{3H}{K^2 M} \frac{(2KNM)^{n+1}}{(n+1)!}$,

$$\left|\frac{\partial z_{n+1}}{\partial x}\right| < 3HM^{n+1}K^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}\left(\frac{x+y}{n+3}+2\right)$$

$< 3HM^{n+1}K^{n}\frac{(x+y)^{n+1}}{(n+1)!} < \frac{3H}{K}\frac{(2KNM)^{n+1}}{(n+1)!}$.

On the right-hand side of the above inequalities, the terms of the development e^{2KNM} appear (with accuracy up to one multiplicative constant). The proven inequalities show that the series (u_n), $(\frac{\partial u_n}{\partial x})$, $(\frac{\partial u_n}{\partial y})$ in the given region uniformly converge to the functions, which we will denote

$$u(x,y) = \lim_{n\to\infty} u_n(x,y), \qquad v(x,y) = \lim_{n\to\infty}\frac{\partial u_n}{\partial x}, \qquad w(x,y) = \lim_{n\to\infty}\frac{\partial u_n}{\partial y}.$$

If we take that in (5) and (6) $n \to \infty$, we have that is:

$$u(x,y) = u_1(x,y) + \int_0^y\int_0^x (a(\xi,\eta)\, v + b(\xi,\eta)\, w + c\,(\xi,\eta)u)d\,\xi d\eta,$$

$$v(x,y) = \frac{\partial u_1}{\partial x} + \int_0^y (\mathrm{a}(x,\eta)\, v + b(x,\eta)\, w + c\,(x,\eta)u)d\eta,$$

$$\mathrm{w}(x,y) = \frac{\partial u_1}{\partial y} + \int_0^x (\mathrm{a}(\,\xi,y)\, v + b(\,\xi,y)\, w + c\,(\,\xi,y)u)d\,\xi. \qquad (7)$$

From (7) we get v = u_x, w=u_y, from where we conclude that the required function u satisfies the integro-differential equation:

$$u(x,y) = A(x) + B(y) - A(0) + \int_0^y\int_y^x f(\xi,\eta)d\xi d\eta +$$

$$\int_0^y\int_y^x (a(\xi,\eta)u\xi + b(\xi,\eta)u\eta + c(\xi,\eta)u)\; d\,\xi d\eta. \qquad (8)$$

That every solution of equation (8) satisfies (1) and (2) is verified directly by differentiation.
Let us now prove that the set Gursat problem has a unique solution. In contrast, let there be two identical solutions $(x,y)\to U_i(x,y), i=1,2$, of the given problem. Let's observe the function:

$$(x,y)\to U(x,y) = U_1(x,y) - U_2(x,y).$$

This function satisfies the integro-differential equation:

$$U(x,y) = \int_0^y\int_0^x \left(aU_x + bU_y + cU\right)d\xi d\eta.$$

This equation is homogeneous. Let Q>0 be such a constant that

$$|U(x,y)| < Q, \qquad |U_x(x,y)| < Q, \qquad |U_y(x,y)| < Q.$$

za $0 \le x \le N, 0 \le y \le N$. Based on the rating we performed for the series (z_n), we have:

$$|U(x,y)| < 3QM^{n+1}K^n\frac{(x+y)^{n+2}}{(n+2)!} < \frac{3Q}{K^2M}\frac{(2KNM)^{n+2}}{(n+2)!},$$

for each n. From there it follows against the assumption that it is:

U(x,y)=0, tj., $U_1(x,y) = (x,y)$.

The established contraindication proves the uniqueness of the solution to Gursat's problem.
Gursat's method

Gursat's method

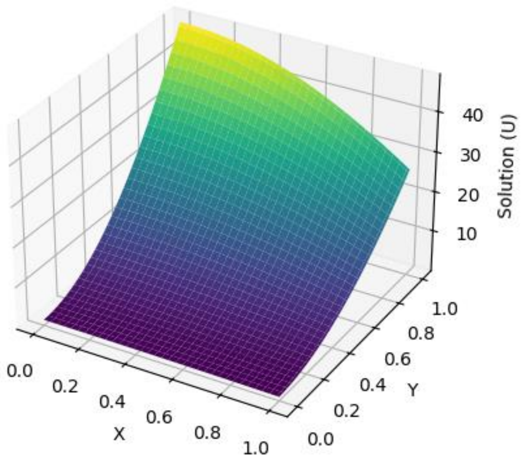

Figure 4: Gursat's Method

The figure shows the solution using Gursat's method for a hyperbolic differential equation. The method uses successive approximations to numerically solve the problem. The solution is obtained by integrating the initial conditions with additional iterations. The label "Gursatov Method" above the image indicates exactly this numerical technique that is used to obtain the displayed 3D graph of the solution.

7. Discussion and Conclusion

This paper studied a variety of numerical methods, including Volterian, Riemannian, and the telegraph equation, with an emphasis on solving hyperbolic differential equations that model the behavior of the wire. We analyzed the results obtained by applying these methods to a specific problem, showing them through 3D graphics.

In the discussion, we highlighted the advantages and disadvantages of each method. For example, Volterra's method may be numerically stable but require a certain number of iterations to converge. The Riemann method, on the other hand, can be faster, but is sensitive to the discretization of space. We used the telegraph equation to model the behavior of the wire under the influence of a changing electric field. Through the implementation and analysis of the results, we noticed that Gursat's method converges towards a solution as the number of iterations increases. This numerical technique shows promising results in solving certain problems and can be applied to a wide range of hyperbolic equations.

The discussion of the paper also includes consideration of the advantages and limitations of Gursat's method. This method can be useful in situations where an analytical solution is not available or difficult to perform. However, the parameters, such as the number of iterations, need to be carefully chosen to achieve convergence. In conclusion, the paper provides insight into the application of numerical methods for solving hyperbolic differential equations and explores the potential of Gursat's method in this context. Further research and comparison with other numerical techniques could contribute to a better understanding of the effectiveness of Gursat's method in different scenarios.

The conclusion of this paper indicates that the choice of a particular method depends on the specific application and characteristics of the problem. We hope that this research will contribute to the understanding of the advantages and limitations of different numerical techniques in the context of solving hyperbolic differential equations, especially in the domain of modeling the behavior of some continuous and discrete systems.

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