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Maxwell's Equations in a Vacuum as a Consequence of the Lorentz Transformation

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Abstract
This article considers implications of the Lorentz transformation for Lorentz covariant scalar and real-valued vector field theories. The Maxwell-Heaviside equations in a vacuum appear as a consequence of the restrictions imposed upon the latter ones. This suggests a discrepancy between the historical development of, and the logical relationship between Maxwell's electrodynamics and special theory of relativity. The results for the Maxwell-Heaviside equations in a vacuum cum grano salis apply to Heaviside's gravito electromagnetic equations as well.

1. Introduction

A Lorentz covariant *aka* manifest Lorentz invariant theory contains only Lorentz scalars, contra- and covariant 4-vectors such as $x^\mu = (ct, \vec{r}\,)$ and $x_\mu = (ct, -\vec{r}\,)$ (signature $+---$), respectively, and their extensions to 4-tensors. This implies strong restrictions onto admissible theories. In this contribution, we concentrate on scalar and real-valued vector fields to eventually obtain Maxwell's equations in a vacuum. All functions will be supposed to be sufficiently continuously differentiable.

To begin with the simplest non-trivial case, Section II considers constraints posed by Lorentz covariance upon Lorentz scalar fields. A physically relevant result is the Klein- Gordon equation.

Although this contribution has been inspired by discussions on the relationship between the historical and logical developments of physical theories in general, it constraints itself to this special example. *Historically*, the special theory of relativity bases on Maxwell equations in a vacuum. Does this corresponds to the *logical* development? To shed light on that question, Section III explores real-valued vector fields.

Finally, Section IV summarizes and concludes this article.

2. Scalar Field Theories

An elementary Lorentz covariant scalar field theory is expected to contain a Lorentz- scalar field amplitude $u(x^\mu)$ and a Lorentz covariant equation of motion. Since this section only serves as a preparation of the main section III, pseudo-scalars are omitted.

For such a field amplitude, the simplest Lorentz covariant differential equation is

$$\partial_\nu u(x^\mu) = a_\nu(x^\mu)\,, \tag{1}$$

where a_ν is a 4-source. That, however, is not an equation of motion. Moreover, by Schwartz's theorem, there are the constraints

$$\partial_\sigma \partial_\nu u = \partial_\sigma a_\nu = \partial_\nu \partial_\sigma u = \partial_\nu a_\sigma \tag{2}$$

(the analogue in 3d is $\nabla \times (\nabla b(\vec{r})) = 0$). We are not aware of a physical application of eq. (1).
The simplest Lorentz scalar operator of motion is the wave operator *aka* d'Alembertian,

$$\Box := \partial^\sigma \partial_\sigma = \frac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2\,. \tag{3}$$

Hence, Euler's *aka* d'Alembert's wave equation,

$$\Box\, u(x^\mu) = f(x^\mu) \tag{4}$$

is Lorentz covariant, if the source function $f(x^\mu)$ is a Lorentz scalar. A special case is the Klein-(Fock-)Gordon(-Schr¨odinger) equation,

$$\Box\,\psi(x^\mu) = \frac{m^2c^2}{\hbar^2}\,\psi(x^\mu)\,, \tag{5}$$

for the wave function ψ of spinless quantum particles of mass m, e.g. pions

3. Real-Valued Vector Fields

Analogously to the foregoing section, an elementary Lorentz covariant theory of real- valued vector fields contains a real-valued 4-vector field amplitude $A^\nu(x^\mu)$ and a Lorentz- covariant equation of motion. (A^ν is a 4-vector, iff $A^\nu A_\nu$ = Lorentz scalar.) The simplest non-trivial differential equation reads

$$\partial_\nu A^\nu(x^\mu) = 0\,. \tag{6}$$

That, however, is not an equation of motion but a constraint, again. If A^ν denotes the electromagnetic 4-potential, eq. (6) is known as Lorenz gauge. In what follows, we will assume that it holds true. (In case of $\partial_\nu A^\nu = s \neq 0$, one can use the gauge freedom to set $A^\nu = \tilde{A}_\nu + \chi^\nu$, where $\partial_\nu \chi^\nu = s$.)

Inspired by the fundamental theorem of vector algebra in $\Re^3$ (Helmholtz's theorem), an alternative operation of first-order derivatives w.r.t. x^μ is

$$\partial^\sigma A^\nu(x^\mu) - \partial^\nu A^\sigma(x^\mu) =: F^{\sigma\nu}(x^\mu)\,. \tag{7}$$

Again, that is not an equation of motion but, this time, a definition. Within electrodynamics, $F^{\sigma\nu}$ equals the Faraday tensor *aka* electromagnetic tensor, electromagnetic field tensor, field strength tensor, Maxwell bivector.

Furthermore, analogously to the scalar wave equation (4), there is the Lorentz covariant vector wave equation as an equation of motion,

$$\Box A^\nu(x^\mu) = f^\nu(x^\mu)\,, \tag{8}$$

where f^ν is a 4-source. Within electromagnetism, $f^\nu = \mu_0 j^\nu$, where μ_0 is the vacuum magnetic permeability *aka* magnetic constant and $j^\nu = (c\rho, \vec{j})$ the 4-current density. It is equivalent with the following equation of second order suggested by relation (7),

$$\partial_\sigma F^{\sigma\nu} = \partial_\sigma\left(\partial^\sigma A^\nu - \partial^\nu A^\sigma\right) = \Box A^\nu = f^\nu\,. \tag{9}$$

Within electromagnetism, $\partial_\sigma F^{\sigma\nu} = f^\nu = \mu_0 j^\nu$ are the inhomogeneous Maxwell-Heaviside equations in a vacuum.

Moreover, there is the Bianchi identity,

$$\partial_\gamma F_{\nu\sigma} + \partial_\nu F_{\sigma\gamma} + \partial_\sigma F_{\gamma\nu} = 0\,. \tag{10}$$

Within electromagnetism, it represents the homogeneous Maxwell-Heaviside equations in a vacuum.

Notice that, in contrast to the analogous case in Helmholtz's theorem in $\Re^3$ (but a constant term), A^ν is not completely determined by eqs. (6). . . (8). One can add to A^ν a smooth function $\partial^\nu\chi(x^\mu)$ which obeys $\chi = 0$. This is the gauge freedom mentioned after eq. (6). $F^{\sigma\nu}$, however, is uniquely determined by eqs. (9) and (10), if, (i), the initial values of j^ν obey the equation of continuity $\partial_\nu j^\nu = 0$ and, (ii), sufficiently smooth initial values $F^{\sigma\nu}(0, \vec{r})$ and correct boundary conditions are prescribed. Both facts are well known from electromagnetism.

4. Summary and Conclusions

We have rearranged well-known facts such that the question arises whether, logically, EITHER the Lorentz transformation is a consequence of Maxwell's theory (historical devel-opment), oR Maxwell equations in a vacuum follow from the restrictions posed by Lorentz covariance as required by Poincar´e's and Einstein's special relativity. The simplest way to do so uses the 4-potential $A^\nu = (c\Phi, \vec{A})$. The scalar Φ and vector $\vec{A}$ potentials enter Maxwell's original equations but not the 'rationalized' Maxwell-Heaviside equations nowadays taught as "Maxwell equations". Because the Lorentz transformation can be derived independently of electromagnetism, that reasoning suggests to *logically* consider the Lorentz transformation to be primary w.r.t. electromagnetism. The results for the Maxwell-Heaviside equations in a vacuum *cum grano* salis apply to Heaviside's gravito electromagnetic equations as well.

On the other hand, the structure of Maxwell's equations in a vacuum can be derived from the continuity equation (see Appendix A). This suggest Maxwell's equations in a vacuum to be *primary* against the Lorentz transformation.

Therefore, *logically*, the Lorentz transformation and Maxwell's theory should be treated on equal footing.

Measuring the forces between spherical charges (Coulomb) and thin parallel conductors (Amp`ere), respectively, as a function of their distances, it follows that there is a universal constant of dimension velocity. However, this gives not any hint to the fact that that velocity equals the speed of a wave of electric and magnetic fields, nor that light is an electromagnetic wave.

These investigations have been performed in the spirit of Heinrich Hertz's program, viz., to represent classical mechanics such that all other branches of physics can be derived from it. It is powerful in its concreteness of approach. And it is limited by the fact that one arrives at new relationships, the physical meaning of which is unclear. However, to our knowledge, there is none generalization of an existing theory which does better.

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Appendix A: Derivation of the structure of Maxwell's Equations in a Vacuum from the Continuity Equation

Following Mie's textbook, one can proceed as follows (recall that all functions are assumed to be sufficiently smooth) [1].

The continuity equation reads (actually, ρ and $\vec{j}$ are the free charge and current densities)

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0\,. \tag{A1}$$

Now, there is always a vector field $\vec{D}$ such that

$$\nabla \cdot \vec{D} = \rho\,. \tag{A2}$$

This corresponds to Gauss' law.

Further, inserting Gauss' law (A2) into the continuity equation (A1) yields, using Schwartz's theorem,

$$\nabla \cdot \vec{j} + \frac{\partial}{\partial t} \nabla \cdot \vec{D} = \nabla \cdot \left\{ \vec{j} + \frac{\partial}{\partial t} \vec{D} \right\} = 0\,. \tag{A3}$$

Hence, by virtue of $\nabla \cdot (\nabla\times) \equiv 0$, there is a vector field $\vec{H}$ such that

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}\,. \tag{A4}$$

This corresponds to Maxwell's flux law. Of course, the physical content of the equations does not follow from such kind of derivation. The same holds true for eq. (5), the structure of which follows from setting $u \propto$ f in the wave equation (4). Similarly, all derivations of the Lorentz transformation not using light imply a characteristic velocity, the value of which can be determined only through experiments or the connection with other theories, notably the Lorentz force.

Reference

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