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General Divisibility Algorithms

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Abstract

In my paper, a divisibility rule named the Vezir function is described, which has been generalized for every integer and encompasses all previously established divisibility rules. Subsequently, the paper discusses other areas where this function is instrumental. These include simplifying ratios, conducting rationality tests, generating prime numbers, and more. Towards the end of the paper, a set of hypotheses is also presented. Among these, the most significant is the novel approach involving the application of the Vezir function for prime number generation.

Keywords: Arithmetic Functions, Discrete Mathematics, Primes, Number Theory.

1. Introduction

I would like to start our discussion with the question, what are integers or natural numbers? Natural numbers have been a tool for measuring a reality that humans have used for centuries. Integers, on the other hand, are an abstraction that includes a bit more of the abstract aspect at a point where they depart from reality. What are the characteristics of natural numbers then? Of course, one of them is whether they are divisible by each other or not. Naturally, there are other features, but the focus of this article is mostly on this aspect. When one natural number is divided by another natural number resulting in another natural number, we call this divisibility, and by using this divisibility, we generate some special numbers, such as prime numbers. I will touch on this topic shortly, but first, let's talk about the rules. What are the rules of divisibility, or in other words, the rules of being divisible? Of course, the rules of divisibility, just like natural numbers, have always been intriguing throughout history. Because imagine having such a power that directly gives you the divisibility of these natural numbers, which you compulsively use throughout your life. It would be nice, of course. And for centuries, many people have researched this and derived many divisibility rules. But there was a problem; none of them were generalized. Until a recent article summarized this. Of course, similar rules are also written in ancient texts, but the first known article written on this subject is "General Divisibility Criteria" by A. A. Grinberg and S. Luryi. This article generally talks about the most basic general divisibility rule. In my article, there is a group of formulas that includes all divisibility rules and also the function of these formulas in finding prime numbers. In addition, some simplification methods related to the formulas are also mentioned. These formulas are divided into two groups: "Fil" and "Vezir," and with the help of both, all divisibility rules can be used, and prime numbers can be found. Additionally, a few hypotheses provide important data.

Lemma 1.1. [1] [2] $x = S_0(x,m) + S_1(x,m)10^m$

Proof. By the definition, we said that $S_1(x,m)$ is the part that x's digits are greater than the m'th digit, in the same meaning but different definition we can say that we put zeros all the digits less than m'th digit. Then we divide it by $10m$. So, by this definition, we can say that this also can be defined as $\frac{x-S_0(x,m)}{10^m}$. Because $x - S_0(x,m)$ is like putting zeros all the digits less than m'th digit and dividing it with 10^m by the definition. Hence, we have the following:

$$\frac{x - S_0(x,m)}{10^m} = S_1(x,m)$$
$$x - S_0(x,m) = S_1(x,m)10^m$$
$$x = S_0(x,m) + S_1(x,m)10^m$$

Lemma 1.2. $a = D_0(x,y) + D_1(x,y)10^y + D_2(x,y)10^{2y} + D_3(x,y)10^{3y} \ldots$

Proof. By the definition, $D_i(x,y)$ is the part that is between the iy and $(i+1)$y'th digits. So, in other saying first we should delete the part that all the $< iy$'th part and then we should delete all the $> iy$'th part. This will be possible if we get the part that x's digits greater than iy'th digit. Then we get its digits less than y'th digits. Because we already deleted the iy digit and all the parts have y digits. But we can write these with our $S()$ function too. For the part that x's digits are greater than iy'th digit: $S_1(x,iy)$. Than its digits less than y'th digits: $S_0(S_1(x,iy),y)$. By the Lemma 1.1, we can rewrite equation as: $S_1(x,iy) - S_1(S_1(x,iy),y)10^y$. $S_1(S_1(x,iy),y)$ is the part that x's digits greater than iy'th digits less than y'th digits. This will add up to this: $S_1(S_1(x,iy),y) = S_1(x,(i+1)y)$. Hence, we get: $D_i(x,y) = S_1(x,iy) - S_1(x,(i+1)y)10^y$. So, by our lemma's equation: $D_0(x,y) +$

$D_1(x,y)10_y + D_2(x,y)10^{2y} + D_3(x,y)10^{3y}$ ...

$$\sum_{j=0}^{\left\lfloor \frac{Decimal(x)}{y} \right\rfloor} D_j(x,y)10^{jy}$$

$$\sum_{j=0}^{\left\lfloor \frac{Decimal(x)}{y} \right\rfloor} (S_1(x,jy) - S_1(x,(j+1)y)10^y)10^{jy}$$

$$\sum_{j=0}^{\left\lfloor \frac{Decimal(x)}{y} \right\rfloor} S_1(x,jy)10^{jy} - S_1(x,(j+1)y)10^{(j+1)y}$$

$$\sum_{j=0}^{\left\lfloor \frac{Decimal(x)}{y} \right\rfloor} S_1(x,jy)10^{jy} - \sum_{j=0}^{\left\lfloor \frac{Decimal(x)}{y} \right\rfloor} S_1(x,(j+1)y)10^{(j+1)y}$$

$S_0 : \mathbf{Z} \to \mathbf{Z}$ where $S_0(x,m)$ is x's digits less than the m'th digit.
$S_1 : \mathbf{Z} \to \mathbf{Z}$ where $S_1(x,m)$ is x's digits greater than the m'th digit.
$D : \mathbf{Z} \to \mathbf{Z}$ where $D_i(x, y)$ is when the digits of x are divided into y parts, the i'th part.
$\lfloor\rfloor : \mathbf{R} \to \mathbf{R}$ where $\lfloor x \rfloor$ is the closest integer to x that smaller than x.
$Decimal : \mathbf{Z} \to \mathbf{Z}$ where $Decimal\ (x)$ is the decimal count of x.

$$S_1(x,0) + \sum_{j=0}^{\left\lfloor \frac{Decimal(x)}{y} \right\rfloor} S_1(x,(j+1)y)10^{(j+1)y} - \sum_{j=0}^{\left\lfloor \frac{Decimal(x)}{y} \right\rfloor} S_1(x,(j+1)y)10^{(j+1)y}$$

$$S_1(x,0)$$

$$x$$

Lemma 1.3. *$\forall x, n :\in \mathbf{N}$ if for $\forall m \in \mathbf{Z}^+$ that $m \leq n$, $^{78}Gcd(x, P_m) = 1$ and $x < P_{n+1}^2$, then a is a prime number.*

Proof. In this proof, we should know that if the natural number x has no other same divisors other than 1 with all the equal or smaller primes than P_n. Then this means the minimum prime divisor that can be in x's divisors is the first prime that is bigger than P_n. And it is P_{n+1}. But the minimum integer that is a multiple of P_{n+1} and not a prime and usable in this situation is P^2_{n+1}. Then this means if $x < P^2_{n+1}$, x must be a prime.

Lemma 1.4. *$\forall x, n \in \mathbf{N}$ if $^9Gcd(x, P_n\#) = 1$ and $x < P_{n+1}^2$, then x is a prime number.*

Proof. If a $Gcd(x, P_n\#) = 1$ then x has no other same divisors than 1 with all the equal or smaller primes than P_n. From the Lemma 1.3, also if $x < P^2_{n+1}$, x must be a prime.

2. Simple General Divisibility Algorithms

2.1. First General Divisibility Rule: Vezir Function

Definition 2.1. $vz : \mathbf{Z} \to \mathbf{Z}$ where $\forall x, y :\in \mathbf{Z}$, $\forall m :\in \mathbf{N}$:

$$vz_m(x,y) = S_0(x,m)S_1(y,m) - S_1(x,m)S_0(y,m)$$

Lemma 2.2.

$$\frac{x}{y} \in \mathbf{Z} \Rightarrow \frac{vz_m(x,y)}{y} \in \mathbf{Z}$$

Proof. By the definition,

$$vz_m(x,y) = S_0(x,m)S_1(y,m) - S_1(x,m)S_0(y,m)$$

Making a change of variable using Lemma 1.1,

$$vz_m(x,y) = (x - S_1(x,m)10^m)S_1(y,m) - S_1(x,m)(y - S_1(y,m)10^m)$$

$$vz_m(x,y) = S_1(y,m)x - S_1(x,m)S_1(y,m)10^m + S_1(x,m)S_1(y,m)10^m - S_1(x,m)y$$

$$vz_m(x,y) = S_1(y,m)x - S_1(x,m)y$$

Dividing both sides by y,

$$\frac{vz_m(x,y)}{y} = \frac{S_1(y,m)x - S_1(x,m)y}{y}$$

$|| : \mathbf{R} \to \mathbf{R}$ where $|x|$ is the positive version of x.
P : set of elements that has no other same divisors that are smaller than it other than 1.
$Gcd : \mathbf{Z} \to \mathbf{Z}$ where $Gcd(x,y)$ is the greatest common divisor of x and y.
$P\# : \mathbf{N} \to \mathbf{N}$ where $P_n\#$ multiple of all the primes that equal or smaller than P_n

$$\frac{vz_m(x,y)}{y} = S_1(y,m)\frac{x}{y} - S_1(x,m)$$

then in this setup, there is only one none-integer part and it is $\frac{x}{y}$. So: $\frac{x}{y} \in \mathbf{Z} \Rightarrow \frac{vz_m(x,y)}{y} \in \mathbf{Z}$.

Lemma 2.3.

$$\Big(Gcd(y, S_1(y,m)) = 1\Big) \Rightarrow \left(\left(\frac{vz_m(x,y)}{y} \in \mathbf{Z}\right) \iff \left(\frac{x}{y} \in \mathbf{Z}\right)\right)$$

Proof. From the first proof,

$$\frac{vz_m(x,y)}{y} = S_1(y,m)\frac{x}{y} - S_1(x,m)$$

If *Gcd(b, S_1(y,m))* = 1, there will be 2 cases:

$\frac{x}{y} \notin \mathbf{Z}$**:** Then $S_1(y,m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{vz_m(x,y)}{y} \notin \mathbf{Z}$

$\frac{x}{y} \in \mathbf{Z}$**:** Then $S_1(y,m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{vz_m(x,y)}{y} \in \mathbf{Z}$

If we try to come from right hand side to left, there will be 2 cases too:

$\frac{vz_m(x,y)}{y} \notin \mathbf{Z}$**:** Then only possibility is $S_1(y,m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{x}{y} \notin \mathbf{Z}$

$\frac{vz_m(x,y)}{y} \in \mathbf{Z}$**:** Then only possibility is $S_1(y,m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{x}{y} \in \mathbf{Z}$ because $Gcd(y, S_1(y,m)) = 1$, in equation $S_1(y,m)$ and y won't be simplified.

So all the possibilities in the if and only if equation will be accurate.

Example 2.4. Normally in number 21's divisibility rule, we should look at if it's divisible by 3 and then 7. This is very complicated. But the reason is that its second digit has no same divisors except 1 with 21. It is very easy to use in the Vezir algorithm. In Table 1 in the first column, some numbers are used with 21's algorithm. In other columns, there are results of the algorithm and in the first row there are the number of reusing of the algorithm and all of the example tables have the same system at:

Example 2.5. Normally 31 is a prime number so also it is so big that can't have a rule. But in the Vezir algorithm it has a very easy rule that can make numbers smaller a lot.

Example 2.6. 101 is a very big number for getting some divisibility rule but also it has 3 digits that can be a possibility for us in the Vezir algorithm in lower variables. Our lower variable is 1 or 2. Both of them are great but if we choose to get a small result, lower variable

2 is better.

2.2. Vezir Function For Smaller Numbers:

Lemma 2.7. *In the Vezir algorithm, we should get* $Gcd(y, S_1(y,m)) = 1$ *for using it in any number. So if y is smaller than* 10 *then it won't be able to be* $Gcd(y, S_1(y,m)) = 1$. *Then we should use its multiples. For any natural number k, yk number also will be usable in Vezir algorithm for y:*

$$\frac{x}{y} \in \mathbf{Z} \Rightarrow \frac{vz_m(x, yk)}{y} \in \mathbf{Z}$$

Algorithm Reapply Count ↦	1	2	3	4	5	6
126	0	0	0	0	0	0
11.214	1.113	105	0	0	0	0
93.450	9.345	9.24	84	0	0	0
398.034	39.795	3.969	378	21	0	0
194.481	19.446	1.932	189	0	0	0
693	63	0	0	0	0	0
2.310	231	21	0	0	0	0
21	0	0	0	0	0	0
441	42	0	0	0	0	0
9.261	924	84	0	0	0	0

Table 1

Algorithm Reapply Count ↦	1	2	3	4	5	6
279	0	0	0	0	0	0
31.248	3.100	310	31	0	0	0
2.687.328	268.708	26.846	2666	248	0	0
340.101	34.007	3.379	310	31	0	0
62	0	0	0	0	0	0
651	62	0	0	0	0	0
31	0	0	0	0	0	0
961	93	0	0	0	0	0
29.791	2.976	279	0	0	0	0
923.521	92.349	9.207	899	62	0	0
28.629.151	2.862.912	286.285	28.613	2.852	279	0

Table 2

Proof. in Lemma 2.2 we saw that,

$$vz_m(x, yk) = S_1(yk, m)x - S_1(x, m)yk$$

$$\frac{vz_m(x, yk)}{y} = S_1(yk, m)\frac{x}{y} - S_1(x, m)\frac{yk}{y}$$

$$\frac{vz_m(x, yk)}{y} = S_1(yk, m)\frac{x}{y} - S_1(x, m)k$$

Algorithm Reapply Count $\mapsto$	1	2	3	4	5
1.111	0	0	0	0	0
101	0	0	0	0	0
10.201	101	0	0	0	0
1.030.301	10.302	101	0	0	0
104.060.401	1.040.603	10.403	101	0	0
10.510.100.501	105.101.004	1.051.006	10.504	101	0

Table 3

Here we came back to our first proof. Also, in here we should think like if $\frac{x}{y} \in \mathbf{Z}$, then $\frac{vz_m(x,yk)}{y} \in \mathbf{Z}$.

In this theorem, we also saw that the Vezir function's result gave us the same divisors of x and y.

Lemma 2.8.

$$\left(Gcd(y, S_1(yk,m)) = 1\right) \Rightarrow \left(\left(\frac{vz_m(x,yk)}{y} \in \mathbf{Z}\right) \iff \left(\frac{x}{y} \in \mathbf{Z}\right)\right)$$

Proof. Same as Lemma 2.3 proof we will see 2 possibilities for each side:

$\frac{x}{y} \notin \mathbf{Z}$: Then $S_1(yk,m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{vz_m(x,yk)}{y} \notin \mathbf{Z}$ because Gcd(y, S1(y, m)) = 1, in equation, S1(y, m) and y won't be simplified.
$\frac{x}{y} \in \mathbf{Z}$: Then $S_1(yk,m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{vz_m(x,yk)}{y} \in \mathbf{Z}$

If we try to come from the right-hand side to the left, there will be 2 cases too:

$\frac{vz_m(x,yk)}{y} \notin \mathbf{Z}$: Then only possibility is $S_1(yk,m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{x}{y} \notin \mathbf{Z}$
$\frac{vz_m(x,yk)}{y} \in \mathbf{Z}$: Then only possibility is $S_1(yk,m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{x}{y} \in \mathbf{Z}$ because $Gcd(y, S_1(yk,m)) = 1$, in equation, $S_1(yk,m)$ and y won't be simplified.

So, all the possibilities in the if and only if equation will be accurate.

2.3. Multiple Times Applied Vezir Function

Definition 2.9. For future usage, we will use a function called multiple Vezir: $\forall k :\in \mathbf{Z}^+$

$$vz_m(x,y)^{(k)} = \underbrace{vz_m(\ldots(vz_m(x,y)\ldots),y)}_{\text{k pieces}}$$

$$vz_m(x,y)^{(0)} = x$$

2.4. Connection Between Vezir Function and Greatest Common Divisor

Lemma 2.10.

$$Gcd(vz_m(x,y)) = Gcd(x,y)$$

Proof. We know that for some integer of x, applying it Vezir with some integer y will give us a value that multiple of common divisors x and y. By the proof in Lemma 2.7. This means Vezir will give us all the common divisors with some different multiple that is not a divisor for y. So, this will prove our claim.

2.5. More Generalized Version of Vezir Algorithm

Definition 2.11. Vezir function already has 3 variables but it's nothing compared to the most generalized version of it. But first, we have Fil function. It has 5 variables and it can make some newborn divisibility rules.

$$fl : \mathbf{Z} \rightarrow \mathbf{Z} \text{ where } \forall a, b, c, n :\in \mathbf{Z}, \forall m :\in \mathbf{N}$$
$$fl_m^n(a, b, c) = S_0(a, m)S_1(bc, m) - S_1(a, m)(S_0(bc, m) - nb)$$

Lemma 2.12.

$$\frac{a}{b} \in \mathbf{Z} \Rightarrow \frac{fl_m^n(a, b, c)}{b} \in \mathbf{Z}$$

Proof.

$$fl_m^n(a, b, c)$$
$$S_0(a, m)S_1(bc, m) - S_1(a, m)(S_0(bc, m) - nb)$$
$$S_0(a, m)S_1(bc, m) - S_1(a, m)S_0(bc, m) + S_1(a, m)nb$$
$$vz_m(a, bc) + S_1(a, m)nb$$

If we try to divide by b:

$$\frac{vz_m(a, bc)}{b} + \frac{S_1(a, m)nb}{b}$$
$$\frac{vz_m(a, bc)}{b} + S_1(a, m)n$$

then all of the formulas turned into the same function with some additions as the Vezir function. Our claim will be proved as same as Vezir's. If $\frac{vz_m(a,bc)}{b} \in \mathbf{Z}$ than $fl_m^n(a, b, c) \in \mathbf{Z}$. But we proved that the only possibility that $\frac{vz_m(a,bc)}{b} \in \mathbf{Z}$ is $\frac{a}{b} \in \mathbf{Z}$. This proves our claim. □

Lemma 2.13.

$$\left(Gcd(b, S_1(b, m)) = 1\right) \Rightarrow \left(\left(\frac{fl_m^n(a, b, c)}{b} \in \mathbf{N}\right) \iff \left(\frac{a}{b} \in \mathbf{N}\right)\right)$$

Proof. Same as other proof when it is an if and only if the situation, we will see 2 possibilities for each 2 sides. But now we don't need much work to prove it, just because it has a Vezir function in it. We are looking for only the non-integer part so we can delete the integer part in the function. When deleting it we only got the $\frac{vz_m(a,bc)}{b}$ part. But we proved it before. So, our claim is proved.

Example 2.14. 98 is a very big and uncomfortable number to work with. This means it is very hard for a divisibility rule. But if we use $fl_2^{-1}(a, 98, 2)$ then it will be much easier, $fl_2^{-1}(a, 98, 2) = S_0(a, 2) + 2S_1(a, 2)$:

Algorithm Reapply Count ↦	1	2	3	4	5
588	98	98	98	98	98
602.112	12.054	294	98	98	98
98	98	98	98	98	98
9.604	196	98	98	98	98
941.192	18.914	392	98	98	98
92.236.816	1.844.752	36.946	784	98	98
9.039.207.968	180.784.226	3.615.710	72.324	1.470	98

Table 4

Example 2.15. Normally 7 has a really simple divisibility rule. It rule can be found in Fil as this example. $fl_1^{-1}(a, 7, 8) = 5S_0(a, 1) + S_1(a, 1)$:

Algorithm Reapply Count ↦	1	2	3	4	5
21	7	35	28	42	14
2.310	231	28	42	14	21
45.360	4.536	483	63	21	7
7	35	28	42	14	21
49	49	49	49	49	49
343	49	49	49	49	49
2.401	245	49	49	49	49
16.807	1.715	196	49	49	49
117.649	11.809	1.225	147	49	49
823.543	82.369	8.281	833	98	49
5.764.801	576.485	57.673	5.782	588	98

Table 5

Example 2.16. Normally 3 and 9 have very good design and simple divisibility rules and we will find them in later the article but for now, there is a simple rule too.

For 3, $fl_1^{-1}(a,3,4) = S_0(a,1) + S_1(a,1)$ same as 9, $fl_1^{-1}(a,9,2) = S_0(a,1) + S_1(a,1)$.

Here are some examples for 3:

Algorithm Reapply Count ↦	1	2	3	4	5	6	7	8
21	3	3	3	3	3	3	3	3
2.310	231	24	6	6	6	6	6	6
691.200	69.120	6.912	693	72	9	9	9	9
3	3	3	3	3	3	3	3	3
9	9	9	9	9	9	9	9	9
27	9	9	9	9	9	9	9	9
81	9	9	9	9	9	9	9	9
243	27	9	9	9	9	9	9	9
729	81	9	9	9	9	9	9	9
2.178	225	27	9	9	9	9	9	9
6.534	657	72	9	9	9	9	9	9
19.602	1.962	198	27	9	9	9	9	9
58.806	5.886	594	63	9	9	9	9	9

Table 6

2.6. Multiple Times Applied Fil Function

Definition 2.17. For future usage, we will use a function called multiple Fil: $\forall k \in$ **Z+**

$$fl_m^n(a,b,c)^{(k)} = \underbrace{fl_m^n(\ldots(fl_m^n(a,b,c)\ldots),b,c)}_{\text{k pieces}}$$

$$fl_m^n(a,b,c)^{(0)} = a$$

2.7. Connection Between Fil Function and Greatest Common Divisor

Definition 2.18. We know that Fil has a Vezir inside, also in the part that has no Vezir there is a y factor so this equation will be accurate:

$$Gcd(fl_m^n(a,b,c),b) = Gcd(a,b)$$

3. Ratio Simplifying with General Divisibility Rules

Definition 3.1. $Svz : \mathbf{Z} \to \mathbf{Z}$ where $\forall a,b :\in \mathbf{Z}$, $\forall m :\in \mathbf{N}$:

$$Svz_m(a,b) = \frac{S_1(a,m) + \frac{vz_m(a,b)}{b}}{S_1(b,m)}$$

Lemma 3.2.

$$Svz_m(a,b) = \frac{a}{b}$$

Proof.

$$Svz_m(a,b)$$

$$\frac{S_1(a,m) + \frac{vz_m(a,b)}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m) + \frac{S_1(b,m)a - S_1(a,m)b}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m) + S_1(b,m)\frac{a}{b} - S_1(a,m)}{S_1(b,m)}$$

$$\frac{S_1(b,m)\frac{a}{b}}{S_1(b,m)}$$

$$\frac{a}{b}$$

3.1. Infinite Series Version of Simplifying Function

Lemma 3.3.

$$Svz_m(a,b) = \sum_{n=0}^{\infty} \frac{S_1(vz_m(a,b)^{(n)},m)}{S_1(b,m)^{n+1}}$$

Proof. In this proof, we will open out simplifying function's ratio part:

$$Svz_m(a,b)$$

$$\frac{S_1(a,m) + \frac{vz_m(a,b)}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{\frac{vz_m(a,b)}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{\frac{S_1(vz_m(a,b),m) + \frac{vz_m(a,b)^{(2)}}{b}}{S_1(b,m)}}{S_1(b,m)}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{S_1(vz_m(a,b),m)}{S_1(b,m)^2} + \frac{\frac{vz_m(a,b)^{(2)}}{b}}{S_1(b,m)^2}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{S_1(vz_m(a,b),m)}{S_1(b,m)^2} + \frac{\frac{S_1(vz_m(a,b)^{(2)},m) + \frac{vz_m(a,b)^{(3)}}{b}}{S_1(b,m)}}{S_1(b,m)^2}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{S_1(vz_m(a,b),m)}{S_1(b,m)^2} + \frac{S_1(vz_m(a,b)^{(2)},m)}{S_1(b,m)^3} \dots$$

$$\sum_{n=0}^{\infty} \frac{S_1(vz_m(a,b)^{(n)},m)}{S_1(b,m)^{n+1}}$$

3.2. Inverse Function of Vezir Function

Definition 3.4. $vz^{(-1)} : \mathbf{Z} \to \mathbf{Z}$ where $\forall a, b, n :\in \mathbf{Z}$, $\forall m :\in \mathbf{N}$: $vz_m(a,b)_n^{(-1)}$ is the n'th value of inverse function of Vezir. This is because there is no one answer to the inverse function of Vezir.

Lemma 3.5,

$$vz_m(a,b)_n^{(-1)} = \frac{nb+a}{S_1(b,m)}$$

Proof. We will start by putting the value of $vz_m(a,b)_n^{(-1)}$ to simplifying function:

$$\frac{vz_m(a,b)_n^{(-1)}}{b} = \frac{S_1(vz_m(a,b)_n^{(-1)}, m) + \frac{vz_m(vz_m(a,b)_n^{(-1)}, b)}{b}}{S_1(b,m)}$$

$$\frac{vz_m(a,b)_n^{(-1)}}{b} = \frac{S_1(vz_m(a,b)_n^{(-1)}, m) + \frac{a}{b}}{S_1(b,m)}$$

$$vz_m(a,b)_n^{(-1)} = \frac{S_1(vz_m(a,b)_n^{(-1)}, m)b + a}{S_1(b,m)}$$

Here we can see the only unknown part on the left-hand side is $S_1(vz_m(a,b)n\ ,m)$. This is because we can choose whatever we want so with a little bit change of variable. We can say that,

$$vz_m(a,b)_n^{(-1)} = \frac{nb+a}{S_1(b,m)}$$

3.3. Multiple Times Applied Inverse Vezir Function

Definition 3.6. For future usage, we will use a function called multiple inverse Vezir:

$$\forall k :\in \mathbf{Z}^+, \forall R \subset \mathbf{Z}$$

$$vz_m(x,y)_R^{(-k)} = \underbrace{vz_m(\ldots(vz_m(x,y)_{R_{\lfloor \frac{k}{2} \rfloor}}^{(-1)} \ldots)_{R_{k-1}}^{(-1)}, y)_{R_k}^{(-1)}}_{\text{k pieces}}$$

Here we can also find the Fil function version of all of them but they will be the same proofs as these proofs so there is no need to prove non-necessary functions.

3.4. Infinite Repeated Inverse Function of Vezir

Lemma 3.7.

$$\lim_{h\to\infty} vz_m(a,b)_R^{-h} = \sum_{i=0}^{\infty} \frac{R_i b}{S_1(b,m)^{i+1}}$$

Proof.

$$\lim_{h\to\infty} vz_m(a,b)_R^{(-h)}$$

$$\lim_{h\to\infty} \frac{bR_1 + \frac{bR_2 + \frac{bR_3 + \ldots}{S_1(b,m)}}{S_1(b,m)}}{S_1(b,m)}$$

$$\lim_{h\to\infty} \frac{bR_1}{S_1(b,m)} + \frac{bR_2}{S_1(b,m)^2} + \frac{bR_3}{S_1(b,m)^3} + \ldots$$

$$\lim_{h\to\infty} \sum_{i=0}^{h} \frac{bR_i}{S_1(b,m)^{i+1}}$$

$$\sum_{i=0}^{\infty} \frac{bR_i}{S_1(b,m)^{i+1}}$$

Conjecture: The pattern of R set is the pattern of multiple times applied Vezir function of infinitely applied reverse Vezir function.

4. Full Versions of General Divisibility Rules

4.1. Full Version of Vezir Algorithm

Definition 4.1. Normally, we use *S*() function in Vezir algorithm. But we can also use the *D*() and *Decimal*() functions. And the formula will be like this: $\forall a,b \in \mathbf{Z}, \forall m \in \mathbf{N}$,

$$Mvz_m(a,b) = \sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) S_1(b,m)^{Decimal(a)-i} S_0(b,m)^i$$

Lemma 4.2.

$$\frac{a}{b} \in \mathbf{N} \Rightarrow \frac{Mvz_m(a,b)}{b} \in \mathbf{N}$$

Proof.

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) S_1(b,m)^{Decimal(a)-i} S_0(b,m)^i$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m)(b - 10^m S_1(b,m))^i S_1(b,m)^{Decimal(a)-i}$$

11 12 13

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right) S_1(b,m)^{Decimal(a)-i} \right) \right) \tag{4.1}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j S_1(b,m)^{Decimal(a)-i} \right) \right) \right) \tag{4.2}$$

$\forall x : |x| < 1 | \sum_{i=0}^{\infty} x^{-i} = \frac{1}{1-\frac{1}{x}}$ [4]

$\forall x, y \in \mathbf{R}, \forall n \in \mathbf{N}^+ | (x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ [5]

$\forall x, y \in \mathbf{N} | \binom{x}{y} = \frac{x!}{y!(x-y)!}$

$\forall x \in \mathbf{N} | x! = x(x-1)(x-2)\cdots 3 \cdot 2 \cdot 1$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j S_1(b,m)^{Decimal(a)-i} \right) + S_1(b,m)^{Decimal(a)-i}(-10^m S_1(b,m))^i \right) \right) \tag{4.3}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j S_1(b,m)^{Decimal(a)-i} \right) + S_1(b,m)^{Decimal(a)}(-10^m)^i \right) \right) \tag{4.4}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j S_1(b,m)^{Decimal(a)-i} \right) \right) \right) \tag{4.5}$$

$$+\sum_{i=0}^{Decimal(a)-1}(-1)^i D_i(a,m) S_1(b,m)^{Decimal(a)}(-10^m)^i$$

$$\begin{aligned}\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big(b^{i-j}(-10^m S_1(b,m))^j \\ S_1(b,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\ +\sum_{i=0}^{Decimal(a)-1} D_i(a,m) S_1(b,m)^{Decimal(a)} 10^{mi}\end{aligned} \tag{4.6}$$

$$\begin{aligned}\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big(b^{i-j}(-10^m S_1(b,m))^j \\ S_1(b,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\ +S_1(b,m)^{Decimal(a)}\sum_{i=0}^{Decimal(a)-1} D_i(a,m) 10^{mi}\end{aligned} \tag{4.7}$$

$$\begin{aligned}\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big(b^{i-j}(-10^m S_1(b,m))^j \\ S_1(b,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\ +S_1(b,m)^{Decimal(a)} a\end{aligned} \tag{4.8}$$

$$\begin{aligned}b\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big(b^{i-j-1}(-10^m S_1(b,m))^j \\ S_1(b,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\ +aS_1(b,m)^{Decimal(a)}\end{aligned} \tag{4.9}$$

Here we have two sides of the equation one of them is the right side of the plus sign which has a factor called a and other is the left side of the plus sign which has a factor called y. Also, other factors are integers. So, if we try to divide it by y:

$$\begin{aligned}\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big(b^{i-j-1}(-10^m S_1(b,m))^j \\ S_1(b,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\ +\frac{aS_1(b,m)^{Decimal(a)}}{b}\end{aligned} \tag{4.10}$$

In here if $\frac{a}{b} \in \mathbf{Z}$ then the whole equation will be an integer. This proves our claim.

Lemma 4.3.

$$\Big(Gcd(b, S_1(b,m)) = 1\Big) \Rightarrow \left(\left(\frac{Mvz_m(a,b)}{b} \in \mathbf{N}\right) \iff \left(\frac{a}{b} \in \mathbf{N}\right)\right)$$

Proof. Same as all of the if and only if proofs we see a $Gcd(b,S_1(b,m)) = 1$ situation. In here only thing we should think is that if $Gcd(b,S_1(b,m)) = 1$ then for every natural number x, $Gcd(b,S_1(b,m)^x) = 1$. So, all the proof is coming from Lemma 2.3.

Example 4.4. In 101 we see the same result as the normal 101 divisibility rule:

Algorithm Reapply Count ↦	1	2	3	4	5	6	7	8
101	0	0	0	0	0	0	0	0
10.201	0	0	0	0	0	0	0	0
1.030.301	11	0	0	0	0	0	0	0
104.060.401	0	0	0	0	0	0	0	0

Table 7

Example 4.5. In 11 we found the same result as the normal 11 divisibility rule:

Algorithm Reapply Count ↦	1	2	3	4	5	6	7	8
564.537.600	0	0	0	0	0	0	0	0
209	11	0	0	0	0	0	0	0
1.331	0	0	0	0	0	0	0	0
14.641	0	0	0	0	0	0	0	0
161.051	0	0	0	0	0	0	0	0
1.771.561	0	0	0	0	0	0	0	0
19.487.171	0	0	0	0	0	0	0	0
40909	22	0	0	0	0	0	0	0

Table 8

4.2. Full Version of Fil Algorithm

Definition 4.6. Normally, we use $S()$ function in Fil algorithm. But we can also use the $D()$ and *Decimal* functions. And the formula will be like this: $\forall a, b, c, n \in \mathbb{Z}, \forall m \in \mathbb{N}$,

$$Mfl_m^n(a,b,c) = \sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) S_1(bc,m)^{Decimal(a)-i} (S_0(bc,m) - bn)^i$$

Lemma 4.7.

$$\frac{a}{b} \in \mathbf{N} \Rightarrow \frac{Mfl_m^n(a,b,c)}{b} \in \mathbf{N}$$

Proof.

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) S_1(bc,m)^{Decimal(a)-i} (S_0(bc,m) - bn)^i$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) (bc - bn - 10^m S_1(bc,m))^i S_1(bc,m)^{Decimal(a)-i}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i} \binom{i}{j} \left((bc-bn)^{i-j} (-10^m S_1(bc,m))^j \right) S_1(bc,m)^{Decimal(a)-i} \right) \right) \tag{4.11}$$

$$
\begin{aligned}
&\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i}\binom{i}{j}\Big((bc-bn)^{i-j}(-10^mS_1(bc,m))^j \\
&\qquad S_1(bc,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg)
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big((bc-bn)^{i-j}(-10^mS_1(bc,m))^j \\
&\qquad S_1(bc,m)^{Decimal(a)-i}\Big) \\
&\qquad +S_1(bc,m)^{Decimal(a)-i}(-10^mS_1(bc,m))^i\Bigg)\Bigg)
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
&\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big((bc-bn)^{i-j}(-10^mS_1(bc,m))^j \\
&\qquad S_1(bc,m)^{Decimal(a)-i}\Big) \\
&\qquad +S_1(bc,m)^{Decimal(a)}(-10^m)^i\Bigg)\Bigg)
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
&\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big((bc-bn)^{i-j}(-10^mS_1(bc,m))^j \\
&\qquad S_1(bc,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\
&\qquad +\sum_{i=0}^{Decimal(a)-1}(-1)^iD_i(a,m)S_1(bc,m)^{Decimal(a)}(-10^m)^i
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
&\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big((bc-bn)^{i-j}(-10^mS_1(bc,m))^j \\
&\qquad S_1(bc,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\
&\qquad +\sum_{i=0}^{Decimal(a)-1}D_i(a,m)S_1(bc,m)^{Decimal(a)}10^{mi}
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
&\sum_{i=0}^{Decimal(a)-1}(-1)^i\Bigg(D_i(a,m)\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}\Big((bc-bn)^{i-j}(-10^mS_1(bc,m))^j \\
&\qquad S_1(bc,m)^{Decimal(a)-i}\Big)\Bigg)\Bigg) \\
&\qquad +S_1(bc,m)^{Decimal(a)}\sum_{i=0}^{Decimal(a)-1}D_i(a,m)10^{mi}
\end{aligned} \tag{4.17}
$$

$$\sum_{i=0}^{Decimal(a)-1}(-1)^i\left(D_i(a,m)\left(\sum_{j=0}^{i-1}\binom{i}{j}\left((bc-bn)^{i-j}(-10^mS_1(b,m))^j S_1(bc,m)^{Decimal(a)-i}\right)\right)\right) + S_1(bc,m)^{Decimal(a)}a \tag{4.18}$$

$$(bc-bn)\sum_{i=0}^{Decimal(a)-1}(-1)^i\left(D_i(a,m)\left(\sum_{j=0}^{i-1}\binom{i}{j}\left((bc-bn)^{i-j-1}(-10^mS_1(bc,m))^j S_1(bc,m)^{Decimal(a)-i}\right)\right)\right) + aS_1(bc,m)^{Decimal(a)} \tag{4.19}$$

Here we have two sides of the equation one of them is the right side of the plus sign which has a factor called a and the other is the left side of the plus sign which has a factor called *b*. Also, other factors is integers. So, if we try to divide it by *b*:

$$\frac{(bc-bn)}{b}\sum_{i=0}^{Decimal(a)-1}(-1)^i\left(D_i(a,m)\left(\sum_{j=0}^{i-1}\binom{i}{j}\left((bc-bn)^{i-j-1}(-10^mS_1(bc,m))^j S_1(bc,m)^{Decimal(a)-i}\right)\right)\right) + \frac{aS_1(bc,m)^{Decimal(a)}}{b} \tag{4.20}$$

$$(c-n)\sum_{i=0}^{Decimal(a)-1}(-1)^i\left(D_i(a,m)\left(\sum_{j=0}^{i-1}\binom{i}{j}\left((bc-bn)^{i-j-1}(-10^mS_1(b,m))^j S_1(bc,m)^{Decimal(a)-i}\right)\right)\right) + \frac{aS_1(bc,m)^{Decimal(a)}}{b} \tag{4.21}$$

In here if $\frac{a}{b} \in \mathbf{Z}$ then the whole equation will be an integer. This proves our claim.

Lemma 4.8.

$$\left(Gcd(b, S_1(bc,m)) = 1\right) \Rightarrow \left(\left(\frac{Mfl_m^n(a,b,c)}{b} \in \mathbf{N}\right) \iff \left(\frac{a}{b} \in \mathbf{N}\right)\right)$$

Proof. Same as all of the if and only if proofs we see a $Gcd(b,S_1(bc,m)) = 1$ situation. In here only thing we should think is that if $Gcd(b,S_1(b,m)) = 1$ then for every natural number x, $Gcd(b,S_1(bc,m)^x) = 1$. So, all the proof is coming from Lemma 2.3.

Example 4.9. In 3 and 9 we get the same result as the normal divisibility rule.
In 3:

Algorithm Reapply Count ↦	1	2	3	4	5	6	7	8
9	9	9	9	9	9	9	9	9
42	6	6	6	6	6	6	6	6
84.240	18	9	9	9	9	9	9	9
81	9	9	9	9	9	9	9	9
729	18	9	9	9	9	9	9	9
6.561	18	9	9	9	9	9	9	9
59.049	18	9	9	9	9	9	9	9
531.441	18	9	9	9	9	9	9	9
4.782.969	45	9	9	9	9	9	9	9
43.046.721	27	9	9	9	9	9	9	9
39.999.999	57	12	3	3	3	3	3	3

Table 9

5. Finding Prime Numbers with General Divisibility Rules

5.1. Finding Prime Numbers with Vezir Function

Lemma 5.1. *Vezir is a divisibility rule for every number. Also, it is a Gcd() simpler. With these truths, we can use to find prime numbers like this by using Lemma 1.4:*

$\forall a, n \in \mathbf{N}, \forall m \in \mathbf{N}$ *that* $Gcd(P_n\#, S_1(P_n\#, m)) = 1$. *If* $Gcd(a, P_n\#) = 1, |vz_m(a, P_n\#)| < P_{n+1}^2$ *then,*

$$|vz_m(a, P_n\#)| \in \mathbf{P}$$

Proof. We know that $Gcd(a, P_n\#) = Gcd(vz_m(a, P_n\#))$ from **Lemma 1.1**. And also using the knowledge of Lemma 1.4. we can say that If $Gcd(a, P_n\#) = 1, |vz_m(a, P_n\#)| < P_{n+1}^2$ then, $|vz_m(a, P_n\#)| \in \mathbf{P}$. So proof finishes by itself.

5.2. Finding Prime Numbers with Fil Function:

Lemma 5.2. *Vezir is a divisibility rule for every number. Also, it is a Gcd() simpler. With these truths, we can use to find prime numbers like this by using Lemma 1.4:*

$\forall a, c, n, k \in \mathbf{N}, \forall m \in \mathbf{N}$ *that* $Gcd(P_n\#, S_1(P_n\#, m)) = 1$. *If* $Gcd(a, P_n\#) = 1, |fl_m^n(a, P_n\#, c)| < P_{n+1}^2$ *then,*

$$|fl_m^n(a, P_n\#, c)| \in \mathbf{P}$$

Proof. We know that $Gcd(a, P_n\#) = Gcd(fl_m^n(a, P_n\#, c))$ from Lemma 1.1. And also using the knowledge of Lemma 1.4. we can say that If $Gcd(a, P_n\#) = 1, |fl_m^n(a, P_n\#, c)| < P_{n+1}^2$ then, $|fl_m^n(a, P_n\#, c)| \in \mathbf{P}$. So proof finishes by itself.

Example 5.3. We can use the $P_n\#$ type of numbers. For example, 2310:

Algorithm Reapply Count $\mapsto$	$\lvert vz_2(a, 2310)\rvert \downarrow$
703	1
101	13
401	17
503	19
1	23
403	29
1307	31
601	37
1103	41
701	47
2011	53
103	59
1007	61
1409	67
907	71
2213	79
1711	83
2113	89
1109	97
1201	97
607	101
$\vdots$	$\vdots$

Table 10

Example 5.4. We can use not only the $P_n\#$ type of numbers. Also, we can use their multiples. For example, 120:

Algorithm Reapply Count $\mapsto$	$\lvert vz_2(a,120)\rvert \downarrow$
101	19
103	17
107	13
113	7
127	7
131	11
137	17
133	13
139	19
143	23
149	29
151	31
157	37
161	41
163	43
167	47

Table 11

5.3. Checkmate Theorem

Lemma 5.5.

$$fl_m^1(a,b,1) - vz_m(a,b) = S_1(a,m)b$$

Proof.

$$fl_m^1(a,b,1) - vz_m(a,b)$$
$$vz_m(a,b) + S_1(a,m)b - vz_m(a,b)$$
$$S_1(a,m)b$$

5.4. Main Conjecture

For $\exists a,b,m \in \mathbf{N}$, $flm^1(a,b,1)$ and $vz_m(a,b)$ will be positive and by using the properties in Lemma 5.1 and Lemma 5.2 we can make them prime numbers. Also, in their properties, y must be an even number. Also using the checkmate theorem in Lemma 5.5, we can get their difference set to $S_1(a,m)b$. So, if we choose a and y correctly we can find 2 prime numbers that have a difference of $S_1(a,m)b$. The question is, can we find infinitely many? There is no full proof in here but all the things are connecting to this solution. The last job is to find a formula for it with these properties. The main conjecture is we can find a general formula with these functions [1-5].

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