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Research Article

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Beyond the Riemann Hypothesis [v2]

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Abstract

The functional equation of real variable that Riemann used in his paper was subjected to elementary operations. And I obtained a lot of complex functional equations that the Riemann zeta function follows respectively. Here, functional equation transformations were the main methods for obtaining the complex functional equations. Half of those are equivalent to the complete symmetric functional equation that the Riemann Xi function follows, and one of those has an origin symmetry with correction terms. From the origin symmetric functional equation including correction terms, the representation containing the leading term of the zeta function for any complex number was obtained. And the Riemann hypothesis was proved by applying reduction to absurdity. Moreover the general representation containing the leading term of the zeta function for any odd number of 3 or more was also obtained.

By suitably combining those functional equations, I observed a new explicit formula for the zeta function.

The Riemann hypothesis was again proven using the deductive method. And two types of general representations for the zeta function for any odd number of either 3 or 7, or more, were also obtained from the explicit formula. In total, three types of general representations for the zeta function for any odd number of either 3 or 7, or more, were discovered.

Conversely, I defined a new function, named the Chi function, for the left side of the origin symmetric functional equation that includes corrective terms. The Chi function is similar to the Riemann Xi function and exhibits origin symmetry. Furthermore, I defined a new function, the eta function, which is similar to the zeta function. The eta function's pole and trivial zeros are the same as those of the zeta function. Furthermore, the Chi and eta functions have the same non-trivial zeros on the imaginary axis. And I proposed a generalized Riemann hypothesis for the eta function that states that all non-trivial zeros lie on the imaginary axis. Since I was able to discover the explicit formula for the eta function, the deductive method was used to prove the generalized Riemann hypothesis for the eta function.

As you know, there are different types of transformations between the prime numbers and the non-trivial zeros of the zeta function. I discovered that there are comparable transformations between the prime numbers and the non-trivial zeros of the eta function. Based on the results of numerical experiments, I proposed some conjectures referring to relationships between the prime numbers and non-trivial zeros of the eta function.

So, let's take a journey to the sister planets Zeta and Eta. The author is only a guide who invites you there.

1. Introduction

When considering the representations for the Riemann zeta function ζ (S) for any odd number of 3 or more, especially ζ (3), ζ (5), and ζ (7), a lot of series representations are known. Conversely, each representation for the Riemann zeta function for any odd number of 9 or more, which gives 1 as remainder when divided by 4 (even any series representation) is unknown. This situation seems to be quite similar to the situation at the end era when the Fermat's last theorem was once called the Fermat's conjecture. This was my impression when I first started research. The initial research motive was to obtain a unified representation for the Riemann zeta function for any odd number of 3 or more, excluding the classification of remainders when divided by 4. I discovered the new functional equation that the Riemann zeta function follows using a technique of functional equation transformation, which is newly devised in my research process. This resulted in the unified representation, which does not require classification for any odd number of 3 or more. After solving the initial research problem and three years later, I observed that the newly discovered functional equation has origin symmetry, including correction terms, when the domain of definition is expanded to the whole complex plane. Moreover, I had a flash of inspiration that the Riemann hypothesis would be solved by creating simultaneous equations consisting of the complete and the origin symmetric functional equations with correction terms.

Here, the complete symmetric functional equation that the Riemann zeta function follows is shown

(1.1) $$\pi^{-\frac{S}{2}}\Gamma\left(\frac{S}{2}\right)\zeta\left(S\right)=\pi^{-\frac{1-S}{2}}\Gamma\left(\frac{1-S}{2}\right)\zeta\left(1-S\right),\quad S\in\mathbb{C}\backslash\{0,1\}.$$

Riemann used the following functional equation derived from the transformation formula (automorphic) for the theta function to show the complete symmetric functional equation:

(1.2) $$\sum_{n=1}^{\infty}e^{-\pi n^2 x}+\frac{1}{2}=\frac{1}{\sqrt{x}}\left(\sum_{n=1}^{\infty}e^{-\frac{\pi n^2}{x}}+\frac{1}{2}\right),\quad x>0.$$

Following Riemann, I approached the Riemann hypothesis using the functional equation (1.2) as a starting point "[1, 2]." In this article, I assumed that the complex variable of the Riemann zeta function was θ, excluded referring to pioneering research results. In addition, when both sides are infinite at the same time and the equality is established, as in functional equation (1.1), to show poles explicitly for other functional equations as well, the condition is excluded from the conditions for the equality to be established. Preliminary knowledge is described in Section 3, and additional knowledge is described in Section 19 as a supporting information, with the results used in Section 4 and after if necessary.

2. Preparations

I develop arguments by assuming that the theta function's functional equation derived from the transformation formula (automorphic) is known. Substituting x with x^2 for the positive continuous variable x,

(2.1) $$\sum_{n=1}^{\infty}e^{-\pi n^2 x^2}+\frac{1}{2}=\frac{1}{x}\left(\sum_{n=1}^{\infty}e^{-\frac{\pi n^2}{x^2}}+\frac{1}{2}\right),\quad x>0.$$

By multiplying both sides by x, the following functional equation is obtained:

(2.2) $$\sum_{n=1}^{\infty}xe^{-\pi n^2 x^2}+\frac{x}{2}=\sum_{n=1}^{\infty}e^{-\frac{\pi n^2}{x^2}}+\frac{1}{2},\quad x>0.$$

Both sides of the functional equation (2.2) can be differentiated infinitely many times, indicating that the functional equation (2.2) belongs to the C^{∞} class. Although it is equivalent to the functional equation (2.2), the following functional equation is constructed as a means to find truth:

(2.3) $$\frac{1}{x}=\left(1+2\sum_{m=1}^{\infty}e^{-\pi m^2 x^2}\right)\Big/\left(1+2\sum_{m=1}^{\infty}e^{-\frac{\pi m^2}{x^2}}\right),\quad x>0.$$

For convenience, I define the denominator of the right side of the functional equation (2.3) as the function $W(x)$.

(2.4) $$W\left(x\right):=1+2\sum_{m=1}^{\infty}e^{-\frac{\pi m^2}{x^2}},\quad x>0.$$

The functional equation of first-order differentiation is obtained by differentiating both sides of the functional equation (2.2) by x.

(2.5) $$\sum_{n=1}^{\infty}e^{-\pi n^2 x^2}-2\pi\sum_{n=1}^{\infty}n^2x^2e^{-\pi n^2 x^2}+\frac{1}{2}=2\pi\sum_{n=1}^{\infty}\frac{n^2}{x^3}e^{-\frac{\pi n^2}{x^2}},\quad x>0.$$

Differentiation of both sides of the functional equation (2.5) by x provides the following functional equation of second-order differentiation:

(2.6) $$l^{(2)}\left(x\right)=r^{(2)}\left(x\right),\quad x>0.$$

Where

$$l^{(2)}\left(x\right):=\frac{d}{dx}\left(\sum_{n=1}^{\infty}e^{-\pi n^2x^2}-2\pi\sum_{n=1}^{\infty}n^2x^2e^{-\pi n^2x^2}+\frac{1}{2}\right)=2\pi\sum_{n=1}^{\infty}\left(-3n^2x+2\pi n^4x^3\right)e^{-\pi n^2x^2}.$$

$$r^{(2)}\left(x\right):=\frac{d}{dx}\left(2\pi\sum_{n=1}^{\infty}\frac{n^2}{x^3}e^{-\frac{\pi n^2}{x^2}}\right)=2\pi\sum_{n=1}^{\infty}\left(-3n^2x^{-4}+2\pi n^4x^{-6}\right)e^{-\frac{\pi n^2}{x^2}}.$$

The order of differentiation is indicated by the numeral in the right-shoulder parentheses.
I decide that the functional equation (2.6) is called the second-order $\mathrm{I_r}$ type functional equation. Suffix r means "real." This rule will be followed in the future.
Formally, the functional equation of the second-order $\mathrm{I_r}$ type is

$$l_{[2]}(x) = r_{[2]}(x), \quad x > 0. \tag{2.7}$$

Where

$$l_{[2]}(x) := l^{(2)}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x + 2\pi n^4 x^3\right) e^{-\pi n^2 x^2}.$$
$$r_{[2]}(x) := r^{(2)}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x^{-4} + 2\pi n^4 x^{-6}\right) e^{-\frac{\pi n^2}{x^2}}.$$

The numeral in the suffix $[*]$ shows the order of the functional equation. The right side of the second-order $\mathrm{I_r}$ type functional equation is considered to be as follows:

$$r_{[2]}(x) = 2\pi \sum_{n=1}^{\infty} \frac{1}{x} \times \left(-3n^2 x^{-3} + 2\pi n^4 x^{-5}\right) e^{-\frac{\pi n^2}{x^2}}. \tag{2.8}$$

The functional equation's relation (2.3) is introduced into the second-order $\mathrm{I_r}$ type functional equation, and both sides are multiplied by $W(x)$ to obtain the following functional equation:

$$a_{[2]}(x) + f_{[2]}(x) = b_{[2]}(x) + g_{[2]}(x), \quad x > 0. \tag{2.9}$$

Where

$$a_{[2]}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x + 2\pi n^4 x^3\right) e^{-\pi n^2 x^2}.$$
$$f_{[2]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-6n^2 x + 4\pi n^4 x^3\right) e^{-\pi\left(n^2 x^2 + \frac{m^2}{x^2}\right)}.$$
$$b_{[2]}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x^{-3} + 2\pi n^4 x^{-5}\right) e^{-\frac{\pi n^2}{x^2}}.$$
$$g_{[2]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-6n^2 x^{-3} + 4\pi n^4 x^{-5}\right) e^{-\pi\left(m^2 x^2 + \frac{n^2}{x^2}\right)}.$$

I decide that the functional equation (2.9) is called the second-order $\mathrm{II_r}$ type functional equation.
Differentiation of both sides of the functional equation (2.6) by x provides the following third-order differentiation functional equation:

$$l^{(3)}(x) = r^{(3)}(x), \quad x > 0. \tag{2.10}$$

Where

$$l^{(3)}(x) := \frac{d}{dx} l^{(2)}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 + 12\pi n^4 x^2 - 4\pi^2 n^6 x^4\right) e^{-\pi n^2 x^2}.$$
$$r^{(3)}(x) := \frac{d}{dx} r^{(2)}(x) = 2\pi \sum_{n=1}^{\infty} \left(12n^2 x^{-5} - 18\pi n^4 x^{-7} + 4\pi^2 n^6 x^{-9}\right) e^{-\frac{\pi n^2}{x^2}}.$$

To obtain the following third-order $\mathrm{I_r}$ type functional equation, multiply both sides of the functional equation (2.10) with x:

$$l_{[3]}(x) = r_{[3]}(x), \quad x > 0. \tag{2.11}$$

Where

$$l_{[3]}(x) := x l^{(3)}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x + 12\pi n^4 x^3 - 4\pi^2 n^6 x^5\right) e^{-\pi n^2 x^2}.$$
$$r_{[3]}(x) := x r^{(3)}(x) = 2\pi \sum_{n=1}^{\infty} \left(12n^2 x^{-4} - 18\pi n^4 x^{-6} + 4\pi^2 n^6 x^{-8}\right) e^{-\frac{\pi n^2}{x^2}}.$$

The functional equation's relation (2.3) is introduced into the third-order $\mathrm{I_r}$ type functional equation, and both sides are multiplied by $W(x)$ to obtain the following functional equation:

$$a_{[3]}(x) + f_{[3]}(x) = b_{[3]}(x) + g_{[3]}(x), \quad x > 0. \tag{2.12}$$

Where

$$a_{[3]}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x + 12\pi n^4 x^3 - 4\pi^2 n^6 x^5\right) e^{-\pi n^2 x^2}.$$

$$f_{[3]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-6n^2 x + 24\pi n^4 x^3 - 8\pi^2 n^6 x^5\right) e^{-\pi\left(n^2 x^2 + \frac{m^2}{x^2}\right)}.$$

$$b_{[3]}(x) = 2\pi \sum_{n=1}^{\infty} \left(12n^2 x^{-3} - 18\pi n^4 x^{-5} + 4\pi^2 n^6 x^{-7}\right) e^{-\frac{\pi n^2}{x^2}}.$$

$$g_{[3]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(24n^2 x^{-3} - 36\pi n^4 x^{-5} + 8\pi^2 n^6 x^{-7}\right) e^{-\pi\left(m^2 x^2 + \frac{n^2}{x^2}\right)}.$$

I decide that the functional equation (2.12) is called the third-order $\mathrm{II_r}$ type functional equation. The differentiation of both sides of functional equation (2.10) by x provides the fourth-order differentiation functional equation:

$$l^{(4)}(x) = r^{(4)}(x), \quad x > 0. \tag{2.13}$$

Where

$$l^{(4)}(x) := \frac{d}{dx} l^{(3)}(x) = 2\pi \sum_{n=1}^{\infty} \left(30\pi n^4 x - 40\pi^2 n^6 x^3 + 8\pi^3 n^8 x^5\right) e^{-\pi n^2 x^2}.$$

$$r^{(4)}(x) := \frac{d}{dx} r^{(3)}(x) = 2\pi \sum_{n=1}^{\infty} \left(-60n^2 x^{-6} + 150\pi n^4 x^{-8} - 72\pi^2 n^6 x^{-10} + 8\pi^3 n^8 x^{-12}\right) e^{-\frac{\pi n^2}{x^2}}.$$

Both sides of the functional equation (2.13) are multiplied by x^2 and both sides of the third-order $\mathrm{I_r}$ type functional equation are added to each other to obtain the following fourth-order $\mathrm{I_r}$ type functional equation:

$$l_{[4]}(x) = r_{[4]}(x), \quad x > 0. \tag{2.14}$$

Where

$$l_{[4]}(x) := x^2 l^{(4)}(x) + l_{[3]} = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x + 42\pi n^4 x^3 - 44\pi^2 n^6 x^5 + 8\pi^3 n^8 x^7\right) e^{-\pi n^2 x^2}.$$

$$r_{[4]}(x) := x^2 r^{(4)}(x) + r_{[3]} = 2\pi \sum_{n=1}^{\infty} \left(-48n^2 x^{-4} + 132\pi n^4 x^{-6} - 68\pi^2 n^6 x^{-8} + 8\pi^3 n^8 x^{-10}\right) e^{-\frac{\pi n^2}{x^2}}.$$

The functional equation's relation (2.3) is introduced into the fourth-order $\mathrm{I_r}$ type functional equation, and both sides are multiplied by $W(x)$ to obtain the following functional equation:

$$a_{[4]}(x) + f_{[4]}(x) = b_{[4]}(x) + g_{[4]}(x), \quad x > 0. \tag{2.15}$$

Where

$$a_{[4]}(x) = 2\pi \sum_{n=1}^{\infty} \left(-3n^2 x + 42\pi n^4 x^3 - 44\pi^2 n^6 x^5 + 8\pi^3 n^8 x^7\right) e^{-\pi n^2 x^2}.$$

$$f_{[4]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-6n^2 x + 84\pi n^4 x^3 - 88\pi^2 n^6 x^5 + 16\pi^3 n^8 x^7\right) e^{-\pi\left(n^2 x^2 + \frac{m^2}{x^2}\right)}.$$

$$b_{[4]}(x) = 2\pi \sum_{n=1}^{\infty} \left(-48n^2 x^{-3} + 132\pi n^4 x^{-5} - 68\pi^2 n^6 x^{-7} + 8\pi^3 n^8 x^{-9}\right) e^{-\frac{\pi n^2}{x^2}}.$$

$$g_{[4]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-96n^2 x^{-3} + 264\pi n^4 x^{-5} - 136\pi^2 n^6 x^{-7} + 16\pi^3 n^8 x^{-9}\right) e^{-\pi\left(m^2 x^2 + \frac{n^2}{x^2}\right)}.$$

I decide that the functional equation (2.15) is called the fourth-order $\mathrm{II_r}$ type functional equation.

Differentiation of both sides of the functional equation (2.13) by x provides the following fifth-order differentiation functional equation:

$$l^{(5)}(x) = r^{(5)}(x), \quad x > 0. \tag{2.16}$$

Where

$$l^{(5)}(x) := \frac{d}{dx} l^{(4)}(x) = 2\pi \sum_{n=1}^{\infty} \begin{pmatrix} 30\pi n^4 - 180\pi^2 n^6 x^2 \\ + 120\pi^3 n^8 x^4 - 16\pi^4 n^{10} x^6 \end{pmatrix} e^{-\pi n^2 x^2}.$$

$$r^{(5)}(x) := \frac{d}{dx} r^{(4)}(x) = 2\pi \sum_{n=1}^{\infty} \begin{pmatrix} 360 n^2 x^{-7} - 1320\pi n^4 x^{-9} + 1020\pi^2 n^6 x^{-11} \\ - 240\pi^3 n^8 x^{-13} + 16\pi^4 n^{10} x^{-15} \end{pmatrix} e^{-\frac{\pi n^2}{x^2}}.$$

Both sides of the functional equation (2.16) are multiplied by x^3 and both sides of the fourth-order $\mathrm{I_r}$ type functional equation are added to each other to obtain the following fifth-order $\mathrm{I_r}$ type functional equation:

$$l_{[5]}(x) = r_{[5]}(x), \quad x > 0. \tag{2.17}$$

Where

$$l_{[5]}(x) := x^3 l^{(5)}(x) + l_{[4]} = 2\pi \sum_{n=1}^{\infty} \begin{pmatrix} -3n^2 x + 72\pi n^4 x^3 - 224\pi^2 n^6 x^5 \\ + 128\pi^3 n^8 x^7 - 16\pi^4 n^{10} x^9 \end{pmatrix} e^{-\pi n^2 x^2}.$$

$$r_{[5]}(x) := x^3 r^{(5)}(x) + r_{[4]} = 2\pi \sum_{n=1}^{\infty} \begin{pmatrix} 312 n^2 x^{-4} - 1188\pi n^4 x^{-6} + 952\pi^2 n^6 x^{-8} \\ - 232\pi^3 n^8 x^{-10} + 16\pi^4 n^{10} x^{-12} \end{pmatrix} e^{-\frac{\pi n^2}{x^2}}.$$

The functional equation's relation (2.3) is introduced into the fifth-order $\mathrm{I_r}$ type functional equation, and both sides are multiplied by $W(x)$ to obtain the following functional equation:

$$a_{[5]}(x) + f_{[5]}(x) = b_{[5]}(x) + g_{[5]}(x), \quad x > 0. \tag{2.18}$$

Where

$$a_{[5]}(x) = 2\pi \sum_{n=1}^{\infty} \begin{pmatrix} -3n^2 x + 72\pi n^4 x^3 - 224\pi^2 n^6 x^5 \\ + 128\pi^3 n^8 x^7 - 16\pi^4 n^{10} x^9 \end{pmatrix} e^{-\pi n^2 x^2}.$$

$$f_{[5]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix} -6n^2 x + 144\pi n^4 x^3 - 448\pi^2 n^6 x^5 \\ + 256\pi^3 n^8 x^7 - 32\pi^4 n^{10} x^9 \end{pmatrix} e^{-\pi\left(n^2 x^2 + \frac{m^2}{x^2}\right)}.$$

$$b_{[5]}(x) = 2\pi \sum_{n=1}^{\infty} \begin{pmatrix} 312 n^2 x^{-3} - 1188\pi n^4 x^{-5} + 952\pi^2 n^6 x^{-7} \\ - 232\pi^3 n^8 x^{-9} + 16\pi^4 n^{10} x^{-11} \end{pmatrix} e^{-\frac{\pi n^2}{x^2}}.$$

$$g_{[5]}(x) = 2\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix} 624 n^2 x^{-3} - 2376\pi n^4 x^{-5} + 1904\pi^2 n^6 x^{-7} \\ - 464\pi^3 n^8 x^{-9} + 32\pi^4 n^{10} x^{-11} \end{pmatrix} e^{-\pi\left(m^2 x^2 + \frac{n^2}{x^2}\right)}.$$

I decide that the functional equation (2.18) is called the fifth-order $\mathrm{II_r}$ type functional equation.

3 Preliminary Knowledge

3.1 The Gamma Function Γ (Z)

The gamma function $\Gamma(Z)$ is an absolutely convergent function under the condition of $\mathrm{Re}(Z) > 0$ "[3]."
The following equation provides its definition:

$$\Gamma(Z) := \int_0^{\infty} x^{Z-1} e^{-x}\, dx, \quad \mathrm{Re}(Z) > 0. \tag{3.1}$$

The formula can be used to calculate the value of the gamma function for any positive integer.

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}. \tag{3.2}$$

The two formulae can be used to calculate value of the gamma function for any positive half-integer.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{3.3}$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}, \quad n \in \mathbb{N}. \tag{3.4}$$

Table shows the values for the gamma function for positive half-integers

n	0	1	2	3	4	5	6	7	8
$\frac{2n+1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$
$\Gamma\left(\frac{2n+1}{2}\right)$	$\sqrt{\pi}$	$\frac{\sqrt{\pi}}{2}$	$\frac{3\sqrt{\pi}}{4}$	$\frac{15\sqrt{\pi}}{8}$	$\frac{105\sqrt{\pi}}{16}$	$\frac{945\sqrt{\pi}}{32}$	$\frac{10395\sqrt{\pi}}{64}$	$\frac{135135\sqrt{\pi}}{128}$	$\frac{2027025\sqrt{\pi}}{256}$

Table. 3.1

Analytic continuation extends the domain of definition of the gamma function Γ (*Z*) to the whole complex plane. The gamma function has simple poles at Z = 0,−1,−2, · · · , i.e.,

$$\frac{1}{\Gamma(1-n)} = 0, \quad n \in \mathbb{N}. \tag{3.5}$$

Residue of the gamma function at *Z* = 0 is shown as follows:

$$\text{Res}\,(0; \Gamma(Z)) = 1. \tag{3.6}$$

Residue of the gamma function at *Z* = −*n* is shown as follows:

$$\text{Res}\,(-n; \Gamma(Z)) = \frac{(-1)^n}{n}, \quad n \in \mathbb{N}. \tag{3.7}$$

Except for the poles, the gamma function is regular in the whole complex plane.
In the entire complex plane, the gamma function has no zero, i.e.,

$$\Gamma(Z) \neq 0, \quad Z \in \mathbb{C}. \tag{3.8}$$

The value of the gamma function for any negative half-integer can be obtained by the formula.

$$\Gamma\left(-\frac{2n-1}{2}\right) = (-1)^n \frac{2^n\sqrt{\pi}}{(2n-1)!!}, \quad n \in \mathbb{N}. \tag{3.9}$$

Table shows the values for the gamma function for negative half-integers

n	1	2	3	4	5	6	7	8	9
$-\frac{2n-1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$	$-\frac{9}{2}$	$-\frac{11}{2}$	$-\frac{13}{2}$	$-\frac{15}{2}$	$-\frac{17}{2}$
$\Gamma\left(-\frac{2n-1}{2}\right)$	$-2\sqrt{\pi}$	$\frac{4\sqrt{\pi}}{3}$	$-\frac{8\sqrt{\pi}}{15}$	$\frac{16\sqrt{\pi}}{105}$	$-\frac{32\sqrt{\pi}}{945}$	$\frac{64\sqrt{\pi}}{10395}$	$-\frac{128\sqrt{\pi}}{135135}$	$\frac{256\sqrt{\pi}}{2027025}$	$-\frac{512\sqrt{\pi}}{34459425}$

Table. 3.2

The difference formula for the gamma function is shown as follows:

$$\Gamma(1+Z) = Z\,\Gamma(Z), \quad Z \in \mathbb{C}\backslash\{-1,-2,-3,\cdots\}. \tag{3.10}$$

The reciprocal formula for the gamma function is shown as follows:

$$\Gamma(Z)\,\Gamma(1-Z) = \frac{\pi}{\sin(\pi Z)}, \quad Z \in \mathbb{C}\backslash\mathbb{Z}. \tag{3.11}$$

3.2 The Riemann Zeta Function *ζ* (*S*)

The Riemann zeta function *ζ* (*S*) is an absolutely convergent function under the condition of Re (*S*) > 1 ”[3].”
Its definition is provided by the following equation, and it has the Euler product representation:

$$\zeta(S) := \sum_{n=1}^{\infty} \frac{1}{n^S} = \prod_{\text{p:prim}} \left(1-p^{-S}\right)^{-1}, \quad \text{Re}\,(S) > 1. \tag{3.12}$$

In this case, the product of the right side crosses all prime numbers.
The Bernoulli numbers B_n are provided by the following equation of definition:

$$\frac{x}{e^x-1} =: \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi. \tag{3.13}$$

Table shows the values for the Bernoulli numbers B_n

k	0	1	2	3	4	5	6	7	8	9	10
B_{2k}	1	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$
B_{2k+1}	$-\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0

Table. 3.3

The formula can be used to calculate the exact value of the Riemann zeta function for any even number of 2 or more.

$$\zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}, \quad n \in \mathbb{N}. \tag{3.14}$$

Table shows the exact values for the Riemann zeta function for even numbers of 2 or more

n	1	2	3	4	5	6	7	8
$2n$	2	4	6	8	10	12	14	16
$\zeta(2n)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	$\frac{691\,\pi^{12}}{638512875}$	$\frac{2\,\pi^{14}}{18243225}$	$\frac{3617\,\pi^{16}}{325641566250}$

Table. 3.4

In 1979, R. Apery proved that $\zeta(3)$ is an irrational number using the following series representation "[4]":

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}}. \tag{3.15}$$

The following series representation for $\zeta(5)$ is adopted from Ramanujan's note book "[4, 5]":

$$\zeta(5) = \frac{\pi^5}{294} - \frac{72}{35} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{2\pi n} - 1)} - \frac{2}{35} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{2\pi n} + 1)}. \tag{3.16}$$

The following series representations for $\zeta(3)$ and $\zeta(7)$ are adopted from Ramanujan's note book, too:

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3 (e^{2\pi n} - 1)}, \quad \zeta(7) = \frac{19\pi^7}{56700} - 2\sum_{n=1}^{\infty} \frac{1}{n^7 (e^{2\pi n} - 1)}, \quad etc. \tag{3.17}$$

By analytic continuation, the domain of definition of the Riemann zeta function $\zeta(S)$ is expanded to the whole complex plane. The Riemann zeta function $\zeta(S)$ has only a single simple pole at $S = 1$, i.e.,

$$\frac{1}{\zeta(1)} = 0. \tag{3.18}$$

The Riemann zeta function is regular in the whole complex plane except the pole.
The residue of the Riemann zeta function at $S = 1$ is as follows:

$$\mathrm{Res}(1; \zeta(S)) = 1. \tag{3.19}$$

The value of the Riemann zeta function at $S = 0$ is shown as follows:

$$\zeta(0) = -\frac{1}{2}. \tag{3.20}$$

The Riemann zeta function has trivial zeros for any even number of -2 or less, i.e.,

$$\zeta(-2n) = 0, \quad n \in \mathbb{N}. \tag{3.21}$$

The value of the Riemann zeta function for any odd number of -1 or less can be obtained using the formul

$$\zeta(1 - 2n) = -\frac{B_{2n}}{2n}, \quad n \in \mathbb{N}. \tag{3.22}$$

Table shows the exact values for the Riemann zeta function for odd numbers of -1 or less

n	1	2	3	4	5	6	7	8	9
$1-2n$	-1	-3	-5	-7	-9	-11	-13	-15	-17
$\zeta(1-2n)$	$-\frac{1}{12}$	$\frac{1}{120}$	$-\frac{1}{252}$	$\frac{1}{240}$	$-\frac{1}{132}$	$\frac{691}{32760}$	$-\frac{1}{12}$	$\frac{3617}{8160}$	$-\frac{43867}{14364}$

Table. 3.5

3.3 The Riemann Xi Function $\xi(S)$

The Riemann Xi function $\xi(S)$ is defined as follows:

$$\xi(S) := \frac{S(S-1)}{2}\pi^{-\frac{S}{2}}\Gamma\left(\frac{S}{2}\right)\zeta(S), \quad S \in \mathbb{C}. \tag{3.23}$$

The Riemann Xi function's functional equation exhibits point symmetry, which at $S = 1/2$.

$$\xi(S) = \xi(1-S), \quad S \in \mathbb{C}. \tag{3.24}$$

The complete symmetric functional equation, of course, has the same symmetry.
The value of the Riemann Xi function at S=0 is shown as follows:

$$\xi(0) = \frac{1}{2}. \tag{3.25}$$

The Riemann Xi function has an infinite product representation called the Hadamard product.

$$\xi(S) = \xi(0) \prod_{\substack{\zeta(\rho)=0 \\ \rho:\text{ non-trivial zero}}} \left(1 - \frac{S}{\rho}\right), \quad S \in \mathbb{C}. \tag{3.26}$$

In this case, the infinite product crosses all non-trivial zeros of the Riemann zeta function.
This infinite product is absolutely convergent.

3.4 The Riemann Hypothesis for the Riemann Zeta Function

It is well known that all non-trivial zeros of the Riemann zeta function lie in the domain called the critical strip, excluding both side lines on the boundary.

$$\text{Critical strip} = \{S \in \mathbb{C} : 0 < \operatorname{Re}(S) < 1\}. \tag{3.27}$$

The Riemann hypothesis asserts that all non-trivial zeros of the Riemann zeta function lie on the critical line.

$$\text{Critical line} = \{S \in \mathbb{C} : \operatorname{Re}(S) = 1/2\}. \tag{3.28}$$

In this article, the Riemann zeta function is hereafter abbreviated to the zeta function.

3.5 The modified Bessel Function of the Second kind $K_\nu(Z)$

The modified Bessel function of the second kind $K_\nu(Z)$ is defined as an independent solution of the following modified Bessel differential equation "[3], [6]":

$$Z^2 y'' + Z y' - \left(Z^2 + \nu^2\right) y = 0, \quad (\nu,\, Z \in \mathbb{C}). \tag{3.29}$$

$K_\nu(Z)$ has a branch cut discontinuity in the complex z plane running from $-\infty$ to 0.
The origin symmetry concerning the index ν is shown as follows:

$$K_\nu(Z) = K_{-\nu}(Z), \quad (\nu \in \mathbb{C},\, Z \in \mathbb{C}\backslash\{0\}). \tag{3.30}$$

The recurrence formula is shown as follows:

$$\frac{Z}{2} K_{\nu+1}(Z) = \nu K_\nu(Z) + \frac{Z}{2} K_{\nu-1}(Z), \quad (\nu \in \mathbb{C},\, Z \in \mathbb{C}\backslash\{0\}). \tag{3.31}$$

The integral representation is shown as follows "[7]":

$$\mathrm{K}_{\nu}(Z)=\frac{1}{2}\int_{0}^{\infty}x^{\nu-1}e^{-\frac{Z}{2}\left(x+\frac{1}{x}\right)}\,dx,\quad(\nu\in\mathbb{C},\ Z\in\mathbb{C}\backslash\{0\}). \tag{3.32}$$

When the index ν is not an integer, the modified Bessel function of the second kind is given by the following equation using the modified Bessel function of the first kind, $\mathrm{I}_{\nu}(Z)$:

$$\mathrm{K}_{\nu}(Z)=\frac{\pi}{\sin(\nu\pi)}\left(\mathrm{I}_{-\nu}(Z)-\mathrm{I}_{\nu}(Z)\right),\quad(\nu\in\mathbb{C}\backslash\{\pm n:n\in\mathbb{N}\},\ Z\in\mathbb{C}\backslash\{0\}). \tag{3.33}$$

In this case, the function $\mathrm{I}_{\nu}(Z)$ is given as follows:

$$\mathrm{I}_{\nu}(Z)=\sum_{\mu=0}^{\infty}\frac{(2Z)^{2\mu+\nu}}{\mu!\,\Gamma(\mu+\nu+1)},\quad(\nu\in\mathbb{C},\ Z\in\mathbb{C}\backslash\{0\}). \tag{3.34}$$

$\mathrm{I}_{\nu}(Z)$ has also a branch cut discontinuity in the complex z plane running from $-\infty$ to 0.
When the index ν is a positive integer, the function $\mathrm{K}_{\nu}(Z)$ is given as follows:

$$\begin{aligned}\mathrm{K}_{n}(Z)&=\lim_{\nu\to n}\mathrm{K}_{\nu}(Z)\\&=(-1)^{n+1}\mathrm{I}_{n}(Z)\log\left(\frac{Z}{2}\right)+\frac{1}{2}\sum_{\mu=0}^{n-1}(-1)^{\mu}\frac{(n-\mu-1)!}{\mu!}\left(\frac{Z}{2}\right)^{2\mu-n}\\&\quad+\frac{(-1)^{n}}{2}\sum_{\mu=0}^{\infty}\frac{\psi(\mu+n+1)+\psi(\mu+1)}{(n+\mu)!\,\mu!}\left(\frac{Z}{2}\right)^{n+2\mu},\quad(n\in\mathbb{N},\ Z\in\mathbb{C}\backslash\{0\}).\end{aligned} \tag{3.35}$$

Here, the Psi function $\psi(Z)$ is defined as the logarithmic differentiation of the gamma function, i.e.,

$$\psi(Z):=\frac{\Gamma'(Z)}{\Gamma(Z)}=\frac{d}{dZ}\log(\Gamma(Z)),\quad Z\in\mathbb{C}\backslash\{0,-1,-2,\cdots\}. \tag{3.36}$$

The Psi function is an analytic function in the whole complex plane except at $Z=0,-1,-2,\cdots$.
When Z is any integer of 2 or more,

$$\psi(n+1)=-\gamma+\sum_{k=1}^{n}\frac{1}{k},\quad n\in\mathbb{N}. \tag{3.37}$$

When $Z=1$,

$$\psi(1)=-\gamma. \tag{3.38}$$

In this case, the constant γ is called Euler's constant, and its definition and approximate value are given as follows:

$$\gamma:=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log(n)\right)\simeq 0.57721566490. \tag{3.39}$$

When the index $\nu=0$, the asymptotic formula for the function $\mathrm{K}_0(x)$ is given as follows "[8]":

$$\mathrm{K}_0(x)=(-\gamma+\log(2)-\log(x))+\frac{1}{4}(1-\gamma+\log(2)-\log(x))\,x^2+O(x)^4,\quad x>0. \tag{3.40}$$

The formula for the function $\mathrm{K}_{\nu}(Z)$ at the index $\nu=1/2$ is given as follows:

$$\mathrm{K}_{\frac{1}{2}}(Z)=\sqrt{\frac{\pi}{2Z}}\,e^{-Z},\quad Z\in\mathbb{C}\backslash\{0\}. \tag{3.41}$$

The formula for the function $\mathrm{K}_{\nu}(Z)$ at the index ν for any positive half-integer is given as follows:

$$\mathrm{K}_{\frac{2n+1}{2}}(Z)=\sqrt{\frac{\pi}{2Z}}\,e^{-Z}\sum_{k=0}^{n}\frac{(n+k)!}{k!\,(n-k)!\,(2Z)^{k}},\quad(n\in\mathbb{N},\ Z\in\mathbb{C}\backslash\{0\}). \tag{3.42}$$

Table shows the formulae for the function $K_\nu(Z)$ at the index ν for half-integers

ν	$K_\nu(Z)$
$\pm\frac{1}{2}$	$\sqrt{\frac{\pi}{2Z}}\,e^{-Z}$
$\pm\frac{3}{2}$	$\sqrt{\frac{\pi}{2Z}}\,\left(1+\frac{1}{Z}\right)e^{-Z}$
$\pm\frac{5}{2}$	$\sqrt{\frac{\pi}{2Z}}\,\left(1+\frac{3}{Z}+\frac{3}{Z^2}\right)e^{-Z}$
$\pm\frac{7}{2}$	$\sqrt{\frac{\pi}{2Z}}\,\left(1+\frac{6}{Z}+\frac{15}{Z^2}+\frac{15}{Z^3}\right)e^{-Z}$
$\pm\frac{9}{2}$	$\sqrt{\frac{\pi}{2Z}}\,\left(1+\frac{10}{Z}+\frac{45}{Z^2}+\frac{105}{Z^3}+\frac{105}{Z^4}\right)e^{-Z}$

Table. 3.6

3.6 The Divisor Sigma Function $\sigma_\theta(n)$

When d is a divisor of any positive integer n, the sum of the divisor sigma function $\sigma_\theta(n)$ runs all divisors, and the defining equation of the divisor sigma function $\sigma_\theta(n)$ is shown as follows:

$$\sigma_\theta(n) := \sum_{d\,|\,n} d^\theta, \quad (\theta\in\mathbb{C},\, n\in\mathbb{N})\,. \tag{3.43}$$

When $\theta = 0$, the divisor sigma function is equivalent to the number of divisors function $\mu(n)$.

$$\sigma_0(n) = \sum_{d\,|\,n} 1 = \mu(n)\,, \quad \theta\in\mathbb{C}. \tag{3.44}$$

When $n = 1$, the divisor sigma function takes the constant 1 independent of the complex variable θ.

$$\sigma_{-\theta}(1) = 1, \quad \theta\in\mathbb{C}. \tag{3.45}$$

In the following discussions, a, b, and c are assumed to be three different arbitrary prime numbers.
The magnitude correlation of a, b, and c does not matter.

$$\sigma_{-\theta}(a) = 1+\frac{1}{a^\theta}, \quad \theta\in\mathbb{C}. \tag{3.46}$$

$$\sigma_{-\theta}\left(a^2\right) = 1+\frac{1}{a^\theta}+\frac{1}{a^{2\theta}}, \quad \theta\in\mathbb{C}. \tag{3.47}$$

$$\sigma_{-\theta}\left(a^3\right) = 1+\frac{1}{a^\theta}+\frac{1}{a^{2\theta}}+\frac{1}{a^{3\theta}}, \quad \theta\in\mathbb{C}. \tag{3.48}$$

$$\sigma_{-\theta}(a\cdot b) = 1+\frac{1}{a^\theta}+\frac{1}{b^\theta}+\frac{1}{a^\theta\cdot b^\theta} = \left(1+\frac{1}{a^\theta}\right)\left(1+\frac{1}{b^\theta}\right), \quad \theta\in\mathbb{C}. \tag{3.49}$$

$$\begin{aligned}\sigma_{-\theta}(a\cdot b\cdot c) &= 1+\frac{1}{a^\theta}+\frac{1}{b^\theta}+\frac{1}{c^\theta}+\frac{1}{a^\theta\cdot b^\theta}+\frac{1}{b^\theta\cdot c^\theta}+\frac{1}{c^\theta\cdot a^\theta}+\frac{1}{a^\theta\cdot b^\theta\cdot c^\theta}\\ &= \left(1+\frac{1}{a^\theta}\right)\left(1+\frac{1}{b^\theta}\right)\left(1+\frac{1}{c^\theta}\right), \quad \theta\in\mathbb{C}.\end{aligned} \tag{3.50}$$

$$\begin{aligned}\sigma_{-\theta}\left(a^2\cdot b\right) &= 1+\frac{1}{a^\theta}+\frac{1}{b^\theta}+\frac{1}{a^\theta\cdot b^\theta}+\frac{1}{a^{2\theta}}+\frac{1}{a^{2\theta}\cdot b^\theta}\\ &= \left(1+\frac{1}{a^\theta}+\frac{1}{a^{2\theta}}\right)\left(1+\frac{1}{b^\theta}\right), \quad \theta\in\mathbb{C}.\end{aligned} \tag{3.51}$$

When considering the product of two different giant prime numbers, there is a difficulty in factorizing the product. In addition, the divisor sigma function $\sigma_\theta(n)$ has the same difficulty.

4. Functional Equation Transformation, Part1

I consider multiplying both sides of the second-order I_Γ type functional equation by the kernel function

$$x^{\theta}e^{-\pi\alpha^2x^2},\quad (x,\,\alpha>0,\,\theta\in\mathbb{C})$$

to perform the improper integral on the open interval (0,∞) and to provide the operation for taking the leftsided limit of α → +0.
To ensure the validity of the above series of operations, it is necessary to evaluate the convergence of the integral and the boundedness of the limit value.
The operation for taking the left-sided limit is carried out at the proper stage of the calculation.
Moreover, the domain of definition is determined sequentially at appropriate stages.
The statements above also apply in Sections 5 and 8, except that the transformation sources are different.
The following are the operations of the functional equation transformation for the second-order I_Γ type functional equation:

(4.1)
$$\begin{aligned} L_{[2]}(\theta) &:= \lim_{\alpha\to+0}\int_0^{\infty} l_{[2]}(x)\,x^{\theta}e^{-\pi\alpha^2x^2}dx \\ &= 2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\left(-3n^2x^{\theta+1}+2\pi n^4x^{\theta+3}\right)e^{-\pi\left(n^2+\alpha^2\right)x^2}dx. \end{aligned}$$

(4.2)
$$\begin{aligned} R_{[2]}(\theta) &:= \lim_{\alpha\to+0}\int_0^{\infty} r_{[2]}(x)\,x^{\theta}e^{-\pi\alpha^2x^2}dx \\ &= 2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\left(-3n^2x^{\theta-4}+2\pi n^4x^{\theta-6}\right)e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx. \end{aligned}$$

For the integral of $L_{[2]}(\theta)$, I perform the variable transformation

$$x=\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}.$$

(4.3)
$$\begin{aligned} L_{[2]}(\theta) &= 2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\left(\begin{aligned} &-3n^2\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+1}{2}} \\ &+2\pi n^4\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+3}{2}}\end{aligned}\right)e^{-y}\frac{1}{2}\left(\frac{1}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}y^{-\frac{1}{2}}dy \\ &= \pi^{-\frac{\theta}{2}}\lim_{\alpha\to+0}\int_0^{\infty}\left(-3\sum_{n=1}^{\infty}n^2\left(\frac{1}{n^2+\alpha^2}\right)^{\frac{\theta}{2}+1}y^{\frac{\theta}{2}}+2\sum_{n=1}^{\infty}n^4\left(\frac{1}{n^2+\alpha^2}\right)^{\frac{\theta}{2}+2}y^{\frac{\theta}{2}+1}\right)e^{-y}dy. \end{aligned}$$

Using the left-sided limit of the positive real variable α,

(4.4)
$$\begin{aligned} L_{[2]}(\theta) &= \pi^{-\frac{\theta}{2}}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{\theta}}\left(-3y^{\frac{\theta}{2}}+2y^{\frac{\theta}{2}+1}\right)e^{-y}dy \\ &= \pi^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{\theta}}\left(-3\int_0^{\infty}y^{\frac{\theta}{2}}e^{-y}dy+2\int_0^{\infty}y^{\frac{\theta}{2}+1}e^{-y}dy\right). \end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta)>1$ can be added to the above result because it is a convergent function under the condition.

(4.5)
$$L_{[2]}(\theta)=\pi^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{\theta}}\left(-3\,\Gamma\left(1+\frac{\theta}{2}\right)+2\,\Gamma\left(2+\frac{\theta}{2}\right)\right),\quad \mathrm{Re}(\theta)>1.$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

(4.6)
$$\begin{aligned} L_{[2]}(\theta) &= \pi^{-\frac{\theta}{2}}\zeta(\theta)\left(-3\cdot\frac{\theta}{2}\Gamma\left(\frac{\theta}{2}\right)+2\cdot\left(1+\frac{\theta}{2}\right)\frac{\theta}{2}\Gamma\left(\frac{\theta}{2}\right)\right) \\ &= \pi^{-\frac{\theta}{2}}\zeta(\theta)\cdot\frac{\theta}{2}(\theta-1)\,\Gamma\left(\frac{\theta}{2}\right),\quad \theta\in\mathbb{C}. \end{aligned}$$

Therefore,

$$L_{[2]}(\theta) = \frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta), \quad \theta \in \mathbb{C}. \tag{4.7}$$

The function $L_{[2]}(\theta)$ is a convergent function in the whole complex plane.
For the integral of $R_{[2]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned} R_{[2]}(\theta) &= 2\pi \lim_{\alpha\to+0}\int_{\infty}^{0}\sum_{n=1}^{\infty}\left(-3n^2\left(\frac{1}{y}\right)^{\theta-4} + 2\pi n^4\left(\frac{1}{y}\right)^{\theta-6}\right)e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}y^{-2}dy \\ &= 2\pi \lim_{\alpha\to+0}\int_{0}^{\infty}\sum_{n=1}^{\infty}\left(-3n^2y^{2-\theta} + 2\pi n^4 y^{4-\theta}\right)e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}dy. \end{aligned} \tag{4.8}$$

Assuming that the integral and the sum can be interchanged, for the integrals of $R_{[2]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{\alpha x}{n}\right)^{\frac{1}{\frac{1}{2}}}.$$

$$\begin{aligned} R_{[2]}(\theta) &= 2\pi \lim_{\alpha\to+0}\int_{0}^{\infty}\sum_{n=1}^{\infty}\left(-3n^2\left(\frac{\alpha x}{n}\right)^{\frac{2-\theta}{2}} + 2\pi n^4\left(\frac{\alpha x}{n}\right)^{\frac{4-\theta}{2}}\right)e^{-\pi n\alpha\left(x+\frac{1}{x}\right)}\frac{1}{2}\left(\frac{\alpha}{n}\right)^{\frac{1}{2}}x^{-\frac{1}{2}}dx \\ &= 2\pi \lim_{\alpha\to+0}\alpha^{\frac{3-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{1+\theta}{2}}\left(\begin{array}{l} -3\cdot\frac{1}{2}\int_{0}^{\infty}x^{\frac{3-\theta}{2}-1}e^{-\frac{2\pi n\alpha}{2}\left(x+\frac{1}{x}\right)}dx \\ \quad +2(\pi n\alpha)\cdot\frac{1}{2}\int_{0}^{\infty}x^{\frac{5-\theta}{2}-1}e^{-\frac{2\pi n\alpha}{2}\left(x+\frac{1}{x}\right)}dx \end{array}\right). \end{aligned} \tag{4.9}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$R_{[2]}(\theta) = 2\pi \lim_{\alpha\to+0}\alpha^{\frac{3-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{1+\theta}{2}}\left(-3\,\mathrm{K}_{\frac{3-\theta}{2}}(2\pi n\alpha) + 2(\pi n\alpha)\,\mathrm{K}_{\frac{5-\theta}{2}}(2\pi n\alpha)\right). \tag{4.10}$$

Because any modified Bessel function of the second kind of the sum converges absolutely, the assumed exchange is justified.
The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$\begin{aligned} R_{[2]}(\theta) &= 2\pi \lim_{\alpha\to+0}\alpha^{\frac{3-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{1+\theta}{2}}\left(-3\,\mathrm{K}_{\frac{3-\theta}{2}}(2\pi n\alpha) + 2\left(\frac{3-\theta}{2}\mathrm{K}_{\frac{3-\theta}{2}}(2\pi n\alpha) + (\pi n\alpha)\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha)\right)\right) \\ &= 2\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}\left(-\theta(\pi n\alpha)\,\mathrm{K}_{\frac{3-\theta}{2}}(2\pi n\alpha) + 2(\pi n\alpha)^2\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha)\right) \\ &= 2\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}\left(-\theta\left(\begin{array}{l}\frac{1-\theta}{2}\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha) \\ \quad +(\pi n\alpha)\,\mathrm{K}_{\frac{-1-\theta}{2}}(2\pi n\alpha)\end{array}\right) + 2(\pi n\alpha)^2\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha)\right) \\ &= \lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}\left(\begin{array}{l}\theta(\theta-1)\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha) \\ -2(\pi n\alpha)\,\theta\,\mathrm{K}_{\frac{1+\theta}{2}}(2\pi n\alpha) \\ \quad +4(\pi n\alpha)^2\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha)\end{array}\right). \end{aligned} \tag{4.11}$$

For convenience, I separate the result of equation (4.11) into three parts adding the condition $\mathrm{Re}(\theta) < 0$.

$$C_{[2]-1}(\theta) := \theta(\theta-1)\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha), \quad \mathrm{Re}(\theta) < 0. \tag{4.12}$$

$$C_{[2]-2}(\theta) := -2\theta\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)\,\mathrm{K}_{\frac{1+\theta}{2}}(2\pi n\alpha), \quad \mathrm{Re}(\theta) < 0. \tag{4.13}$$

$$C_{[2]-3}(\theta) := 4\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)^2\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha), \quad \mathrm{Re}(\theta) < 0. \tag{4.14}$$

Additionally, when the complex variable θ is not an odd number of -1 or less, the modified Bessel function of the second kind is given by the modified Bessel functions of the first kind. For the first part,

$$\begin{aligned}
C_{[2]-1}(\theta) &= \theta(\theta-1)\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha) \\
&= \theta(\theta-1)\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}\frac{\pi}{2\sin\left(\frac{1-\theta}{2}\pi\right)}\left(\mathrm{I}_{\frac{-1+\theta}{2}}(2\pi n\alpha)-\mathrm{I}_{\frac{1-\theta}{2}}(2\pi n\alpha)\right) \\
&= \frac{\theta(\theta-1)\pi}{2\sin\left(\frac{1-\theta}{2}\pi\right)}\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{-1+\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{-1+\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{1-\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{1-\theta}{2}+1\right)}\right) \\
&= \frac{\theta(\theta-1)\pi}{2\sin\left(\frac{1-\theta}{2}\pi\right)}\pi^{-\frac{1-\theta}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}n^{\theta-1}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu}}{\mu!\,\Gamma\left(\mu+\frac{-1+\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+1-\theta}}{\mu!\,\Gamma\left(\mu+\frac{1-\theta}{2}+1\right)}\right) \\
&= \frac{\theta(\theta-1)\pi}{2\sin\left(\frac{1-\theta}{2}\pi\right)}\pi^{-\frac{1-\theta}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(\begin{aligned}&\frac{(\pi n\alpha)^{0}}{0!\,\Gamma\left(1-\frac{1-\theta}{2}\right)}\\&+\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+2}}{(\mu+1)!\,\Gamma\left(\mu+\frac{1+\theta}{2}+1\right)}\\&-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+1-\theta}}{\mu!\,\Gamma\left(\mu+\frac{1-\theta}{2}+1\right)}\end{aligned}\right), \\
&\quad (\mathrm{Re}(\theta)<0,\ \theta\in\mathbb{C}\backslash\{1-2p:p\in\mathbb{N}\}).
\end{aligned}\tag{4.15}$$

Now, using the following fact for calculation:

$$\lim_{\alpha\to+0}(\pi n\alpha)^{\beta}=\begin{cases}1 & (\beta=0,\ n\in\mathbb{N}),\\ 0 & (\mathrm{Re}(\beta)>0,\ n\in\mathbb{N}).\end{cases}\tag{4.16}$$

Therefore

$$C_{[2]-1}(\theta)=\frac{\theta(\theta-1)}{2}\pi^{-\frac{1-\theta}{2}}\frac{\pi}{\sin\left(\frac{1-\theta}{2}\pi\right)\Gamma\left(1-\frac{1-\theta}{2}\right)}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}},\quad (\mathrm{Re}(\theta)<0,\ \theta\in\mathbb{C}\backslash\{1-2p:p\in\mathbb{N}\}).\tag{4.17}$$

By applying the reciprocal formula for the gamma function,

$$C_{[2]-1}(\theta)=\frac{\theta(\theta-1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}},\quad (\mathrm{Re}(\theta)<0,\ \theta\in\mathbb{C}\backslash\{1-2p:p\in\mathbb{N}\}).\tag{4.18}$$

For the second part,

$$\begin{aligned}
C_{[2]-2}(\theta) &= -2\theta\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)\mathrm{K}_{\frac{1+\theta}{2}}(2\pi n\alpha) \\
&= -2\theta\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)\frac{\pi}{2\sin\left(\frac{1+\theta}{2}\pi\right)}\left(\mathrm{I}_{-\frac{1+\theta}{2}}(2\pi n\alpha)-\mathrm{I}_{\frac{1+\theta}{2}}(2\pi n\alpha)\right) \\
&= -\frac{\theta\pi}{\sin\left(\frac{1+\theta}{2}\pi\right)}\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)\left(\begin{aligned}&\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\frac{1+\theta}{2}}}{\mu!\,\Gamma\left(\mu-\frac{1+\theta}{2}+1\right)}\\&-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{1+\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{1+\theta}{2}+1\right)}\end{aligned}\right) \\
&= -\frac{\theta\pi}{\sin\left(\frac{1+\theta}{2}\pi\right)}\pi^{-\frac{1-\theta}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+1-\theta}}{\mu!\,\Gamma\left(\mu-\frac{1+\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+2}}{\mu!\,\Gamma\left(\mu+\frac{1+\theta}{2}+1\right)}\right) \\
&= 0,\quad (\mathrm{Re}(\theta)<0,\ \theta\in\mathbb{C}\backslash\{1-2p:p\in\mathbb{N}\}).
\end{aligned}\tag{4.19}$$

For the third part,

$$
\begin{aligned}
C_{[2]-3}(\theta) &= 4\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)^2\,\mathrm{K}_{\frac{1-\theta}{2}}(2\pi n\alpha)\\
&= 4\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)^2\frac{\pi}{2\sin\left(\frac{1-\theta}{2}\pi\right)}\left(\mathrm{I}_{\frac{-1+\theta}{2}}(2\pi n\alpha)-\mathrm{I}_{\frac{1-\theta}{2}}(2\pi n\alpha)\right)\\
&= \frac{2\pi}{\sin\left(\frac{1-\theta}{2}\pi\right)}\lim_{\alpha\to+0}\alpha^{\frac{1-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{-1+\theta}{2}}(\pi n\alpha)^2\begin{pmatrix}\displaystyle\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{-1+\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{-1+\theta}{2}+1\right)}\\ \displaystyle-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{1-\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{1-\theta}{2}+1\right)}\end{pmatrix}\\
&= \frac{2\pi}{\sin\left(\frac{1-\theta}{2}\pi\right)}\pi^{-\frac{1-\theta}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+2}}{\mu!\,\Gamma\left(\mu+\frac{-1+\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+3-\theta}}{\mu!\,\Gamma\left(\mu+\frac{1-\theta}{2}+1\right)}\right)\\
&= 0,\quad \left(\mathrm{Re}(\theta)<0,\ \theta\in\mathbb{C}\backslash\{1-2p:p\in\mathbb{N}\}\right).
\end{aligned}
\tag{4.20}
$$

By combining the above results,

$$
R_{[2]}(\theta)=\frac{\theta(\theta-1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}},\quad \left(\mathrm{Re}(\theta)<0,\ \theta\in\mathbb{C}\backslash\{1-2p:p\in\mathbb{N}\}\right).
\tag{4.21}
$$

When the complex variable θ is any odd number of -1 or less, for the first part,

$$
\begin{aligned}
C_{[2]-1}(1-2p) &= (1-2p)(-2p)\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}\,\mathrm{K}_p(2\pi n\alpha)\\
&= 2p(2p-1)\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}\begin{pmatrix}(-1)^{p+1}\,\mathrm{I}_p(2\pi n\alpha)\log(\pi n\alpha)\\ \displaystyle+\frac{1}{2}\sum_{\mu=0}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu-p}\\ \displaystyle+\frac{(-1)^p}{2}\sum_{\mu=0}^{\infty}\frac{\psi(\mu+p+1)+\psi(\mu+1)}{(p+\mu)!\,\mu!}(\pi n\alpha)^{p+2\mu}\end{pmatrix}\\
&= 2p(2p-1)\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}\begin{pmatrix}\displaystyle(-1)^{p+1}\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+p}}{\mu!\,\Gamma(\mu+p+1)}\log(\pi n\alpha)\\ \displaystyle+\frac{1}{2}\sum_{\mu=0}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu-p}\\ \displaystyle+\frac{(-1)^p}{2}\sum_{\mu=0}^{\infty}\frac{\psi(\mu+p+1)+\psi(\mu+1)}{(p+\mu)!\,\mu!}(\pi n\alpha)^{p+2\mu}\end{pmatrix}\\
&= \frac{p(2p-1)}{\pi^p}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\begin{pmatrix}\displaystyle\sum_{\mu=0}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu}\\ \displaystyle+(-1)^{p+1}\sum_{\mu=0}^{\infty}\frac{2\log(\pi n\alpha)-\psi(\mu+p+1)-\psi(\mu+1)}{\mu!\,\Gamma(\mu+p+1)}(\pi n\alpha)^{2\mu+2p}\end{pmatrix}\\
&= \frac{p(2p-1)}{\pi^p}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\sum_{\mu=0}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu},\quad p\in\mathbb{N}.
\end{aligned}
\tag{4.22}
$$

When the complex variable θ is any odd number of -3 or less,

$$
\begin{aligned}
C_{[2]-1}(1-2p) &= \frac{p(2p-1)}{\pi^p}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\left((-1)^0\frac{(p-1)!}{0!}(\pi n\alpha)^0+\sum_{\mu=1}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu}\right)\\
&= \frac{p!\,(2p-1)}{\pi^p}\sum_{n=1}^{\infty}\frac{1}{n^{2p}},\quad p\in\mathbb{N}\backslash\{1\}.
\end{aligned}
\tag{4.23}
$$

When the complex variable θ is equal to -1,

$$C_{[2]-1}(-1) = \frac{1}{\pi}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{\mu=0}^{0}(-1)^{\mu}\frac{(1-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu} = \frac{1}{\pi}\cdot\frac{\pi^2}{6} = \frac{\pi}{6}. \tag{4.24}$$

For the result of equation (4.23), 1 is substituted directly for p.

$$\left.\frac{p!\,(2p-1)}{\pi^p}\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\right|_{p=1} = \frac{1!\cdot 1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi}{6}. \tag{4.25}$$

This result is the same as the result of equation (4.24); hence, they can be combined.

$$C_{[2]-1}(1-2p) = \frac{p!\,(2p-1)}{\pi^p}\sum_{n=1}^{\infty}\frac{1}{n^{2p}},\quad p\in\mathbb{N}. \tag{4.26}$$

For the second part,

$$C_{[2]-2}(1-2p) = -2(1-2p)\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)\,\mathrm{K}_{1-p}(2\pi n\alpha),\quad p\in\mathbb{N}. \tag{4.27}$$

When the complex variable θ is any odd number of -3 or less,

$$\begin{aligned}
C_{[2]-2}(1-2p) &= 2(2p-1)\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)\,\mathrm{K}_{p-1}(2\pi n\alpha)\\
&= 2(2p-1)\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)\begin{pmatrix}(-1)^p\,\mathrm{I}_{p-1}(2\pi n\alpha)\log(\pi n\alpha)\\ +\dfrac{1}{2}\displaystyle\sum_{\mu=0}^{p-2}(-1)^{\mu}\frac{(p-\mu-2)!}{\mu!}(\pi n\alpha)^{2\mu-p+1}\\ +\dfrac{(-1)^{p-1}}{2}\displaystyle\sum_{\mu=0}^{\infty}\frac{\psi(\mu+p)+\psi(\mu+1)}{(p+\mu-1)!\,\mu!}(\pi n\alpha)^{p+2\mu-1}\end{pmatrix}\\
&= 2(2p-1)\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)\begin{pmatrix}(-1)^p\displaystyle\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+p-1}}{\mu!\,\Gamma(\mu+p)}\log(\pi n\alpha)\\ +\dfrac{1}{2}\displaystyle\sum_{\mu=0}^{p-2}(-1)^{\mu}\frac{(p-\mu-2)!}{\mu!}(\pi n\alpha)^{2\mu-p+1}\\ +\dfrac{(-1)^{p-1}}{2}\displaystyle\sum_{\mu=0}^{\infty}\frac{\psi(\mu+p)+\psi(\mu+1)}{(p+\mu-1)!\,\mu!}(\pi n\alpha)^{p+2\mu-1}\end{pmatrix}\\
&= \frac{2p-1}{\pi^p}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\begin{pmatrix}\displaystyle\sum_{\mu=0}^{p-2}(-1)^{\mu}\frac{(p-\mu-2)!}{\mu!}(\pi n\alpha)^{2\mu+2}\\ +(-1)^p\displaystyle\sum_{\mu=0}^{\infty}\frac{2\log(\pi n\alpha)-\psi(\mu+p)-\psi(\mu+1)}{\mu!\,\Gamma(\mu+p)}(\pi n\alpha)^{2\mu+2p}\end{pmatrix}\\
&= 0,\quad p\in\mathbb{N}\backslash\{1\}.
\end{aligned} \tag{4.28}$$

When the complex variable θ is equal to -1,

$$\begin{aligned}
C_{[2]-2}(-1) &= -2(-1)\lim_{\alpha\to+0}\alpha\sum_{n=1}^{\infty}n^{-1}(\pi n\alpha)\,\mathrm{K}_0(2\pi n\alpha)\\
&= \frac{2}{\pi}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^2}(\pi n\alpha)^2\left((-\gamma-\log(\pi n\alpha))+\frac{1}{4}(1-\gamma-\log(\pi n\alpha))(2\pi n\alpha)^2+O(2\pi n\alpha)^4\right)\\
&= 0.
\end{aligned} \tag{4.29}$$

Here, I used the asymptotic formula for the function $\mathrm{K}_0(x)$.

For the third part,

$$
\begin{aligned}
C_{[2]-3}(1-2p) &= 4\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)^2\,\mathrm{K}_p(2\pi n\alpha)\\
&= 4\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)^2\left(\begin{aligned}&(-1)^{p+1}\,\mathrm{I}_p(2\pi n\alpha)\log(\pi n\alpha)\\&\quad+\frac{1}{2}\sum_{\mu=0}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu-p}\\&\quad+\frac{(-1)^p}{2}\sum_{\mu=0}^{\infty}\frac{\psi(\mu+p+1)+\psi(\mu+1)}{(p+\mu)!\,\mu!}(\pi n\alpha)^{p+2\mu}\end{aligned}\right)\\
&= 4\lim_{\alpha\to+0}\alpha^p\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)^2\left(\begin{aligned}&(-1)^{p+1}\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+p}}{\mu!\,\Gamma(\mu+p+1)}\log(\pi n\alpha)\\&\quad+\frac{1}{2}\sum_{\mu=0}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu-p}\\&\quad+\frac{(-1)^p}{2}\sum_{\mu=0}^{\infty}\frac{\psi(\mu+p+1)+\psi(\mu+1)}{(p+\mu)!\,\mu!}(\pi n\alpha)^{p+2\mu}\end{aligned}\right)\\
&= \frac{2}{\pi^p}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\left(\begin{aligned}&\sum_{\mu=0}^{p-1}(-1)^{\mu}\frac{(p-\mu-1)!}{\mu!}(\pi n\alpha)^{2\mu+2}\\&\quad+(-1)^{p+1}\sum_{\mu=0}^{\infty}\frac{2\log(\pi n\alpha)-\psi(\mu+p+1)-\psi(\mu+1)}{\mu!\,\Gamma(\mu+p+1)}(\pi n\alpha)^{2\mu+2p+2}\end{aligned}\right)\\
&= 0,\quad p\in\mathbb{N}.
\end{aligned}
\tag{4.30}
$$

For the right side of equation (4.21), $(1-2p)$ is substituted directly for θ.

$$
\begin{aligned}
\left.\frac{\theta(\theta-1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\right|_{\theta=1-2p} &= \frac{(1-2p)(-2p)}{2}\pi^{-p}\Gamma(p)\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\\
&= \frac{p!\,(2p-1)}{\pi^p}\sum_{n=1}^{\infty}\frac{1}{n^{2p}},\quad p\in\mathbb{N}.
\end{aligned}
\tag{4.31}
$$

This result is the same as the result of equation (4.26); hence, they can be combined.

$$
R_{[2]}(\theta) = \frac{\theta(\theta-1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}},\quad \mathrm{Re}(\theta)<0.
\tag{4.32}
$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$
R_{[2]}(\theta) = \frac{\theta(\theta-1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\zeta(1-\theta),\quad \theta\in\mathbb{C}.
\tag{4.33}
$$

The function $R_{[2]}(\theta)$ is also a convergent function in the whole complex plane. Thus, the following equation holds true:

$$
L_{[2]}(\theta) = R_{[2]}(\theta),\quad \theta\in\mathbb{C}.
\tag{4.34}
$$

Therefore

$$
\frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta) = \frac{\theta(\theta-1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\zeta(1-\theta),\quad \theta\in\mathbb{C}.
\tag{4.35}
$$

I decide that functional equation (4.35) is called the second-order $\mathrm{I_c}$ type functional equation. Suffix c means "complex." This rule is also hereafter applicable.
It is equivalent to the functional equation that the Riemann Xi function follows.

5. Functional Equation Transformation, Part2

The following are the operations of the functional equation transformation for the second-order $\mathrm{II_r}$ type functional equation:

$$A_{[2]}(\theta) := \lim_{\alpha\to+0}\int_0^\infty a_{[2]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-3n^2x^{\theta+1}+2\pi n^4x^{\theta+3}\right)e^{-\pi\left(n^2+\alpha^2\right)x^2}dx. \tag{5.1}$$

$$F_{[2]}(\theta) := \lim_{\alpha\to+0}\int_0^\infty f_{[2]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\left(-6n^2x^{\theta+1}+4\pi n^4x^{\theta+3}\right)e^{-\pi\left(\left(n^2+\alpha^2\right)x^2+\frac{m^2}{x^2}\right)}dx. \tag{5.2}$$

$$B_{[2]}(\theta) := \lim_{\alpha\to+0}\int_0^\infty b_{[2]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-3n^2x^{\theta-3}+2\pi n^4x^{\theta-5}\right)e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx. \tag{5.3}$$

$$G_{[2]}(\theta) := \lim_{\alpha\to+0}\int_0^\infty g_{[2]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\left(-6n^2x^{\theta-3}+4\pi n^4x^{\theta-5}\right)e^{-\pi\left(\left(m^2+\alpha^2\right)x^2+\frac{n^2}{x^2}\right)}dx. \tag{5.4}$$

Immediately, the following obvious result is obtained:

$$A_{[2]}(\theta) = L_{[2]}(\theta) = \frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta),\quad \theta\in\mathbb{C}. \tag{5.5}$$

For the integral of $B_{[2]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$B_{[2]}(\theta) = 2\pi\lim_{\alpha\to+0}\int_\infty^0\sum_{n=1}^\infty\left(-3n^2\left(\frac{1}{y}\right)^{\theta-3}+2\pi n^4\left(\frac{1}{y}\right)^{\theta-5}\right)e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}\left(-y^{-2}\right)dy$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-3n^2y^{1-\theta}+2\pi n^4y^{3-\theta}\right)e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}dy. \tag{5.6}$$

Assuming that the integral and the sum can be interchanged, for the integrals of $B_{[2]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{\alpha x}{n}\right)^{\frac{1}{2}}.$$

$$B_{[2]}(\theta) = 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-3n^2\left(\frac{\alpha x}{n}\right)^{\frac{1-\theta}{2}}+2\pi n^4\left(\frac{\alpha x}{n}\right)^{\frac{3-\theta}{2}}\right)e^{-\pi n\alpha\left(x+\frac{1}{x}\right)}\frac{1}{2}\left(\frac{\alpha}{n}\right)^{\frac{1}{2}}x^{-\frac{1}{2}}dx$$
$$= 2\pi\lim_{\alpha\to+0}\alpha^{\frac{2-\theta}{2}}\sum_{n=1}^\infty n^{\frac{2+\theta}{2}}\left(\begin{array}{l}-3\cdot\frac{1}{2}\int_0^\infty x^{\frac{2-\theta}{2}-1}e^{-\frac{2\pi n\alpha}{2}\left(x+\frac{1}{x}\right)}dx\\ +2\left(\pi n\alpha\right)\cdot\frac{1}{2}\int_0^\infty x^{\frac{4-\theta}{2}-1}e^{-\frac{2\pi n\alpha}{2}\left(x+\frac{1}{x}\right)}dx\end{array}\right). \tag{5.7}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$B_{[2]}(\theta) = 2\pi\lim_{\alpha\to+0}\alpha^{\frac{2-\theta}{2}}\sum_{n=1}^\infty n^{\frac{2+\theta}{2}}\left(-3\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi n\alpha)+2(\pi n\alpha)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi n\alpha)\right). \tag{5.8}$$

Because any modified Bessel function of the second kind of the sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$B_{[2]}(\theta)=2\pi\lim_{\alpha\to+0}\alpha^{\frac{2-\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{2+\theta}{2}}\left(-3\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi n\alpha)+2\left(\frac{2-\theta}{2}\mathrm{K}_{\frac{2-\theta}{2}}(2\pi n\alpha)+(\pi n\alpha)\,\mathrm{K}_{-\frac{\theta}{2}}(2\pi n\alpha)\right)\right)$$

$$=2\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}\left(-(\theta+1)(\pi n\alpha)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi n\alpha)+2(\pi n\alpha)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha)\right)$$

(5.9)

$$=2\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}\left(-(\theta+1)\begin{pmatrix}-\frac{\theta}{2}\mathrm{K}_{-\frac{\theta}{2}}(2\pi n\alpha)\\+(\pi n\alpha)\,\mathrm{K}_{-\frac{2+\theta}{2}}(2\pi n\alpha)\end{pmatrix}+2(\pi n\alpha)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha)\right)$$

$$=\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}\left(\theta(\theta+1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha)-2(\theta+1)(\pi n\alpha)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi n\alpha)+4(\pi n\alpha)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha)\right).$$

For convenience, I separate the result of equation (5.9) into three parts adding the condition $\mathrm{Re}(\theta)<-1$.

(5.10) $$D_{[2]-1}(\theta):=\theta(\theta+1)\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha),\quad \mathrm{Re}(\theta)<-1.$$

(5.11) $$D_{[2]-2}(\theta):=-2(\theta+1)\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi n\alpha),\quad \mathrm{Re}(\theta)<-1.$$

(5.12) $$D_{[2]-3}(\theta):=4\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha),\quad \mathrm{Re}(\theta)<-1.$$

Furthermore, when the complex variable θ is not an even number of -2 or less, the modified Bessel function of the second kind is provided using the modified Bessel functions of the first kind. For the first part,

$$D_{[2]-1}(\theta)=\theta(\theta+1)\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha)$$

$$=\theta(\theta+1)\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}\frac{\pi}{2\sin\left(\frac{\theta}{2}\pi\right)}\left(\mathrm{I}_{-\frac{\theta}{2}}(2\pi n\alpha)-\mathrm{I}_{\frac{\theta}{2}}(2\pi n\alpha)\right)$$

$$=\frac{\theta(\theta+1)\pi}{2\sin\left(\frac{\theta}{2}\pi\right)}\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\frac{\theta}{2}}}{\mu!\,\Gamma\left(\mu-\frac{\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{\theta}{2}+1\right)}\right)$$

(5.13)

$$=\frac{\theta(\theta+1)\pi}{2\sin\left(\frac{\theta}{2}\pi\right)}\pi^{\frac{\theta}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}n^{\theta}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\theta}}{\mu!\,\Gamma\left(\mu-\frac{\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu}}{\mu!\,\Gamma\left(\mu+\frac{\theta}{2}+1\right)}\right)$$

$$=\frac{\theta(\theta+1)\pi}{2\sin\left(\frac{\theta}{2}\pi\right)}\pi^{\frac{\theta}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}n^{\theta}\begin{pmatrix}\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\theta}}{\mu!\,\Gamma\left(\mu-\frac{\theta}{2}+1\right)}\\-\frac{(\pi n\alpha)^{0}}{0!\,\Gamma\left(1+\frac{\theta}{2}\right)}\\-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+2}}{(\mu+1)!\,\Gamma\left(\mu+\frac{\theta}{2}+2\right)}\end{pmatrix}$$

$$=\frac{\theta(\theta+1)}{2}\pi^{\frac{\theta}{2}}\frac{\pi}{\sin\left(-\frac{\theta}{2}\pi\right)\Gamma\left(1-\left(-\frac{\theta}{2}\right)\right)}\sum_{n=1}^{\infty}n^{\theta},\quad \left(\mathrm{Re}(\theta)<-1,\ \theta\in\mathbb{C}\backslash\{-2p:p\in\mathbb{N}\}\right).$$

By applying the reciprocal formula for the gamma function,

(5.14) $$D_{[2]-1}(\theta)=\frac{\theta(\theta+1)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{-\theta}},\quad \left(\mathrm{Re}(\theta)<-1,\ \theta\in\mathbb{C}\backslash\{-2p:p\in\mathbb{N}\}\right).$$

For the second part,

$$\begin{aligned}
D_{[2]-2}(\theta) &= -2(\theta+1)\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi n\alpha)\\
&= -2(\theta+1)\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)\frac{\pi}{2\sin\left(\frac{2+\theta}{2}\pi\right)}\left(\mathrm{I}_{-\frac{2+\theta}{2}}(2\pi n\alpha)-\mathrm{I}_{\frac{2+\theta}{2}}(2\pi n\alpha)\right)\\
&= -\frac{(\theta+1)\pi}{\sin\left(\frac{2+\theta}{2}\pi\right)}\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\frac{2+\theta}{2}}}{\mu!\,\Gamma\left(\mu-\frac{2+\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{2+\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{2+\theta}{2}+1\right)}\right)\\
&= -\frac{(\theta+1)\pi}{\sin\left(\frac{2+\theta}{2}\pi\right)}\pi^{\frac{\theta}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}n^{\theta}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\theta}}{\mu!\,\Gamma\left(\mu-\frac{\theta}{2}\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+2}}{\mu!\,\Gamma\left(\mu+\frac{\theta}{2}+2\right)}\right)\\
&= 0,\quad (\mathrm{Re}(\theta)<-1,\ \theta\in\mathbb{C}\backslash\{-2p:p\in\mathbb{N}\}).
\end{aligned}\tag{5.15}$$

For the third part,

$$\begin{aligned}
D_{[2]-3}(\theta) &= 4\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi n\alpha)\\
&= 4\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)^{2}\frac{\pi}{2\sin\left(\frac{\theta}{2}\pi\right)}\left(\mathrm{I}_{-\frac{\theta}{2}}(2\pi n\alpha)-\mathrm{I}_{\frac{\theta}{2}}(2\pi n\alpha)\right)\\
&= \frac{2\pi}{\sin\left(\frac{\theta}{2}\pi\right)}\pi^{\frac{1}{2}}\lim_{\alpha\to+0}\alpha^{-\frac{\theta}{2}}\sum_{n=1}^{\infty}n^{\frac{\theta}{2}}(\pi n\alpha)^{2}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\frac{\theta}{2}}}{\mu!\,\Gamma\left(\mu-\frac{\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+\frac{\theta}{2}}}{\mu!\,\Gamma\left(\mu+\frac{\theta}{2}+1\right)}\right)\\
&= \frac{2\pi}{\sin\left(\frac{\theta}{2}\pi\right)}\pi^{\frac{1}{2}}\lim_{\alpha\to+0}\sum_{n=1}^{\infty}n^{\theta}\left(\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu-\theta+2}}{\mu!\,\Gamma\left(\mu-\frac{\theta}{2}+1\right)}-\sum_{\mu=0}^{\infty}\frac{(\pi n\alpha)^{2\mu+2}}{\mu!\,\Gamma\left(\mu+\frac{\theta}{2}+1\right)}\right)\\
&= 0,\quad (\mathrm{Re}(\theta)<-1,\ \theta\in\mathbb{C}\backslash\{-2p:p\in\mathbb{N}\}).
\end{aligned}\tag{5.16}$$

By combining the above results,

$$B_{[2]}(\theta)=\frac{\theta(\theta+1)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{-\theta}},\quad (\mathrm{Re}(\theta)<-1,\ \theta\in\mathbb{C}\backslash\{-2p:p\in\mathbb{N}\}).\tag{5.17}$$

When the complex variable θ is any even number of -2 or less, for the first part,

$$\begin{aligned}
D_{[2]-1}(-2p) &= (-2p)(-2p+1)\lim_{\alpha\to+0}\alpha^{p}\sum_{n=1}^{\infty}n^{-p}\,\mathrm{K}_{-p}(2\pi n\alpha)\\
&= C_{[2]-1}(1-2p)\\
&= \frac{p!\,(2p-1)}{\pi^{p}}\sum_{n=1}^{\infty}\frac{1}{n^{2p}},\quad p\in\mathbb{N}.
\end{aligned}\tag{5.18}$$

For the second part,

$$\begin{aligned}
D_{[2]-2}(-2p) &= -2(-2p+1)\lim_{\alpha\to+0}\alpha^{p}\sum_{n=1}^{\infty}n^{-p}(\pi n\alpha)\,\mathrm{K}_{1-p}(2\pi n\alpha)\\
&= C_{[2]-2}(1-2p)\\
&= 0,\quad p\in\mathbb{N}.
\end{aligned}\tag{5.19}$$

For the third part,

$$\begin{aligned}
D_{[2]-3}(-2p) &= 4\lim_{\alpha\to+0}\alpha^{p}\sum_{n=1}^{\infty}n^{-p}(2\pi n\alpha)^{2}\,\mathrm{K}_{-p}(2\pi n\alpha)\\
&= C_{[2]-3}(1-2p)\\
&= 0,\quad p\in\mathbb{N}.
\end{aligned}\tag{5.20}$$

For the right side of equation (5.17), $(-2p)$ is substituted directly for θ.

$$\frac{\theta(\theta+1)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{-\theta}}\bigg|_{\theta=-2p} = \frac{(-2p)(-2p+1)}{2}\pi^{-p}\Gamma(p)\sum_{n=1}^{\infty}\frac{1}{n^{2p}} = \frac{p!\,(2p-1)}{\pi^{p}}\sum_{n=1}^{\infty}\frac{1}{n^{2p}},\quad p\in\mathbb{N}. \tag{5.21}$$

This result is the same as the result of equation (5.18), so one equation can be presented.

$$B_{[2]}(\theta) = \frac{\theta(\theta+1)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{-\theta}},\quad \mathrm{Re}(\theta)<-1. \tag{5.22}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$B_{[2]}(\theta) = \frac{\theta(\theta+1)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta),\quad \theta\in\mathbb{C}. \tag{5.23}$$

The function $B_{[2]}(\theta)$ is also a convergent function in the whole complex plane. And the following origin symmetry holds:

$$A_{[2]}(-\theta) = B_{[2]}(\theta),\quad \theta\in\mathbb{C}. \tag{5.24}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $F_{[2]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{m\,y}{\sqrt{n^2+\alpha^2}}\right)^{\frac{1}{2}}.$$

$$\begin{aligned} &F_{[2]}(\theta) \\ &= 2\pi\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\left(\begin{aligned}&-6n^2\left(\frac{m\,y}{\sqrt{n^2+\alpha^2}}\right)^{\frac{\theta+1}{2}}\\ &+4\pi n^4\left(\frac{m\,y}{\sqrt{n^2+\alpha^2}}\right)^{\frac{\theta+3}{2}}\end{aligned}\right)e^{-\pi m\sqrt{n^2+\alpha^2}\left(y+\frac{1}{y}\right)}\frac{1}{2}\left(\frac{m}{\sqrt{n^2+\alpha^2}}\right)^{\frac{1}{2}}y^{-\frac{1}{2}}dy \\ &= 2\pi\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\left(\begin{aligned}&-6n^2\left(\frac{m}{\sqrt{n^2+\alpha^2}}\right)^{\frac{\theta+2}{2}}\cdot\frac{1}{2}\int_0^{\infty}y^{\frac{\theta}{2}}e^{-\pi m\sqrt{n^2+\alpha^2}\left(y+\frac{1}{y}\right)}dy\\ &+4\pi n^4\left(\frac{m}{\sqrt{n^2+\alpha^2}}\right)^{\frac{\theta+4}{2}}\cdot\frac{1}{2}\int_0^{\infty}y^{\frac{\theta+2}{2}}e^{-\pi m\sqrt{n^2+\alpha^2}\left(y+\frac{1}{y}\right)}dy\end{aligned}\right). \end{aligned} \tag{5.25}$$

Using the left-sided limit of the positive real variable α,

$$\begin{aligned} F_{[2]}(\theta) &= 2\pi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\begin{aligned}&-6n^2\left(\frac{m}{n}\right)^{\frac{\theta+2}{2}}\cdot\frac{1}{2}\int_0^{\infty}y^{\frac{\theta}{2}}e^{-\pi mn\left(y+\frac{1}{y}\right)}dy\\ &+4\pi n^4\left(\frac{m}{n}\right)^{\frac{\theta+4}{2}}\cdot\frac{1}{2}\int_0^{\infty}y^{\frac{\theta+2}{2}}e^{-\pi mn\left(y+\frac{1}{y}\right)}dy\end{aligned}\right) \\ &= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{aligned}&-3\cdot\frac{1}{2}\int_0^{\infty}y^{\frac{\theta+2}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy\\ &+2(\pi mn)\cdot\frac{1}{2}\int_0^{\infty}y^{\frac{\theta+4}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy\end{aligned}\right). \end{aligned} \tag{5.26}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$F_{[2]}(\theta) = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(-3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)+2(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)\right). \tag{5.27}$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied twice.

(5.28)
$$\begin{aligned}
F_{[2]}(\theta) &= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(-3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)+2\left(\frac{\theta+2}{2}\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right)\right)\\
&= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left((\theta-1)(\pi mn)\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)+2(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right)\\
&= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{aligned}(\theta-1)\left(\frac{\theta}{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn)\right)\\ +2(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\end{aligned}\right)\\
&= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\theta(\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)+(\theta-1)(2\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn)+(2\pi mn)^2\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right).
\end{aligned}$$

For the integral of $G_{[2]}(\theta)$, I perform the variable transformation $x=y^{-1}$.

(5.29)
$$\begin{aligned}
G_{[2]}(\theta) &= 2\pi\lim_{\alpha\to+0}\int_{\infty}^{0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(-6n^2\left(\frac{1}{y}\right)^{\theta-3}+4\pi n^4\left(\frac{1}{y}\right)^{\theta-5}\right)e^{-\pi\left(\frac{m^2+\alpha^2}{y^2}+n^2y^2\right)}\left(-y^{-2}\right)dy\\
&= 2\pi\lim_{\alpha\to+0}\int_{0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(-6n^2y^{1-\theta}+4\pi n^4y^{3-\theta}\right)e^{-\pi\left(n^2y^2+\frac{m^2+\alpha^2}{y^2}\right)}dy.
\end{aligned}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $G_{[2]}(\theta)$, I perform the variable transformation

$$y=\left(\frac{\sqrt{m^2+\alpha^2}\,x}{n}\right)^{\frac{1}{2}}.$$

(5.30)
$$\begin{aligned}
&G_{[2]}(\theta)\\
&= 2\pi\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{0}^{\infty}\left(\begin{aligned}-6n^2\left(\frac{\sqrt{m^2+\alpha^2}\,x}{n}\right)^{\frac{1-\theta}{2}}\\ +4\pi n^4\left(\frac{\sqrt{m^2+\alpha^2}\,x}{n}\right)^{\frac{3-\theta}{2}}\end{aligned}\right)e^{-\pi\sqrt{m^2+\alpha^2}\,n\left(x+\frac{1}{x}\right)}\frac{1}{2}\left(\frac{\sqrt{m^2+\alpha^2}}{n}\right)^{\frac{1}{2}}x^{-\frac{1}{2}}dx\\
&= 2\pi\lim_{\alpha\to+0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\begin{aligned}-6n^2\left(\frac{\sqrt{m^2+\alpha^2}}{n}\right)^{\frac{2-\theta}{2}}\cdot\frac{1}{2}\int_{0}^{\infty}x^{-\frac{\theta}{2}}e^{-\pi\sqrt{m^2+\alpha^2}\,n\left(x+\frac{1}{x}\right)}dx\\ +4\pi n^4\left(\frac{\sqrt{m^2+\alpha^2}}{n}\right)^{\frac{4-\theta}{2}}\cdot\frac{1}{2}\int_{0}^{\infty}x^{\frac{2-\theta}{2}}e^{-\pi\sqrt{m^2+\alpha^2}\,n\left(x+\frac{1}{x}\right)}dx\end{aligned}\right).
\end{aligned}$$

Using the left-sided limit of the positive real variable α,

(5.31)
$$\begin{aligned}
G_{[2]}(\theta) &= 2\pi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\begin{aligned}-6n^2\left(\frac{m}{n}\right)^{\frac{2-\theta}{2}}\cdot\frac{1}{2}\int_{0}^{\infty}x^{-\frac{\theta}{2}}e^{-\pi mn\left(x+\frac{1}{x}\right)}dx\\ +4\pi n^4\left(\frac{m}{n}\right)^{\frac{4-\theta}{2}}\cdot\frac{1}{2}\int_{0}^{\infty}x^{\frac{2-\theta}{2}}e^{-\pi mn\left(x+\frac{1}{x}\right)}dx\end{aligned}\right)\\
&= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\left(\begin{aligned}-3\cdot\frac{1}{2}\int_{0}^{\infty}x^{\frac{2-\theta}{2}-1}e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)}dx\\ +2(\pi mn)\cdot\frac{1}{2}\int_{0}^{\infty}x^{\frac{4-\theta}{2}-1}e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)}dx\end{aligned}\right).
\end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

(5.32)
$$G_{[2]}(\theta)=4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\left(-3\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)+2(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\right).$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$\begin{aligned}G_{[2]}(\theta) &= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\left(\begin{aligned}&-3\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\\&+2\left(\frac{2-\theta}{2}\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{-\frac{\theta}{2}}(2\pi mn)\right)\end{aligned}\right)\\
&= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\left(-(\theta+1)(\pi mn)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)+2(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right)\\
&= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\left(\begin{aligned}&-(\theta+1)\left(-\frac{\theta}{2}\mathrm{K}_{-\frac{\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{-\frac{2+\theta}{2}}(2\pi mn)\right)\\&+2(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\end{aligned}\right)\\
&= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\left(\theta(\theta+1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)-(\theta+1)(2\pi mn)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi mn)+(2\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right).\end{aligned}\tag{5.33}$$

The following origin symmetry holds:

$$F_{[2]}(\pm\theta) = G_{[2]}(\mp\theta).\quad\text{(the double sign is in same order)}\tag{5.34}$$

Now, I define the set of lattice points on the hyperbola $x\,y = p$ in the first quadrant as follows:

$$\Lambda_p := \{(m,\,n) : m,\,n\in\mathbb{N},\, m\,n = p\}.\tag{5.35}$$

From the definition,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sum_{(m,\,n)\in\Lambda_p}\left(\frac{n^2}{p}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sum_{(m,\,n)\in\Lambda_p} n^{\theta}p^{-\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sigma_{\theta}(p)\,p^{-\frac{\theta}{2}},\quad \theta\in\mathbb{C}.\tag{5.36}$$

Here, $\sigma_\theta(p)$ is the divisor sigma function. Conversely,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sum_{(m,\,n)\in\Lambda_p}\left(\frac{p}{n^2}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sum_{(m,\,n)\in\Lambda_p} n^{-\theta}p^{\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}},\quad \theta\in\mathbb{C}.\tag{5.37}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sum_{(m,\,n)\in\Lambda_p}\left(\frac{m^2}{p}\right)^{-\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sum_{(m,\,n)\in\Lambda_p} m^{-\theta}p^{\frac{\theta}{2}} = \sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}},\quad \theta\in\mathbb{C}.\tag{5.38}$$

The left sides of equations (5.36) and (5.37) are equivalent due to axial symmetry between the points $(m,\,n)$ and $(n,\,m)$.
And I obtain the following formula with the origin symmetry for the complex variable θ:

$$\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}} = \sigma_{\theta}(p)\,p^{-\frac{\theta}{2}},\quad (p\in\mathbb{N},\,\theta\in\mathbb{C}).\tag{5.39}$$

From equations (5.28) and (5.37),

$$F_{[2]}(\theta) = 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\left(-\theta(1-\theta)+(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)-(1-\theta)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right).\tag{5.40}$$

From equations (5.33) and (5.38),

$$G_{[2]}(\theta) = 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\left(\theta(1+\theta)+(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)-(1+\theta)(2\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\right).\tag{5.41}$$

The same term appears on the right sides of equations (5.40) and (5.41). I define the following equations to reduce it:

$$\hat{F}_{[2]}(\theta) := F_{[2]}(\theta) - 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p).\tag{5.42}$$

$$\hat{G}_{[2]}(\theta) := G_{[2]}(\theta) - 2\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} (2\pi p)^2 \operatorname{K}_{\frac{\theta}{2}}(2\pi p). \tag{5.43}$$

Therefore,

$$\hat{F}_{[2]}(\theta) = -2(1-\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \left(\theta \operatorname{K}_{\frac{\theta}{2}}(2\pi p) + (2\pi p)\operatorname{K}_{\frac{2-\theta}{2}}(2\pi p)\right). \tag{5.44}$$

$$\hat{G}_{[2]}(\theta) = -2(1+\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \left(-\theta \operatorname{K}_{\frac{\theta}{2}}(2\pi p) + (2\pi p)\operatorname{K}_{\frac{2+\theta}{2}}(2\pi p)\right). \tag{5.45}$$

For the right side of equation (5.44), the recurrence formula for the modified Bessel function of the second kind is applied.

$$\begin{aligned} \hat{F}_{[2]}(\theta) &= -2(1-\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \left(\theta \operatorname{K}_{\frac{\theta}{2}}(2\pi p) + 2\left(-\frac{\theta}{2}\operatorname{K}_{-\frac{\theta}{2}}(2\pi p) + (\pi p)\operatorname{K}_{-\frac{2+\theta}{2}}(2\pi p)\right)\right) \\ &= -2(1-\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} (2\pi p)\operatorname{K}_{\frac{2+\theta}{2}}(2\pi p). \end{aligned} \tag{5.46}$$

By changing to the integral representation,

$$\hat{F}_{[2]}(\theta) = -2(1-\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} (2\pi p) \cdot \frac{1}{2}\int_0^{\infty} x^{\frac{2+\theta}{2}-1} e^{-\frac{2\pi p}{2}\left(x+\frac{1}{x}\right)} dx. \tag{5.47}$$

For the integral of $\hat{F}_{[2]}(\theta)$, I perform the variable transformation

$$x = \frac{y^2}{\pi p}.$$

$$\begin{aligned} \hat{F}_{[2]}(\theta) &= -2(1-\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} (2\pi p) \cdot \frac{1}{2}\int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta}{2}} e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \\ &= -2(1-\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} (2\pi p) \cdot \left(\frac{1}{\pi p}\right)^{\frac{\theta+2}{2}} \int_0^{\infty} y^{\theta+1} e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)} dy. \end{aligned} \tag{5.48}$$

Therefore,

$$\hat{F}_{[2]}(\theta) = 4(\theta-1)\pi^{-\frac{\theta}{2}} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \int_0^{\infty} y^{\theta+1} e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)} dy. \tag{5.49}$$

For the right side of equation (5.45), I change to the integral representations.

$$\hat{G}_{[2]}(\theta) = -2(1+\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \begin{pmatrix} -\theta \cdot \frac{1}{2}\int_0^{\infty} x^{\frac{\theta}{2}-1} e^{-\frac{2\pi p}{2}\left(x+\frac{1}{x}\right)} dx \\ +(2\pi p)\cdot \frac{1}{2}\int_0^{\infty} x^{\frac{2+\theta}{2}-1} e^{-\frac{2\pi p}{2}\left(x+\frac{1}{x}\right)} dx \end{pmatrix}. \tag{5.50}$$

For the integrals of $\hat{G}_{[2]}(\theta)$, I perform the variable transformation

$$x = \frac{y^2}{\pi p}.$$

$$\hat{G}_{[2]}(\theta) = -2(1+\theta)\sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \begin{pmatrix} -\theta \cdot \frac{1}{2}\int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta-2}{2}} e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \\ +(2\pi p)\cdot \frac{1}{2}\int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta}{2}} e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \end{pmatrix}. \tag{5.51}$$

Therefore,

$$\hat{G}_{[2]}(\theta) = 2(\theta+1)\pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(\theta\int_0^{\infty} y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy - 2\int_0^{\infty} y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\right). \tag{5.52}$$

The following equations are defined to evaluate the absolute convergence of the functions $\hat{F}_{[2]}(\theta)$ and $\hat{G}_{[2]}(\theta)$:

$$S_{\pm}(\theta) := \sum_{p=1}^{\infty}\sigma_{-\theta}(p)\int_0^{\infty} y^{\pm 1+\mathrm{Re}(\theta)}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy. \quad \text{(the double sign is in same order)} \tag{5.53}$$

For both sides of equation (5.49), the absolute value is taken.

$$\left|\hat{F}_{[2]}(\theta)\right| \le 4|\theta-1|\pi^{-\frac{\mathrm{Re}(\theta)}{2}}|S_{+}(\theta)|. \tag{5.54}$$

Similarly, for both sides of equation (5.52), the absolute value is taken.

$$\left|\hat{G}_{[2]}(\theta)\right| \le 2|\theta+1|\pi^{-\frac{\mathrm{Re}(\theta)}{2}}\left(|\theta|\cdot|S_{-}(\theta)| + 2|S_{+}(\theta)|\right). \tag{5.55}$$

Furthermore, for both sides of equations (5.53), the absolute values are taken.

$$|S_{\pm}(\theta)| \le \sum_{p=1}^{\infty}\sigma_{-\mathrm{Re}(\theta)}(p)\int_0^{\infty} y^{\pm 1+\mathrm{Re}(\theta)}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy. \quad \text{(the double sign is in same order)} \tag{5.56}$$

Now, the following obvious fact is introduced into the inequalities (5.56):

$$\sigma_{-\mathrm{Re}(\theta)}(p) < p^2\zeta(2+\mathrm{Re}(\theta)), \quad (\mathrm{Re}(\theta) > -1,\, p\in\mathbb{N}). \tag{5.57}$$

The following proposition decides that the condition should be $\mathrm{Re}(\theta) > -1$:

$$2+\mathrm{Re}(\theta) > 1 \implies \zeta(2+\mathrm{Re}(\theta)) > 1. \tag{5.58}$$

Under the condition of $\mathrm{Re}(\theta) > -1$, the absolute value of $S_{+}(\theta)$ is written using the absolutely convergent integrals.

$$|S_{+}(\theta)| < \zeta(2+\mathrm{Re}(\theta))\sum_{p=1}^{\infty}p^2\int_0^{\infty} y^{1+\mathrm{Re}(\theta)}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy, \quad \mathrm{Re}(\theta) > -1. \tag{5.59}$$

Because the variable of the zeta function is irrelevant to variable p, the zeta function of the infinite sum is shifted outside.
Under the condition of $\mathrm{Re}(\theta) > 0$, the absolute value of $S_{-}(\theta)$ is also written using the absolutely convergent integrals.

$$|S_{-}(\theta)| < \zeta(2+\mathrm{Re}(\theta))\sum_{p=1}^{\infty}p^2\int_0^{\infty} y^{-1+\mathrm{Re}(\theta)}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy, \quad \mathrm{Re}(\theta) > 0. \tag{5.60}$$

Here, the condition $\mathrm{Re}(\theta) > 0$ is applied to the gamma function described later.
Because each integral converges absolutely, I can change the order of relation between the infinite sum and the integrals for the function $S_{+}(\theta)$. And the infinite sum is written as the limit of the L-th partial sum.

$$|S_{+}(\theta)| < \zeta(2+\mathrm{Re}(\theta))\int_0^{\infty} y^{1+\mathrm{Re}(\theta)}e^{-y^2}\lim_{L\to\infty}\sum_{p=1}^{L}p^2\, e^{-\frac{(\pi p)^2}{y^2}}dy, \quad \mathrm{Re}(\theta) > -1. \tag{5.61}$$

The end point of the partial sum is expanded to L^2 from L after correcting the function in the finite sum.

$$|S_{+}(\theta)| < \zeta(2+\mathrm{Re}(\theta))\int_0^{\infty} y^{1+\mathrm{Re}(\theta)}e^{-y^2}\lim_{L\to\infty}\sum_{p=1}^{L^2}p\, e^{-\frac{\pi^2 p}{y^2}}dy, \quad \mathrm{Re}(\theta) > -1. \tag{5.62}$$

From the result of equation (19.90), the limit value of the sum of L^2 terms is described using the hyperbolic sine.

$$|S_{+}(\theta)| < \frac{\zeta(2+\mathrm{Re}(\theta))}{4}\int_0^{\infty} y^{1+\mathrm{Re}(\theta)}\frac{e^{-y^2}}{\sinh^2\left(\frac{\pi^2}{2y^2}\right)}dy, \quad \mathrm{Re}(\theta) > -1. \tag{5.63}$$

By the same way,

(5.64) $$|S_{-}(\theta)| < \frac{\zeta(2+\mathrm{Re}(\theta))}{4}\int_0^\infty y^{-1+\mathrm{Re}(\theta)}\frac{e^{-y^2}}{\sinh^2\left(\frac{\pi^2}{2y^2}\right)}dy, \quad \mathrm{Re}(\theta) > 0.$$

To evaluate the integrals of inequalities (5.63) and (5.64), I prepare the function $f(x)$.
The defining equation is shown as follows:

(5.65) $$f(x) := \frac{1}{\pi^3}e^{-\frac{x^2}{\pi}} - \frac{e^{-x^2}}{\sinh^2\left(\frac{\pi^2}{2x^2}\right)}, \quad x > 0.$$

Function values at both sides of the open interval are given by taking the limits.

(5.66) $$\lim_{x\to+0} f(x) = \frac{1}{\pi^3}.$$

(5.67) $$\lim_{x\to\infty} f(x) = 0.$$

The function $f(x)$ is differentiated so as to understand its behavior.

(5.68) $$\begin{aligned} f'(x) &= -\frac{2x}{\pi^4}e^{-\frac{x^2}{\pi}} + \frac{2x\,e^{-x^2}}{\sinh^2\left(\frac{\pi^2}{2x^2}\right)} - \frac{2\pi^2\,e^{-x^2}\cosh\left(\frac{\pi^2}{2x^2}\right)}{x^3\sinh^3\left(\frac{\pi^2}{2x^2}\right)} \\ &= \left(\begin{aligned} &-x^4\,e^{-\frac{x^2}{\pi}}\sinh^3\left(\frac{\pi^2}{2x^2}\right) \\ &+\pi^4x^4\,e^{-x^2}\sinh\left(\frac{\pi^2}{2x^2}\right) \\ &-\pi^6\,e^{-x^2}\cosh\left(\frac{\pi^2}{2x^2}\right)\end{aligned}\right) \Big/ \left(\frac{\pi^4}{2}x^3\sinh^3\left(\frac{\pi^2}{2x^2}\right)\right), \quad x > 0.\end{aligned}$$

α_1 and β_1 are presumably the solutions to the equation $f'(x) = 0$.
The numerical solutions are shown as follows:

(5.69) $$f'(\alpha_1) = f'(\beta_1) = 0, \quad (\alpha_1 \simeq 2.12138,\ \beta_1 \simeq 2.59993).$$

(5.70) $$f(\alpha_1) \simeq 0.00141964 > 0.$$

Table shows the increase and decrease in the function

x	$+0$	$\cdots$	α_1	$\cdots$	β_1	$\cdots$	∞
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	$1/\pi^3$ 0.0322515...	$\searrow$	minimum 0.0014196...	$\nearrow$	maximum 0.0019232...	$\searrow$	0

Table. 5.1

From the table of the increase and decrease,

(5.71) $$f(x) > 0, \quad x > 0.$$

Therefore

(5.72) $$\frac{e^{-x^2}}{\sinh^2\left(\frac{\pi^2}{2x^2}\right)} < \frac{1}{\pi^3}e^{-\frac{x^2}{\pi}}, \quad x > 0.$$

This result is introduced into the right side of inequality (5.63).
And for the integral, I perform the variable transformation

$$y = (\pi x)^{\frac{1}{2}}.$$

$$
\begin{aligned}
|S_{+}(\theta)| &< \frac{\zeta(2+\mathrm{Re}(\theta))}{4\pi^3}\int_0^\infty y^{1+\mathrm{Re}(\theta)}e^{-\frac{y^2}{\pi}}dy \\
&= \frac{\zeta(2+\mathrm{Re}(\theta))}{4\pi^3}\int_0^\infty (\pi x)^{\frac{1+\mathrm{Re}(\theta)}{2}}e^{-x}\frac{1}{2}\pi^{\frac{1}{2}}x^{-\frac{1}{2}}dx \\
&= \frac{\zeta(2+\mathrm{Re}(\theta))}{8\pi^3}\pi^{\frac{2+\mathrm{Re}(\theta)}{2}}\int_0^\infty x^{\frac{2+\mathrm{Re}(\theta)}{2}-1}e^{-x}dx, \quad \mathrm{Re}(\theta) > -1.
\end{aligned} \tag{5.73}
$$

Therefore

$$
|S_{+}(\theta)| < \frac{1}{8\pi^3}\pi^{\frac{2+\mathrm{Re}(\theta)}{2}}\Gamma\left(\frac{2+\mathrm{Re}(\theta)}{2}\right)\zeta(2+\mathrm{Re}(\theta)), \quad \mathrm{Re}(\theta) > -1. \tag{5.74}
$$

By the same way,

$$
\begin{aligned}
|S_{-}(\theta)| &< \frac{\zeta(2+\mathrm{Re}(\theta))}{4\pi^3}\int_0^\infty y^{-1+\mathrm{Re}(\theta)}e^{-\frac{y^2}{\pi}}dy \\
&= \frac{\zeta(2+\mathrm{Re}(\theta))}{4\pi^3}\int_0^\infty (\pi x)^{\frac{-1+\mathrm{Re}(\theta)}{2}}e^{-x}\frac{1}{2}\pi^{\frac{1}{2}}x^{-\frac{1}{2}}dx \\
&= \frac{\zeta(2+\mathrm{Re}(\theta))}{8\pi^3}\pi^{\frac{\mathrm{Re}(\theta)}{2}}\int_0^\infty x^{\frac{\mathrm{Re}(\theta)}{2}-1}e^{-x}dx, \quad \mathrm{Re}(\theta) > 0.
\end{aligned} \tag{5.75}
$$

Therefore

$$
|S_{-}(\theta)| < \frac{1}{8\pi^3}\pi^{\frac{\mathrm{Re}(\theta)}{2}}\Gamma\left(\frac{\mathrm{Re}(\theta)}{2}\right)\zeta(2+\mathrm{Re}(\theta)), \quad \mathrm{Re}(\theta) > 0. \tag{5.76}
$$

From inequalities (5.54) and (5.74),

$$
\begin{aligned}
\left|\hat{F}_{[2]}(\theta)\right| &\le 4|\theta-1|\pi^{-\frac{\mathrm{Re}(\theta)}{2}}\cdot|S_{+}(\theta)| \\
&= 4|\theta-1|\pi^{-\frac{\mathrm{Re}(\theta)}{2}}\cdot\frac{1}{8\pi^3}\pi^{\frac{2+\mathrm{Re}(\theta)}{2}}\Gamma\left(\frac{2+\mathrm{Re}(\theta)}{2}\right)\zeta(2+\mathrm{Re}(\theta)) \\
&= \frac{|\theta-1|}{2\pi^2}\Gamma\left(\frac{2+\mathrm{Re}(\theta)}{2}\right)\zeta(2+\mathrm{Re}(\theta)), \quad (\mathrm{Re}(\theta) > -1, \text{ the equality sign holds at } \theta = 1).
\end{aligned} \tag{5.77}
$$

From inequalities (5.55), (5.74), and (5.76),

$$
\begin{aligned}
\left|\hat{G}_{[2]}(\theta)\right| &< 2|\theta+1|\pi^{-\frac{\mathrm{Re}(\theta)}{2}}\left(|\theta|\cdot|S_{-}(\theta)|+2\cdot|S_{+}(\theta)|\right) \\
&= 2|\theta+1|\pi^{-\frac{\mathrm{Re}(\theta)}{2}}\left(\begin{aligned}&|\theta|\cdot\frac{1}{8\pi^3}\pi^{\frac{\mathrm{Re}(\theta)}{2}}\Gamma\left(\frac{\mathrm{Re}(\theta)}{2}\right)\zeta(2+\mathrm{Re}(\theta)) \\ &+2\cdot\frac{1}{8\pi^3}\pi^{\frac{2+\mathrm{Re}(\theta)}{2}}\Gamma\left(\frac{2+\mathrm{Re}(\theta)}{2}\right)\zeta(2+\mathrm{Re}(\theta))\end{aligned}\right) \\
&= \frac{|\theta+1|\left(|\theta|+\pi\,\mathrm{Re}(\theta)\right)}{4\pi^3}\Gamma\left(\frac{\mathrm{Re}(\theta)}{2}\right)\zeta(2+\mathrm{Re}(\theta)), \quad \mathrm{Re}(\theta) > 0.
\end{aligned} \tag{5.78}
$$

From the origin symmetry between the functions $\hat{F}_{[2]}(\theta)$ and $\hat{G}_{[2]}(\theta)$,

$$
\begin{aligned}
\left|\hat{F}_{[2]}(\theta)\right| &= \left|\hat{G}_{[2]}(-\theta)\right| \\
&< \frac{|1-\theta|\left(|\theta|-\pi\,\mathrm{Re}(\theta)\right)}{4\pi^3}\Gamma\left(\frac{-\mathrm{Re}(\theta)}{2}\right)\zeta(2-\mathrm{Re}(\theta)), \quad \mathrm{Re}(\theta) < 0.
\end{aligned} \tag{5.79}
$$

$$
\begin{aligned}
\left|\hat{G}_{[2]}(\theta)\right| &= \left|\hat{F}_{[2]}(-\theta)\right| \\
&\le \frac{|1+\theta|}{2\pi^2}\Gamma\left(\frac{2-\mathrm{Re}(\theta)}{2}\right)\zeta(2-\mathrm{Re}(\theta)), \quad (\mathrm{Re}(\theta) < 1, \text{ the equality sign holds at } \theta = -1).
\end{aligned} \tag{5.80}
$$

The function $\hat{F}_{[2]}(\theta)$ is an absolutely convergent function in the whole complex plane, based on the inequalities (5.77) and (5.79).
The function $\hat{G}_{[2]}(\theta)$ is also an absolutely convergent function in the whole complex plane, based on the inequalities (5.78) and (5.80).
Combining the above results yields the following functional equation:

$$A_{[2]}(\theta) + \hat{F}_{[2]}(\theta) = B_{[2]}(\theta) + \hat{G}_{[2]}(\theta), \quad \theta \in \mathbb{C}. \tag{5.81}$$

Where

$$A_{[2]}(\theta) = \frac{-\theta(1-\theta)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta). \tag{5.82}$$

$$B_{[2]}(\theta) = \frac{\theta(1+\theta)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta). \tag{5.83}$$

$$\hat{F}_{[2]}(\theta) = -2(1-\theta)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + (2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right). \tag{5.84}$$

$$\hat{G}_{[2]}(\theta) = -2(1+\theta)\sum_{p=1}^{\infty}\sigma_{\theta}(p)\,p^{-\frac{\theta}{2}}\left(-\theta\,\mathrm{K}_{-\frac{\theta}{2}}(2\pi p) + (2\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\right). \tag{5.85}$$

Here, in order to emphasize the origin symmetry, equations (3.30) and (5.39) are applied to equation (5.45). In the whole complex plane, all functions on both sides of the functional equation (5.85) are convergent.
It is the origin symmetric functional equation with correction terms, and in this case the functions $\hat{F}_{[2]}(\theta)$ and $\hat{F}_{[2]}(\theta)$ are applicable to the correction terms.
I decide that it is called the second-order II_c type functional equation.
A symmetrical poit of the the second-order II_c type functional equation is shifted to the left "[9, 10]."

6 The First Proof of the Riemann Hypothesis

6.1 Derivation of a Representation Containing the Leading Term of the Zeta Function for any Complex Number

The second-order II_c type functional equation is shown again, but with a bit of difference.

$$A_{[2]}(\theta) = B_{[2]}(\theta) + H_{[2]}(\theta), \quad \theta \in \mathbb{C}. \tag{6.1}$$

where

$$A_{[2]}(\theta) = \frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta). \tag{6.2}$$

$$B_{[2]}(\theta) = \frac{\theta(\theta+1)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta) = \frac{\theta(\theta+1)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta). \tag{6.3}$$

$$H_{[2]}(\theta) := \hat{G}_{[2]}(\theta) - \hat{F}_{[2]}(\theta). \tag{6.4}$$

Equations (5.45) and (5.84) are introduced into the equation (6.4).

$$\begin{aligned} H_{[2]}(\theta) = &-2(1+\theta)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(-\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + (2\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\right) \\ &+ 2(1-\theta)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + (2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{6.5}$$

Therefore,

$$H_{[2]}(\theta) = -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix} -2\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + (\theta-1)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \\ + (\theta+1)(2\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\end{pmatrix}, \quad \theta \in \mathbb{C}. \tag{6.6}$$

The recurrence formula for the modified Bessel function of the second kind is applied to the equation (6.6).

$$\begin{aligned} H_{[2]}(\theta) &= -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l} -2\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + (\theta-1)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \\ + 2(\theta+1)\left(\frac{\theta}{2}\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + (\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\right)\end{array}\right) \\ &= -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\theta(\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + 2\theta(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{6.7}$$

Therefore,

$$H_{[2]}(\theta) = -2\theta\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta \in \mathbb{C}. \tag{6.8}$$

By combining the equations (6.2), (6.3), and (6.8),

$$\begin{aligned} \frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta) &= \frac{\theta(\theta+1)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta) \\ &- 2\theta\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{6.9}$$

Both sides of equation (6.9) are multiplied by

$$\frac{2}{\theta(\theta-1)\Gamma\left(\frac{\theta}{2}\right)}\pi^{\frac{\theta}{2}}$$

to obtain the zeta-hat function.

$$\begin{aligned} \hat{\zeta}(\theta) &= \frac{(\theta+1)\Gamma\left(\frac{1+\theta}{2}\right)}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\sqrt{\pi}}\zeta(1+\theta) \\ &- \frac{4\pi^{\frac{\theta}{2}}}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta \in \mathbb{C}\backslash\{1\}. \end{aligned} \tag{6.10}$$

This is the representation containing the leading term of the zeta function for any complex number.
The above equation also gives the analytic continuation of the zeta function explicitly.

6.2 The First Proof of the Riemann Hypothesis

The Riemann hypothesis is demonstrated using the following equation:

$$\begin{aligned} (\theta-1)\Gamma\left(\frac{\theta}{2}\right)\pi^{-\frac{\theta}{2}}\hat{\zeta}(\theta) &= (\theta+1)\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta) \\ &+ 4\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{6.11}$$

Here, the above equation is obtained from

$$(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\pi^{-\frac{\theta}{2}} \times (\text{equation (6.10)}).$$

Pioneers' previous research established the existence of non-trivial zeros on the critical strip.
It is complicated to describe and insignificant to examine by focusing on this region.
The proof of the reduction to absurdity will be performed by assuming the whole complex plane as the target region, excluding the critical line, and assuming the existence of at least one non-trivial zero $(x+iy)$.

Under the assumption,

$$\hat{\zeta}(x+iy)=0,\quad (x\in\mathbb{R}\backslash\{1/2\},\ y\in\mathbb{R}). \tag{6.12}$$

Under the assumption, the point $(1-x-iy)$ is also the non-trivial zero of the zeta function due to the point symmetry of the Riemann Xi function.

$$\hat{\zeta}(1-x-iy)=0,\quad (x\in\mathbb{R}\backslash\{1/2\},\ y\in\mathbb{R}). \tag{6.13}$$

For equation (6.11), $(x+iy)$ is substituted for θ.

$$\begin{aligned}&(x+iy-1)\,\Gamma\left(\frac{x+iy}{2}\right)\pi^{-\frac{x+iy}{2}}\hat{\zeta}(x+iy)=\\&\quad+(x+iy+1)\,\pi^{-\frac{1+x+iy}{2}}\Gamma\left(\frac{1+x+iy}{2}\right)\zeta(1+x+iy)\\&+4\sum_{p=1}^{\infty}\sigma_{-x-iy}(p)\,p^{\frac{x+iy}{2}}\Big((x+iy-1)\,\mathrm{K}_{\frac{x+iy}{2}}(2\pi p)+2\,(2\pi p)\,\mathrm{K}_{\frac{2-x-iy}{2}}(2\pi p)\Big),\\&(x\in\mathbb{R}\backslash\{1/2\},\ y\in\mathbb{R}).\end{aligned} \tag{6.14}$$

For equation (6.11), $(1-x-iy)$ is substituted for θ.

$$\begin{aligned}&(-x-iy)\,\Gamma\left(\frac{1-x-iy}{2}\right)\pi^{-\frac{1-x-iy}{2}}\hat{\zeta}(1-x-iy)=\\&\quad+(2-x-iy)\,\pi^{-\frac{2-x-iy}{2}}\Gamma\left(\frac{2-x-iy}{2}\right)\zeta(2-x-iy)\\&+4\sum_{p=1}^{\infty}\sigma_{-1+x+iy}(p)\,p^{\frac{1-x-iy}{2}}\Big((-x-iy)\,\mathrm{K}_{\frac{1-x-iy}{2}}(2\pi p)+2\,(2\pi p)\,\mathrm{K}_{\frac{1+x+iy}{2}}(2\pi p)\Big),\\&(x\in\mathbb{R}\backslash\{1/2\},\ y\in\mathbb{R}).\end{aligned} \tag{6.15}$$

According to the assumption, the left side of equation (6.14) equals zero.

$$\begin{aligned}0=&\,(1+x+iy)\,\pi^{-\frac{1+x+iy}{2}}\Gamma\left(\frac{1+x+iy}{2}\right)\zeta(1+x+iy)\\&+4\sum_{p=1}^{\infty}\sigma_{-x-iy}(p)\,p^{\frac{x+iy}{2}}\Big((x-1+iy)\,\mathrm{K}_{\frac{x+iy}{2}}(2\pi p)+2\,(2\pi p)\,\mathrm{K}_{\frac{2-x-iy}{2}}(2\pi p)\Big),\\&(x\in\mathbb{R}\backslash\{1/2\},\ y\in\mathbb{R}).\end{aligned} \tag{6.16}$$

Based on this assumption, the left side of equation (6.15) is also zero.

$$\begin{aligned}0=&\,(2-x-iy)\,\pi^{-\frac{2-x-iy}{2}}\Gamma\left(\frac{2-x-iy}{2}\right)\zeta(2-x-iy)\\&+4\sum_{p=1}^{\infty}\sigma_{x-1+iy}(p)\,p^{\frac{1-x-iy}{2}}\Big((-x-iy)\,\mathrm{K}_{\frac{1-x-iy}{2}}(2\pi p)+2\,(2\pi p)\,\mathrm{K}_{\frac{1+x+iy}{2}}(2\pi p)\Big),\\&(x\in\mathbb{R}\backslash\{1/2\},\ y\in\mathbb{R}).\end{aligned} \tag{6.17}$$

The non-trivial zeros of the zeta function off the critical line are given by the solutions of the simultaneous equations with the qualification consisting of the equations (6.16) and (6.17).
To solve the simultaneous equations with the qualification, I consider the simultaneous equations excluding the qualification.
The following are the simultaneous equations without qualification:

$$\begin{aligned}&(1+x+iy)\,\pi^{-\frac{1+x+iy}{2}}\Gamma\left(\frac{1+x+iy}{2}\right)\zeta(1+x+iy)\\&+4\sum_{p=1}^{\infty}\sigma_{-x-iy}(p)\,p^{\frac{x+iy}{2}}\Big((x-1+iy)\,\mathrm{K}_{\frac{x+iy}{2}}(2\pi p)+2\,(2\pi p)\,\mathrm{K}_{\frac{2-x-iy}{2}}(2\pi p)\Big)=0,\quad (x,\,y\in\mathbb{R}).\end{aligned} \tag{6.18}$$

$$(2-x-iy)\,\pi^{-\frac{2-x-iy}{2}}\Gamma\left(\frac{2-x-iy}{2}\right)\zeta\left(2-x-iy\right) + 4\sum_{p=1}^{\infty}\sigma_{x-1+iy}\left(p\right)p^{\frac{1-x-iy}{2}}\left(\left(-x-iy\right)\mathrm{K}_{\frac{1-x-iy}{2}}\left(2\pi p\right)+2\left(2\pi p\right)\mathrm{K}_{\frac{1+x+iy}{2}}\left(2\pi p\right)\right)=0,\quad \left(x,\,y\in\mathbb{R}\right). \tag{6.19}$$

The non-trivial zeros are determined by solving the simultaneous equations (6.18) and (6.19).
Because the equations are symmetrical,

$$x=\frac{1}{2} \tag{6.20}$$

is immediately determined.
For equation (6.18), 1/2 is substituted for x.

$$\left(\frac{3}{2}+iy\right)\pi^{-\frac{3+2iy}{4}}\Gamma\left(\frac{3+2iy}{4}\right)\zeta\left(\frac{3}{2}+iy\right) + 4\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-iy}\left(p\right)p^{\frac{1+2iy}{4}}\left(\left(-\frac{1}{2}+iy\right)\mathrm{K}_{\frac{1+2iy}{4}}\left(2\pi p\right)+2\left(2\pi p\right)\mathrm{K}_{\frac{3-2iy}{4}}\left(2\pi p\right)\right)=0,\quad y\in\mathbb{R}. \tag{6.21}$$

For equation (6.19), 1/2 is substituted for x.

$$\left(\frac{3}{2}-iy\right)\pi^{-\frac{3-2iy}{4}}\Gamma\left(\frac{3-2iy}{4}\right)\zeta\left(\frac{3}{2}-iy\right) + 4\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}+iy}\left(p\right)p^{\frac{1-2iy}{4}}\left(\left(-\frac{1}{2}-iy\right)\mathrm{K}_{\frac{1-2iy}{4}}\left(2\pi p\right)+2\left(2\pi p\right)\mathrm{K}_{\frac{3+2iy}{4}}\left(2\pi p\right)\right)=0,\quad y\in\mathbb{R}. \tag{6.22}$$

Equations (6.21) and (6.22) are complex conjugates of one another. And I can freely choose either of the equations (6.21) or (6.22) as the determining equation of the real variable y. The existence proof of the real solutions for the determining function is not given. As a result, for the simultaneous equations (6.16) and (6.17), there are no solutions off the critical line. This fact contradicts the assumption of reduction to absurdity. Therefore, the assumption "At least one non-trivial zero lies in the whole complex plane excluding the critical line" is denied, and "There is no non-trivial zero in the whole complex plane excluding the critical line at all" is confirmed. Incidentally, in collaboration with J.E. Littlewood, G.H. Hardy proved in 1914 that an infinite number of non-trivial zeros lie on the critical line "[11, 12]." By combining the time result and Hardy's result, I concluded that all non-trivial zeros of the zeta function lie on the critical line. Therefore, the Riemann hypothesis for the zeta function is proved correct.

6.3 Visualization of the Zeta-Hat Function

Currently, it is known that all real parts of non-trivial zeros of the zeta function are 1/2.
I define that ρ_m is the positive imaginary part of the zeta function's m-th non-trivial zero, to which the number is allocated in order that is closer to the real axis.

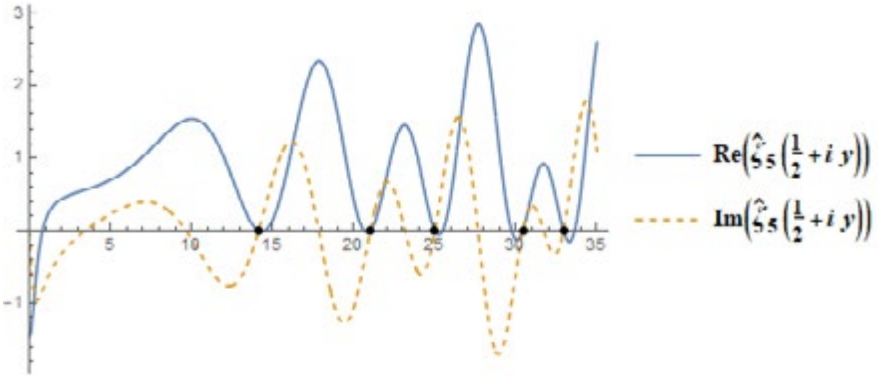
$$\hat{\zeta}\left(\frac{1}{2}+i\rho_m\right):=0,\quad \left(0<\rho_1\leq\rho_2\leq\rho_3\leq\cdots\right). \tag{6.23}$$

$$\{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\}\simeq\{14.134725142, 21.022039639, 25.010857580, 30.424876126, 32.935061588\}. \tag{6.24}$$

The infinite sum that is terminated at a finite term λ is used for the approximate calculation for the zeta function. Now, I decide that the positive integer λ will be referred to as the degree of approximation.
The approximate formula for the zeta-hat function is written as $(\text{zeta-hat})_{\lambda}\,(\theta)$.

$$\hat{\zeta}_{\lambda}\left(\theta\right):=\frac{\left(\theta+1\right)\Gamma\left(\frac{1+\theta}{2}\right)}{\left(\theta-1\right)\Gamma\left(\frac{\theta}{2}\right)\sqrt{\pi}}\zeta\left(1+\theta\right) - \frac{4\pi^{\frac{\theta}{2}}}{\left(\theta-1\right)\Gamma\left(\frac{\theta}{2}\right)}\sum_{p=1}^{\lambda}\sigma_{-\theta}\left(p\right)p^{\frac{\theta}{2}}\left(\left(\theta-1\right)\mathrm{K}_{\frac{\theta}{2}}\left(2\pi p\right)+2\left(2\pi p\right)\mathrm{K}_{\frac{2-\theta}{2}}\left(2\pi p\right)\right),\quad \left(\theta\in\mathbb{C}\backslash\{1\},\,\lambda\in\mathbb{N}\right). \tag{6.25}$$

Graph of $\operatorname{Re}\left(\hat{\zeta}_5\left(\frac{1}{2}+i\,y\right)\right)$ and $\operatorname{Im}\left(\hat{\zeta}_5\left(\frac{1}{2}+i\,y\right)\right)$, $y \in [0, 35]$



Graph. 6.1

The dots on the horizontal axis indicate the positive imaginary parts of non-trivial zeros of the zeta function. Five non-trivial zeros of the zeta function can be found on this interval. Here are the details.

$$\begin{aligned}
&\{\operatorname{Re}\left(\hat{\zeta}_7\left(\frac{1}{2}+i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Re}\left(\hat{\zeta}_7\left(\frac{1}{2}+i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{9.16162\times 10^{-11}, -1.57783\times 10^{-10}\},\\
&\{\operatorname{Im}\left(\hat{\zeta}_7\left(\frac{1}{2}+i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Im}\left(\hat{\zeta}_7\left(\frac{1}{2}+i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{-5.75483\times 10^{-10}, 9.91110\times 10^{-10}\}.
\end{aligned} \tag{6.26}$$

$$\begin{aligned}
&\{\operatorname{Re}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Re}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{-1.91909\times 10^{-10}, 3.05551\times 10^{-10}\},\\
&\{\operatorname{Im}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Im}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{-8.55883\times 10^{-10}, 1.36271\times 10^{-9}\}.
\end{aligned} \tag{6.27}$$

$$\begin{aligned}
&\{\operatorname{Re}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Re}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{5.15602\times 10^{-10}, -3.84471\times 10^{-10}\},\\
&\{\operatorname{Im}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Im}\left(\hat{\zeta}_8\left(\frac{1}{2}+i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{-1.48458\times 10^{-9}, 1.10701\times 10^{-9}\}.
\end{aligned} \tag{6.28}$$

$$\begin{aligned}
&\{\operatorname{Re}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Re}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{-5.73733\times 10^{-10}, 7.61286\times 10^{-10}\},\\
&\{\operatorname{Im}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Im}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{-9.62767\times 10^{-10}, 1.27749\times 10^{-9}\}.
\end{aligned} \tag{6.29}$$

$$\begin{aligned}
&\{\operatorname{Re}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{32\,935\,061\,588}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Re}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{32\,935\,061\,588}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{5.54802\times 10^{-10}, -9.46306\times 10^{-10}\},\\
&\{\operatorname{Im}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{32\,935\,061\,588}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \operatorname{Im}\left(\hat{\zeta}_9\left(\frac{1}{2}+i\left(\frac{32\,935\,061\,588}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\
&\simeq \{-8.57881\times 10^{-10}, 1.46326\times 10^{-9}\}.
\end{aligned} \tag{6.30}$$

7 The General Representation Containing the Leading Term of the Zeta Function for any Odd Number of 3 or More

In this section, the exact values of the zeta function for any even number of 4 or more will be treated as the known values for convenience.

7.1 Derivation of a General Representation Containing the Leading Term of the Zeta Function for any Odd Number of 3 or More

Equations (5.49) and (5.52) are introduced on the right side of equation (6.4).

$$\begin{aligned} H_{[2]}(\theta) =& 2(\theta+1)\pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(\theta\int_0^{\infty} y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy - 2\int_0^{\infty} y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\right) \\ & - 4(\theta-1)\pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\int_0^{\infty} y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy \\ & = -2\theta\pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\begin{pmatrix} 4\int_0^{\infty} y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy \\ -(\theta+1)\int_0^{\infty} y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy \end{pmatrix}, \quad \theta\in\mathbb{C}. \end{aligned} \tag{7.1}$$

Therefore

$$\begin{aligned} \hat{\zeta}(\theta) =& \frac{(\theta+1)\Gamma\left(\frac{1+\theta}{2}\right)}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\sqrt{\pi}}\zeta(1+\theta) + \frac{2}{\theta(\theta-1)\Gamma\left(\frac{\theta}{2}\right)}\pi^{\frac{\theta}{2}}H_{[2]}(\theta) \\ =& \frac{(\theta+1)\Gamma\left(\frac{1+\theta}{2}\right)}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\sqrt{\pi}}\zeta(1+\theta) \\ & - \frac{4}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\begin{pmatrix} 4\int_0^{\infty} y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy \\ -(\theta+1)\int_0^{\infty} y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy \end{pmatrix}, \quad \theta\in\mathbb{C}\backslash\{1\}. \end{aligned} \tag{7.2}$$

For equation (7.2), any odd number of 3 or more is substituted for θ.

$$\begin{aligned} \hat{\zeta}(2k+1) =& \frac{(k+1)\Gamma(k+1)}{k\,\Gamma\left(\frac{2k+1}{2}\right)\sqrt{\pi}}\zeta(2k+2) \\ & - \frac{4}{k\,\Gamma\left(\frac{2k+1}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\begin{pmatrix} 2\int_0^{\infty} y^{2(k+1)}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy \\ -(k+1)\int_0^{\infty} y^{2k}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy \end{pmatrix}, \quad k\in\mathbb{N}. \end{aligned} \tag{7.3}$$

The description in subsection 19.4 gives the general representation for the two integrals in equation (7.3).

$$\int_0^{\infty} y^{2k}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy = \frac{1}{2^{k+1}}\sum_{\mu=0}^{k} a_{k,\mu}(2\pi p)^{\mu}\sqrt{\pi}\,e^{-2\pi p}, \quad (p,k\in\mathbb{N}). \tag{7.4}$$

Where

$$a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}, \quad (k\in\mathbb{N}, \mu=0,1,2,\cdots,k). \tag{7.5}$$

And the following coefficients deserve special mention:

$$a_{k,k} = 1, \quad k\in\mathbb{N}. \tag{7.6}$$

$$a_{k,1} = a_{k,0} = (2k-1)!!, \quad k\in\mathbb{N}. \tag{7.7}$$

The above relations are introduced on the right side of equation (7.3).

$$
\begin{aligned}
\hat{\zeta}\left(2k+1\right) =& \frac{\left(k+1\right)\Gamma\left(k+1\right)}{k\,\Gamma\left(\frac{2k+1}{2}\right)\sqrt{\pi}}\zeta\left(2k+2\right) \\
& - \frac{4}{k\,\Gamma\left(\frac{2k+1}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)\left(\begin{array}{l} 2\cdot\dfrac{1}{2^{k+2}}\displaystyle\sum_{\mu=0}^{k+1}a_{k+1,\mu}(2\pi p)^{\mu}\sqrt{\pi}\,e^{-2\pi p} \\ \quad -\left(k+1\right)\cdot\dfrac{1}{2^{k+1}}\displaystyle\sum_{\mu=0}^{k}a_{k,\mu}(2\pi p)^{\mu}\sqrt{\pi}\,e^{-2\pi p}\end{array}\right) \\
& = \frac{\left(k+1\right)k!}{k\cdot\frac{(2k-1)!!}{2^k}\sqrt{\pi}\cdot\sqrt{\pi}}\zeta\left(2k+2\right) \\
& - \frac{2}{k\cdot\frac{(2k-1)!!}{2^k}\sqrt{\pi}\cdot 2^k}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)\left(\begin{array}{l} a_{k+1,k+1}(2\pi p)^{k+1} \\ \quad +\displaystyle\sum_{\mu=0}^{k}\left(a_{k+1,\mu}-\left(k+1\right)a_{k,\mu}\right)(2\pi p)^{\mu}\end{array}\right)\sqrt{\pi}\,e^{-2\pi p} \\
& = \frac{\left(k+1\right)\left(k-1\right)!\,2^k}{(2k-1)!!\,\pi}\zeta\left(2k+2\right) \\
& - \frac{2}{k\left(2k-1\right)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)\left(\begin{array}{l} a_{k+1,k+1}(2\pi p)^{k+1} \\ \quad +\displaystyle\sum_{\mu=0}^{k}\left(a_{k+1,\mu}-\left(k+1\right)a_{k,\mu}\right)(2\pi p)^{\mu}\end{array}\right)e^{-2\pi p}, \quad k\in\mathbb{N}.
\end{aligned} \tag{7.8}
$$

The following equation is the obvious from the definition of $a_{k,\mu}$:

$$
a_{k+1,k+1} = 1, \quad k\in\mathbb{N}. \tag{7.9}
$$

Therefore

$$
\begin{aligned}
\hat{\zeta}\left(2k+1\right) =& \frac{\left(k+1\right)\left(k-1\right)!\,2^k}{(2k-1)!!\,\pi}\zeta\left(2k+2\right) \\
& - \frac{2}{k\left(2k-1\right)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)\left(\begin{array}{l} (2\pi p)^{k+1} \\ \quad +\displaystyle\sum_{\mu=0}^{k}\left(a_{k+1,\mu}-\left(k+1\right)a_{k,\mu}\right)(2\pi p)^{\mu}\end{array}\right)e^{-2\pi p}, \quad k\in\mathbb{N}.
\end{aligned} \tag{7.10}
$$

For the general representation containing the leading term of the zeta function for any odd number of 3 or more, new coefficients are employed.
The coefficients are determined by defining the following equation:

$$
b_{k,\mu} := a_{k+1,\mu} - \left(k-1\right)a_{k,\mu}, \quad \left(k\in\mathbb{N}, \mu=0,1,2,\cdots,k\right). \tag{7.11}
$$

The coefficients are calculated as follows:

$$
\begin{aligned}
b_{k,\mu} &= \frac{(2k+2-\mu)!}{2^{k+1-\mu}\mu!\,(k+1-\mu)!} - \left(k-1\right)\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!} \\
&= \frac{\mu\left(\mu-1\right)+2k\left(k+1-\mu\right)}{2^{k+1-\mu}\mu!\,(k+1-\mu)!}\left(2k-\mu\right)!, \quad \left(k\in\mathbb{N}, \mu=0,1,2,\cdots,k\right).
\end{aligned} \tag{7.12}
$$

For equation (7.12), 0 and 1 are substituted directly for μ.

$$
\begin{aligned}
b_{k,0} &= \left.\frac{\mu\left(\mu-1\right)+2k\left(k+1-\mu\right)}{2^{k+1-\mu}\mu!\,(k+1-\mu)!}\left(2k-\mu\right)!\right|_{\mu=0} \\
&= \frac{2k\left(k+1\right)}{2^{k+1}0!\,(k+1)!}\left(2k\right)! = \frac{k}{2^k k!}\left(2k\right)! = k\left(2k-1\right)!!, \quad k\in\mathbb{N}.
\end{aligned} \tag{7.13}
$$

$$
\begin{aligned}
b_{k,1} &= \left.\frac{\mu\left(\mu-1\right)+2k\left(k+1-\mu\right)}{2^{k+1-\mu}\mu!\,(k+1-\mu)!}\left(2k-\mu\right)!\right|_{\mu=1} \\
&= \frac{2k^2}{2^k 1!\,k!}\left(2k-1\right)! = \frac{k}{2^{k-1}\left(k-1\right)!}\left(2k-1\right)! = k\left(2k-1\right)!!, \quad k\in\mathbb{N}.
\end{aligned} \tag{7.14}
$$

And for equation (7.12), $(k+1)$ is substituted directly for μ.

(7.15) $$b_{k,k+1} = \frac{\mu(\mu-1)+2k(k+1-\mu)}{2^{k+1-\mu}\mu!(k+1-\mu)!}(2k-\mu)!\bigg|_{\mu=k+1} = \frac{(k+1)k}{2^0(k+1)!\,0!}(k-1)! = 1, \quad k \in \mathbb{N}.$$

As a result, the general representation containing the leading term of the zeta function for any odd number of 3 or more is obtained.
This is also the general representation for the zeta function for any odd number of 3 or more.

(7.16) $$\begin{aligned}\hat{\zeta}(2k+1) =& \frac{(k+1)(k-1)!\,2^k}{(2k-1)!!\,\pi}\zeta(2k+2) \\ &- \frac{2}{k(2k-1)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left(\sum_{\mu=0}^{k+1}b_{k,\mu}(2\pi p)^{\mu}\right)e^{-2\pi p}, \quad k \in \mathbb{N}.\end{aligned}$$

The following expression is equivalent to the above equation.

(7.17) $$\begin{aligned}\hat{\zeta}(2k+1) =& \frac{(k+1)(k-1)!\,2^k}{(2k-1)!!\,\pi}\zeta(2k+2) \\ &- \frac{2(2\pi)^{2k+1}}{k(2k-1)!!}\sum_{p=1}^{\infty}\sigma_{2k+1}(p)\left(\sum_{\mu=0}^{k+1}\frac{b_{k,\mu}}{(2\pi p)^{2k+1-\mu}}\right)e^{-2\pi p}, \quad k \in \mathbb{N}.\end{aligned}$$

Where

(7.18) $$b_{k,\mu} = \frac{\mu(\mu-1)+2k(k+1-\mu)}{2^{k+1-\mu}\mu!(k+1-\mu)!}(2k-\mu)!, \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \cdots, k+1).$$

And the coefficients deserving special mention are as follows:

(7.19) $$b_{k,k+1} = 1, \quad k \in \mathbb{N}.$$

(7.20) $$b_{k,1} = b_{k,0} = k(2k-1)!!, \quad k \in \mathbb{N}.$$

Table of the leading terms and the coefficients of the infinite series

k	1	2	3	4
$\frac{(k+1)(k-1)!\,2^k}{(2k-1)!!\,\pi}\cdot\zeta(2k+2)$	$\frac{2\pi^3}{45}$	$\frac{4\pi^5}{945}$	$\frac{32\pi^7}{70875}$	$\frac{32\pi^9}{654885}$
$-\frac{2}{k(2k-1)!!}$	-2	$-\frac{1}{3}$	$-\frac{2}{45}$	$-\frac{1}{210}$
k	5	6	7	8
$\frac{(k+1)(k-1)!\,2^k}{(2k-1)!!\,\pi}\cdot\zeta(2k+2)$	$\frac{353792\pi^{11}}{67043851875}$	$\frac{1024\pi^{13}}{1806079275}$	$\frac{29630464\pi^{15}}{488950811724375}$	$\frac{179679232\pi^{17}}{27870196268289375}$
$-\frac{2}{k(2k-1)!!}$	$-\frac{2}{4725}$	$-\frac{1}{31185}$	$-\frac{2}{945945}$	$-\frac{1}{8108100}$

Table. 7.1

Table of the coefficients $b_{k,\mu}$

k	$b_{k,9}$	$b_{k,8}$	$b_{k,7}$	$b_{k,6}$	$b_{k,5}$	$b_{k,4}$	$b_{k,3}$	$b_{k,2}$	$b_{k,1}$	$b_{k,0}$
1								1	1	1
2							1	3	6	6
3						1	6	21	45	45
4					1	10	55	195	420	420
5				1	15	120	630	2205	4725	4725
6			1	21	231	1680	8505	29295	62370	62370
7		1	28	406	3906	26775	131670	446985	945945	945945
8	1	36	666	8190	72765	478170	2297295	7702695	16216200	16216200

Table. 7.2

7.2 Another ViewPoint

The equation (7.16) by changing the start point of the finite sum from 0 to 1 is written.

$$\begin{aligned}\hat{\zeta}\left(2k+1\right) =& \frac{\left(k+1\right)\left(k-1\right)!\,2^{k}}{\left(2k-1\right)!!\,\pi}\zeta\left(2k+2\right) \\ & -\frac{2}{k\left(2k-1\right)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)\left(\sum_{\mu=1}^{k+1}b_{k,\mu}(2\pi p)^{\mu}+k\left(2k-1\right)!!\right)e^{-2\pi p}, \quad k\in\mathbb{N}.\end{aligned} \tag{7.21}$$

And the function $E\left(2k+1\right)$ that is the separated term by changing the finite sum's starting is defined as follows:

$$E\left(2k+1\right) := -\frac{2}{k\left(2k-1\right)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)k\left(2k-1\right)!!\,e^{-2\pi p}, \quad k\in\mathbb{N}. \tag{7.22}$$

By simplifying,

$$E\left(2k+1\right) = -2\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)e^{-2\pi p}, \quad k\in\mathbb{N}. \tag{7.23}$$

The first viewpoint is that the general representation for the zeta function for any odd number of 3 or more is obtained.

There are two other viewpoints on the function $E\left(2k+1\right)$.

The second viewpoint is that the function $E\left(2k+1\right)$ includes the zeta function in itself for any odd number of 3 or more, as follows:

$$E\left(2k+1\right) = -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^{2k+1}}e^{-2\pi mn} = -\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)-1}{n^{2k+1}} = -\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^{2k+1}}+\zeta\left(2k+1\right), \quad k\in\mathbb{N}. \tag{7.24}$$

When this relation is introduced into the equation (7.16), the cancellation of the zeta functions is observed on both sides of the equation.

$$\begin{aligned}\hat{\zeta}\left(2k+1\right) =& \frac{\left(k+1\right)\left(k-1\right)!\,2^{k}}{\left(2k-1\right)!!\,\pi}\zeta\left(2k+2\right) \\ & -\frac{2}{k\left(2k-1\right)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)\left(\sum_{\mu=0}^{k}b_{k,\mu+1}(2\pi p)^{\mu}\right)\left(2\pi p\right)e^{-2\pi p} \\ & -\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^{2k+1}}+\zeta\left(2k+1\right), \quad k\in\mathbb{N}.\end{aligned} \tag{7.25}$$

Therefore, the essence of equation (7.16) does not represent the zeta function for any odd number of 3 or more, but the following representation does:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^{2k+1}} =& \frac{\left(k+1\right)\left(k-1\right)!\,2^{k}}{\left(2k-1\right)!!\,\pi}\zeta\left(2k+2\right) \\ & -\frac{2}{k\left(2k-1\right)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}\left(p\right)\left(\sum_{\mu=0}^{k}b_{k,\mu+1}(2\pi p)^{\mu}\right)\left(2\pi p\right)e^{-2\pi p}, \quad k\in\mathbb{N}.\end{aligned} \tag{7.26}$$

The third viewpoint is that the function $E\left(2k+1\right)$ can be written using the infinite series as follows:

$$\begin{aligned}E\left(2k+1\right) &= -\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)-1}{n^{2k+1}} \\ &= -\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\cdot\frac{\cosh\left(\pi n\right)-\sinh\left(\pi n\right)}{\sinh\left(\pi n\right)} \\ &= -\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\cdot\frac{2\,e^{-\pi n}}{e^{\pi n}-e^{-\pi n}} \\ &= -2\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}\left(e^{2\pi n}-1\right)}, \quad k\in\mathbb{N}.\end{aligned} \tag{7.27}$$

From the result of equation (7.24),

$$\zeta(2k+1) = \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} + E(2k+1), \quad k \in \mathbb{N}. \tag{7.28}$$

Ramanujan discovered the explicit formula for the first term on the right side of equation (7.28), where $(2k+1)$ is equivalent to 3 modulo 4, as follows "[13]":

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} = 2^{2k}\pi^{2k+1} \sum_{\mu=0}^{k+1} (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{2k+2-2\mu}}{(2k+2-2\mu)!}, \quad (k \in \mathbb{N},\ 2k+1 \equiv 3 \,(\mathrm{mod}\, 4)). \tag{7.29}$$

By combining the results of equations (7.27), (7.28), and (7.29),

$$\begin{aligned} &\zeta(2k+1) = \\ &2^{2k}\pi^{2k+1} \sum_{\mu=0}^{k+1} (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{2k+2-2\mu}}{(2k+2-2\mu)!} - 2\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}\left(e^{2\pi n}-1\right)}, \quad (k \in \mathbb{N},\ 2k+1 \equiv 3 \,(\mathrm{mod}\, 4)). \end{aligned} \tag{7.30}$$

I also arrived at the same representation for the zeta function as Ramanujan for 3, 7, and 11, and so on.

7.3 Some Examples of Calculation

When $\theta = 3$,
The representations for the zeta-hat(3) are

$$\hat{\zeta}(3) = \frac{2\pi^3}{45} - 2\sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^2 + (2\pi p) + 1\right) e^{-2\pi p}. \tag{7.31}$$

$$\hat{\zeta}(3) = \frac{2\pi^3}{45} - 2(2\pi)^3 \sum_{p=1}^{\infty} \sigma_3(p) \left(\frac{1}{2\pi p} + \frac{1}{(2\pi p)^2} + \frac{1}{(2\pi p)^3}\right) e^{-2\pi p}. \tag{7.32}$$

In contrast, the essence of equation (7.31) is

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} = \frac{2\pi^3}{45} - 2\sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p) + 1\right)(2\pi p)\, e^{-2\pi p}. \tag{7.33}$$

The description in subsection 19.6 shows the representation of equation (7.31) using hyperbolic functions.

$$\begin{aligned} \hat{\zeta}(3) &= \frac{2\pi^3}{45} - 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^3} \left((2\pi mn)^2 + (2\pi mn) + 1\right) e^{-2\pi mn} \\ &= \frac{2\pi^3}{45} - 2\sum_{n=1}^{\infty} \left(\frac{(2\pi n)^2}{n^3} \cdot \frac{\coth(\pi n)}{4\sinh^2(\pi n)} + \frac{(2\pi n)}{n^3} \cdot \frac{1}{4\sinh^2(\pi n)} + \frac{1}{n^3} \cdot \frac{\coth(\pi n) - 1}{2}\right). \end{aligned} \tag{7.34}$$

Therefore,

$$\hat{\zeta}(3) = \frac{2\pi^3}{45} - \sum_{n=1}^{\infty} \left(\frac{2\pi^2 \coth(\pi n)}{n\sinh^2(\pi n)} + \frac{\pi}{n^2\sinh^2(\pi n)} + \frac{\coth(\pi n) - 1}{n^3}\right). \tag{7.35}$$

For equation (7.29), 1 is substituted for k.

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} = 2^2\pi^3 \sum_{\mu=0}^{2} (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{4-2\mu}}{(4-2\mu)!} = 2^2\pi^3 \left(-\frac{2}{4!}\left(-\frac{1}{30}\right) + \left(\frac{1}{2!}\cdot\frac{1}{6}\right)^2\right) = \frac{7\pi^3}{180}. \tag{7.36}$$

Also for equation (7.30), 1 is substituted for k.

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3\left(e^{2\pi n}-1\right)}. \tag{7.37}$$

By combining the results of equations (7.33) and (7.36), I obtained the infinite series that gives the transcendental number "[14]."

$$\sum_{p=1}^{\infty}\sigma_{-3}(p)\left((2\pi p)+1\right)(2\pi p)\,e^{-2\pi p}=\frac{\pi^3}{360}. \tag{7.38}$$

The equations (7.31), (7.32), (7.33), (7.35), (7.37), and (7.38) are equivalent each other.
When $\theta=5$,
The representations for the zeta-hat(5) are

$$\hat{\zeta}(5)=\frac{4\pi^5}{945}-\frac{1}{3}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^3+3(2\pi p)^2+6(2\pi p)+6\right)e^{-2\pi p}. \tag{7.39}$$

$$\hat{\zeta}(5)=\frac{4\pi^5}{945}-\frac{(2\pi)^5}{3}\sum_{p=1}^{\infty}\sigma_{5}(p)\left(\frac{1}{(2\pi p)^2}+\frac{3}{(2\pi p)^3}+\frac{6}{(2\pi p)^4}+\frac{6}{(2\pi p)^5}\right)e^{-2\pi p}. \tag{7.40}$$

The essence of equation (7.39) is

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^5}=\frac{4\pi^5}{945}-\frac{1}{3}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^2+3(2\pi p)+6\right)(2\pi p)\,e^{-2\pi p}. \tag{7.41}$$

The representation of equation (7.39) using hyperbolic functions is as follows:

$$\begin{aligned}\hat{\zeta}(5)&=\frac{4\pi^5}{945}-\frac{1}{3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^5}\left((2\pi mn)^3+3(2\pi mn)^2+6(2\pi mn)+6\right)e^{-2\pi mn}\\&=\frac{4\pi^5}{945}-\frac{1}{3}\sum_{n=1}^{\infty}\left(\begin{aligned}\frac{(2\pi n)^3}{n^5}\cdot\frac{3}{8}\left(\frac{2}{3}+\frac{1}{\sinh^2(\pi n)}\right)\frac{1}{\sinh^2(\pi n)}+\frac{3(2\pi n)^2}{n^5}\cdot\frac{\coth(\pi n)}{4\sinh^2(\pi n)}\\+\frac{6(2\pi n)}{n^5}\cdot\frac{1}{4\sinh^2(\pi n)}+\frac{6}{n^5}\cdot\frac{\coth(\pi n)-1}{2}\end{aligned}\right).\end{aligned} \tag{7.42}$$

Therefore,

$$\hat{\zeta}(5)=\frac{4\pi^5}{945}-\sum_{n=1}^{\infty}\left(\begin{aligned}\left(\frac{2}{3}+\frac{1}{\sinh^2(\pi n)}\right)\frac{\pi^3}{n^2\sinh^2(\pi n)}+\frac{\pi^2\coth(\pi n)}{n^3\sinh^2(\pi n)}\\+\frac{\pi}{n^4\sinh^2(\pi n)}+\frac{\coth(\pi n)-1}{n^5}\end{aligned}\right). \tag{7.43}$$

The equations (7.39), (7.40), (7.41), and (7.43) are equivalent each other.
Hereafter, the principal results are displayed in order without explanation. When $\theta=7$,

$$\hat{\zeta}(7)=\frac{32\pi^7}{70875}-\frac{2}{45}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^4+6(2\pi p)^3+21(2\pi p)^2+45(2\pi p)+45\right)e^{-2\pi p}. \tag{7.44}$$

$$\hat{\zeta}(7)=\frac{32\pi^7}{70875}-\frac{2(2\pi)^7}{45}\sum_{p=1}^{\infty}\sigma_{7}(p)\left(\frac{1}{(2\pi p)^3}+\frac{6}{(2\pi p)^4}+\frac{21}{(2\pi p)^5}+\frac{45}{(2\pi p)^6}+\frac{45}{(2\pi p)^7}\right)e^{-2\pi p}. \tag{7.45}$$

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^7}=\frac{32\pi^7}{70875}-\frac{2}{45}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^3+6(2\pi p)^2+21(2\pi p)+45\right)(2\pi p)\,e^{-2\pi p}. \tag{7.46}$$

$$\hat{\zeta}(7)=\frac{32\pi^7}{70875}-\sum_{n=1}^{\infty}\left(\begin{aligned}\left(\frac{1}{3}+\frac{1}{\sinh^2(\pi n)}\right)\frac{8\pi^4\coth(\pi n)}{15n^3\sinh^2(\pi n)}+\left(\frac{2}{3}+\frac{1}{\sinh^2(\pi n)}\right)\frac{4\pi^3}{5n^4\sinh^2(\pi n)}\\+\frac{14\pi^2\coth(\pi n)}{15n^5\sinh^2(\pi n)}+\frac{\pi}{n^6\sinh^2(\pi n)}+\frac{\coth(\pi n)-1}{n^7}\end{aligned}\right). \tag{7.47}$$

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = 2^6\pi^7 \sum_{\mu=0}^{4} (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{8-2\mu}}{(8-2\mu)!}$$

$$(7.48) \quad = 2^6\pi^7 \left(-\frac{2}{8!}\left(-\frac{1}{30}\right) + 2\left(\frac{1}{2!}\cdot\frac{1}{6}\right)\left(\frac{1}{6!}\cdot\frac{1}{42}\right) - \left(\frac{1}{4!}\left(-\frac{1}{30}\right)\right)^2 \right)$$

$$= \frac{19\,\pi^7}{56700}.$$

$$(7.49) \qquad \zeta(7) = \frac{19\,\pi^7}{56700} - 2\sum_{n=1}^{\infty} \frac{1}{n^7\left(e^{2\pi n}-1\right)}.$$

$$(7.50) \qquad \sum_{p=1}^{\infty} \sigma_{-7}(p) \left((2\pi p)^3 + 6(2\pi p)^2 + 21\,(2\pi p) + 45 \right) (2\pi p)\, e^{-2\pi p} = \frac{11\,\pi^3}{4200}.$$

When $\theta = 9$,

$$(7.51) \quad \hat{\zeta}(9) = \frac{32\,\pi^9}{654885} - \frac{1}{210}\sum_{p=1}^{\infty} \sigma_{-9}(p) \left((2\pi p)^5 + 10(2\pi p)^4 + 55(2\pi p)^3 + 195(2\pi p)^2 + 420\,(2\pi p) + 420 \right) e^{-2\pi p}.$$

$$(7.52) \quad \hat{\zeta}(9) = \frac{32\,\pi^9}{654885} - \frac{(2\pi)^9}{210}\sum_{p=1}^{\infty} \sigma_{9}(p) \left(\frac{1}{(2\pi p)^4} + \frac{10}{(2\pi p)^5} + \frac{55}{(2\pi p)^6} + \frac{195}{(2\pi p)^7} + \frac{420}{(2\pi p)^8} + \frac{420}{(2\pi p)^9} \right) e^{-2\pi p}.$$

$$(7.53) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^9} = \frac{32\,\pi^9}{654885} - \frac{1}{210}\sum_{p=1}^{\infty} \sigma_{-9}(p) \left(\begin{array}{r} (2\pi p)^4 + 10(2\pi p)^3 + 55(2\pi p)^2 \\ + 195\,(2\pi p) + 420 \end{array} \right) (2\pi p)\, e^{-2\pi p}.$$

$$(7.54) \quad \hat{\zeta}(9) = \frac{32\,\pi^9}{654885} - \sum_{n=1}^{\infty} \left(\begin{array}{r} \left(\frac{2}{15} + \frac{1}{\sinh^2(\pi n)} + \frac{1}{\sinh^4(\pi n)}\right) \frac{2\,\pi^5\,\coth(\pi n)}{7\,n^4 \sinh^2(\pi n)} \\ + \left(\frac{1}{3} + \frac{1}{\sinh^2(\pi n)}\right) \frac{4\,\pi^4\,\coth(\pi n)}{7\,n^5 \sinh^2(\pi n)} + \left(\frac{2}{3} + \frac{1}{\sinh^2(\pi n)}\right) \frac{11\,\pi^3}{14\,n^6 \sinh^2(\pi n)} \\ + \frac{13\,\pi^2\,\coth(\pi n)}{14\,n^7 \sinh^2(\pi n)} + \frac{\pi}{n^8 \sinh^2(\pi n)} + \frac{\coth(\pi n) - 1}{n^9} \end{array} \right).$$

When $\theta = 11$,

$$(7.55) \quad \hat{\zeta}(11) = \frac{353792\,\pi^{11}}{67043851875} - \frac{2}{4725}\sum_{p=1}^{\infty} \sigma_{-11}(p) \left(\begin{array}{r} (2\pi p)^6 + 15(2\pi p)^5 + 120(2\pi p)^4 + 630(2\pi p)^3 \\ + 2205(2\pi p)^2 + 4725\,(2\pi p) + 4725 \end{array} \right) e^{-2\pi p}.$$

$$(7.56) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{11}} = \frac{353792\,\pi^{11}}{67043851875} - \frac{2}{4725}\sum_{p=1}^{\infty} \sigma_{-11}(p) \left(\begin{array}{r} (2\pi p)^5 + 15(2\pi p)^4 + 120(2\pi p)^3 \\ + 630(2\pi p)^2 + 2205\,(2\pi p) + 4725 \end{array} \right) (2\pi p)\, e^{-2\pi p}.$$

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{11}} = 2^{10}\pi^{11} \sum_{\mu=0}^{6} (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{12-2\mu}}{(12-2\mu)!}$$

$$(7.57) \quad = 2^{10}\pi^{11} \left(-\frac{2}{12!}\left(-\frac{691}{2730}\right) + 2\left(\frac{1}{2!}\cdot\frac{1}{6}\right)\left(\frac{1}{10!}\cdot\frac{5}{66}\right) - 2\left(\frac{1}{4!}\left(-\frac{1}{30}\right)\cdot\frac{1}{8!}\left(-\frac{1}{30}\right)\right) + \left(\frac{1}{6!}\left(\frac{1}{42}\right)\right)^2 \right)$$

$$= \frac{1453\,\pi^{11}}{425675250}.$$

$$\zeta\left(11\right)=\frac{1453\,\pi^{11}}{425675250}-2\sum_{n=1}^{\infty}\frac{1}{n^{11}\left(e^{2\pi n}-1\right)}. \tag{7.58}$$

$$\sum_{p=1}^{\infty}\sigma_{-11}\left(p\right)\begin{pmatrix}(2\pi p)^{5}+15(2\pi p)^{4}+120(2\pi p)^{3}\\ +630(2\pi p)^{2}+2205\left(2\pi p\right)+4725\end{pmatrix}\left(2\pi p\right)e^{-2\pi p}=\frac{249889\,\pi^{11}}{56756700}. \tag{7.59}$$

When $\theta=13$,

$$\hat{\zeta}\left(13\right)=\frac{1024\,\pi^{13}}{1806079275}-\frac{1}{31185}\sum_{p=1}^{\infty}\sigma_{-13}\left(p\right)\begin{pmatrix}(2\pi p)^{7}+21(2\pi p)^{6}+231(2\pi p)^{5}\\ +1680(2\pi p)^{4}+8505(2\pi p)^{3}+29295(2\pi p)^{2}\\ +62370\left(2\pi p\right)+62370\end{pmatrix}e^{-2\pi p}. \tag{7.60}$$

$$\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^{13}}=\frac{1024\,\pi^{13}}{1806079275}-\frac{1}{31185}\sum_{p=1}^{\infty}\sigma_{-13}\left(p\right)\begin{pmatrix}(2\pi p)^{6}+21(2\pi p)^{5}+231(2\pi p)^{4}\\ +1680(2\pi p)^{3}+8505(2\pi p)^{2}\\ +29295\left(2\pi p\right)+62370\end{pmatrix}\left(2\pi p\right)e^{-2\pi p}. \tag{7.61}$$

When $\theta=15$,

$$\begin{aligned}\hat{\zeta}\left(15\right)&=\frac{29630464\,\pi^{15}}{488950811724375}\\ &-\frac{2}{945945}\sum_{p=1}^{\infty}\sigma_{-15}\left(p\right)\begin{pmatrix}(2\pi p)^{8}+28(2\pi p)^{7}+406(2\pi p)^{6}\\ +3906(2\pi p)^{5}+26775(2\pi p)^{4}+131670(2\pi p)^{3}\\ +446985(2\pi p)^{2}+945945\left(2\pi p\right)+945945\end{pmatrix}e^{-2\pi p}.\end{aligned} \tag{7.62}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^{15}}&=\frac{29630464\,\pi^{13}}{488950811724375}\\ &-\frac{1}{945945}\sum_{p=1}^{\infty}\sigma_{-15}\left(p\right)\begin{pmatrix}(2\pi p)^{7}+28(2\pi p)^{6}+406(2\pi p)^{5}\\ +3906(2\pi p)^{4}+26775(2\pi p)^{3}+131670(2\pi p)^{2}\\ +446985\left(2\pi p\right)+945945\end{pmatrix}\left(2\pi p\right)e^{-2\pi p}.\end{aligned} \tag{7.63}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^{15}}&=2^{14}\pi^{15}\sum_{\mu=0}^{8}(-1)^{\mu+1}\frac{B_{2\mu}}{(2\mu)!}\cdot\frac{B_{16-2\mu}}{(16-2\mu)!}\\ &=2^{14}\pi^{15}\left(\begin{matrix}-\frac{2}{16!}\left(-\frac{3617}{510}\right)+2\left(\frac{1}{2!}\cdot\frac{1}{6}\right)\left(\frac{1}{14!}\cdot\frac{7}{6}\right)-2\left(\frac{1}{4!}\left(-\frac{1}{30}\right)\cdot\frac{1}{12!}\left(-\frac{691}{2730}\right)\right)\\ +2\left(\frac{1}{6!}\cdot\frac{1}{42}\right)\left(\frac{1}{10!}\cdot\frac{5}{66}\right)-\left(\frac{1}{8!}\left(-\frac{1}{30}\right)\right)^{2}\end{matrix}\right)\\ &=\frac{13687\,\pi^{15}}{390769879500}.\end{aligned} \tag{7.64}$$

$$\zeta\left(15\right)=\frac{13687\,\pi^{15}}{390769879500}-2\sum_{n=1}^{\infty}\frac{1}{n^{15}\left(e^{2\pi n}-1\right)}. \tag{7.65}$$

$$\sum_{p=1}^{\infty}\sigma_{-15}\left(p\right)\begin{pmatrix}(2\pi p)^{7}+28(2\pi p)^{6}+406(2\pi p)^{5}\\ +3906(2\pi p)^{4}+26775(2\pi p)^{3}+131670(2\pi p)^{2}\\ +446985\left(2\pi p\right)+945945\end{pmatrix}\left(2\pi p\right)e^{-2\pi p}=\frac{16672807\,\pi^{15}}{1378377000}. \tag{7.66}$$

When $\theta=17$,

$$\begin{aligned}\hat{\zeta}\left(17\right)&=\frac{179679232\,\pi^{17}}{27870196268289375}\\ &-\frac{1}{8108100}\sum_{p=1}^{\infty}\sigma_{-17}\left(p\right)\begin{pmatrix}(2\pi p)^{9}+36(2\pi p)^{8}+666(2\pi p)^{7}+8190(2\pi p)^{6}\\ +72765(2\pi p)^{5}+478170(2\pi p)^{4}+2297295(2\pi p)^{3}\\ +7702695(2\pi p)^{2}+16216200\left(2\pi p\right)+16216200\end{pmatrix}e^{-2\pi p}.\end{aligned} \tag{7.67}$$

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{17}} = \frac{179679232\,\pi^{17}}{27870196268289375} - \frac{1}{8108100}\sum_{p=1}^{\infty}\sigma_{-17}(p)\begin{pmatrix}(2\pi p)^8+36(2\pi p)^7+666(2\pi p)^6+8190(2\pi p)^5\\ +72765(2\pi p)^4+478170(2\pi p)^3+2297295(2\pi p)^2\\ +7702695\,(2\pi p)+16216200\end{pmatrix}(2\pi p)\,e^{-2\pi p}. \tag{7.68}$$

The infinite sum that is terminated at a finite term λ is used for the approximate calculation for the zeta function. The positive integer λ will be referred to as the degree of approximation.
The approximate formula for the zeta function is written as $(\text{zeta-hat})_\lambda\,(2k+1)$.

$$\hat{\zeta}_\lambda(2k+1) := \frac{(k+1)\,(k-1)!\,2^k}{(2k-1)!!\,\pi}\zeta(2k+2) - \frac{2}{k\,(2k-1)!!}\sum_{p=1}^{\lambda}\sigma_{-(2k+1)}(p)\left(\sum_{\mu=0}^{k+1}b_{k,\mu}(2\pi p)^\mu\right)e^{-2\pi p},\quad (k,\lambda\in\mathbb{N}). \tag{7.69}$$

I use the built-in Mathematica functions to perform numerical calculations, for example, DivisorSigma[θ, n] and Zeta[S]. Furthermore, when calculating the values of the function $(\text{zeta-hat})_\lambda\,(2k+1)$, the Mathematica inputs are shown as follows "[8]":

$$\boxed{\text{Alt}} + \boxed{9}\quad \text{(Creating a new input line on a notebook)}$$

$$\mathrm{b}\left[\{\mathrm{k\_},\mu\_\}\right] := \frac{\mu\ (\mu-1)+2\,\mathrm{k}\ (\mathrm{k}+1-\mu)}{2^{\mathrm{k}+1-\mu}\ \mu!\ (\mathrm{k}+1-\mu)!}\ (2\,\mathrm{k}-\mu)!;\ \boxed{\text{Shift}} + \boxed{\text{Enter}}$$

$$\boxed{\text{Alt}} + \boxed{9}\quad \text{(Creating the second input line)}$$

$$\mathrm{zetahat}\left[\{\mathrm{k\_},\lambda\_\}\right] := \frac{(\mathrm{k}+1)\ (\mathrm{k}-1)!\,2^{\mathrm{k}}}{(2\,\mathrm{k}-1)!!\,\pi}\ \mathrm{Zeta}\,[2\,\mathrm{k}+2] - \frac{2}{\mathrm{k}\ (2\,\mathrm{k}-1)!!}$$
$$\times\ \mathrm{Sum}\left[\mathrm{DivisorSigma}\,[-(2\,\mathrm{k}+1)\,,\mathrm{p}]\ (\mathrm{Sum}\,[\mathrm{b}\,[\{\mathrm{k},\mu\}]\ (2\,\pi\,\mathrm{p})^\mu,\{\mu,0,\mathrm{k}+1\}])\ e^{-2\,\pi\,\mathrm{p}},\{\mathrm{p},1,\lambda\}\right];$$
$$\boxed{\text{Shift}} + \boxed{\text{Enter}}$$

$$\boxed{\text{Alt}} + \boxed{9}\quad \text{(Creating the third input line)}$$

$$\mathrm{N}\left[\{\mathrm{zetahat}\,[\{1,1\}]\,,\mathrm{zetahat}\,[\{1,2\}]\,,\mathrm{zetahat}\,[\{1,3\}]\,,\mathrm{zetahat}\,[\{1,4\}]\,,\mathrm{Zeta}\,[3]\},9\right]\ \boxed{\text{Shift}} + \boxed{\text{Enter}}$$

Note: Valid for Mathmatica Ver. 12.0 or later.

By increasing the degree of approximation, it asymptotes the true value of the zeta function for any odd number of 3 or more.

$$\{\hat{\zeta}_1(3),\hat{\zeta}_2(3),\hat{\zeta}_3(3),\hat{\zeta}_4(3),\zeta(3)\} \simeq \{1.20340751, 1.20206199, 1.20205692, 1.20205690, 1.20205690\}. \tag{7.70}$$

$$\{\hat{\zeta}_1(5),\hat{\zeta}_2(5),\hat{\zeta}_3(5),\hat{\zeta}_4(5),\zeta(5)\} \simeq \{1.03998935, 1.03694501, 1.03692783, 1.03692776, 1.03692776\}. \tag{7.71}$$

$$\{\hat{\zeta}_1(7),\hat{\zeta}_2(7),\hat{\zeta}_3(7),\hat{\zeta}_4(7),\zeta(7)\} \simeq \{1.01476857, 1.00840017, 1.00834956, 1.00834928, 1.00834928\}. \tag{7.72}$$

$$\{\hat{\zeta}_1(9),\hat{\zeta}_2(9),\hat{\zeta}_3(9),\hat{\zeta}_4(9),\hat{\zeta}_5(9),\zeta(9)\} \simeq \{1.01392241, 1.00213605, 1.00200927, 1.00200840, 1.00200839, 1.00200839\}. \tag{7.73}$$

By choosing the approximation appropriately, for example, a value can be obtained with 30 decimal places of precision.

$$\{\hat{\zeta}_{12}(3),\zeta(3)\} \simeq \{1.202056903159594285399738161511, 1.202056903159594285399738161511\}. \tag{7.74}$$

(7.75) $$\left\{\hat{\zeta}_{13}(5), \zeta(5)\right\} \simeq \{1.0369277551433699263313654864 57, 1.036927755143369926331365486457\}.$$

(7.76) $$\left\{\hat{\zeta}_{13}(7), \zeta(7)\right\} \simeq \{1.008349277381922826839797549850, 1.008349277381922826839797549850\}.$$

(7.77) $$\left\{\hat{\zeta}_{14}(9), \zeta(9)\right\} \simeq \{1.002008392826082214417852769232, 1.002008392826082214417852769232\}.$$

8. Study of the Zeta Function for any Even Number of 0 or More, and for any Odd Number of -1 or Less

The right side of the second-order I_C type functional equation is replaced with the result of equation (4.8).
The result of equation (4.8) is shown again.

(8.1) $$R_{[2]}(\theta) = 2\pi \lim_{\alpha\to+0} \int_0^\infty \sum_{n=1}^{\infty} \left(-3n^2 y^{2-\theta} + 2\pi n^4 y^{4-\theta}\right) e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)} dy.$$

According to the knowledge in Section 4, the integral and the sum can be interchanged. Therefore,

(8.2) $$R_{[2]}(\theta) = 2\pi \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \int_0^\infty \left(-3n^2 y^{2-\theta} + 2\pi n^4 y^{4-\theta}\right) e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)} dy.$$

For the integrals of $R_{[2]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi} n}.$$

(8.3) $$\begin{aligned} R_{[2]}(\theta) &= 2\pi \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \int_0^\infty \left(-3n^2\left(\frac{x}{\sqrt{\pi}n}\right)^{2-\theta} + 2\pi n^4\left(\frac{x}{\sqrt{\pi}n}\right)^{4-\theta}\right) e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi}n} dx. \\ &= 2\pi^{\frac{\theta-1}{2}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(-3\int_0^\infty x^{2-\theta} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx + 2\int_0^\infty x^{4-\theta} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx\right). \end{aligned}$$

Equation (4.7) is shown again.

(8.4) $$L_{[2]}(\theta) = \frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta), \quad \theta \in \mathbb{C}.$$

Therefore

(8.5) $$\begin{aligned} &\frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \\ &= 2\pi^{\frac{\theta-1}{2}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(-3\int_0^\infty x^{2-\theta} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx + 2\int_0^\infty x^{4-\theta} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx\right), \quad \theta \in \mathbb{C}. \end{aligned}$$

Both sides of equation (8.5) are multiplied by

$$\frac{\pi^{\frac{\theta}{2}}}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)}$$

to obtain the zeta function $\zeta(\theta)$.

(8.6) $$\zeta(\theta) = \frac{2\pi^\theta}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \begin{pmatrix} -3\int_0^\infty x^{2-\theta} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx \\ +2\int_0^\infty x^{4-\theta} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx \end{pmatrix}, \quad \theta \in \mathbb{C}\backslash\{1\}.$$

When the complex variable θ is set to any even number of 4 or less, the representation type1 for the zeta function with the quadrable integrals is obtained.

$$\zeta(2k) = \frac{2\,\pi^{2k}}{(2k-1)\,\Gamma(k+1)\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n^{1-2k}} \left(\begin{array}{l} -\,3\displaystyle\int_0^\infty x^{2(1-k)} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx \\ +\,2\displaystyle\int_0^\infty x^{2(2-k)} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx \end{array} \right), \tag{8.7}$$

$$(k = 2, 1, 0, \cdots).$$

When the complex variable θ is set to any even number of -2 or less, the zeta function has the value zero at these points (the trivial zeros). Furthermore, knowing that no even number of 6 or more can be calculated using equation (8.7), the result of equation of definition (4.2) is presented again.

$$R_{[2]}(\theta) = 2\pi \lim_{\alpha\to+0} \int_0^\infty \sum_{n=1}^{\infty} \left(-3n^2 x^{\theta-4} + 2\pi n^4 x^{\theta-6}\right) e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2 x^2\right)} dx. \tag{8.8}$$

Similarly, the integral and the sum can be interchanged. Therefore,

$$R_{[2]}(\theta) = 2\pi \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \int_0^\infty \left(-3n^2 x^{\theta-4} + 2\pi n^4 x^{\theta-6}\right) e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2 x^2\right)} dx. \tag{8.9}$$

For the integrals of $R_{[2]}(\theta)$, I perform the variable transformation

$$x = \frac{y}{\sqrt{\pi} n}.$$

$$\begin{aligned} R_{[2]}(\theta) &= 2\pi \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \int_0^\infty \left(-3n^2 \left(\frac{y}{\sqrt{\pi} n}\right)^{\theta-4} + 2\pi n^4 \left(\frac{y}{\sqrt{\pi} n}\right)^{\theta-6}\right) e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} \frac{1}{\sqrt{\pi} n} dy. \\ &= 2\pi^{\frac{\theta-1}{2}} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^{\theta-3} \sum_{n=1}^{\infty} n^2 \left(\begin{array}{l} -\,3\displaystyle\int_0^\infty y^{\theta-4} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \\ +\,2(\pi n\alpha)^2 \displaystyle\int_0^\infty y^{\theta-6} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \end{array} \right). \end{aligned} \tag{8.10}$$

From the equation $L_{[2]}(\theta) = R_{[2]}(\theta)$,

$$\begin{aligned} &\frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \\ &= 2\pi^{\frac{\theta-1}{2}} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^{\theta-3} \sum_{n=1}^{\infty} n^2 \left(\begin{array}{l} -\,3\displaystyle\int_0^\infty y^{\theta-4} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \\ +\,2(\pi n\alpha)^2 \displaystyle\int_0^\infty y^{\theta-6} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \end{array} \right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{8.11}$$

Both sides of equation (8.11) are multiplied by

$$\frac{\pi^{\frac{\theta}{2}}}{(\theta-1)\,\Gamma\left(1+\frac{\theta}{2}\right)}$$

to obtain the zeta function $\zeta(\theta)$.

$$\zeta(\theta) = \frac{2\,\pi^{\theta}}{(\theta-1)\,\Gamma\left(1+\frac{\theta}{2}\right)\sqrt{\pi}} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^{\theta-3} \sum_{n=1}^{\infty} n^2 \left(\begin{array}{l} -\,3\displaystyle\int_0^\infty y^{\theta-4} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \\ +\,2(\pi n\alpha)^2 \displaystyle\int_0^\infty y^{\theta-6} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \end{array} \right), \tag{8.12}$$

$$\theta \in \mathbb{C}\backslash\{1\}.$$

When the complex variable θ is set to any even number of 6 or more, I obtained the representation type2 for the zeta function with the quadrable integrals.

$$\zeta(2k) = \frac{2\pi^{2k}}{(2k-1)\Gamma(k+1)\sqrt{\pi}} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^{2k-3} \sum_{n=1}^{\infty} n^2 \left(\begin{array}{l} -3\int_0^\infty y^{2(k-2)} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \\ +2(\pi n\alpha)^2 \int_0^\infty y^{2(k-3)} e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \end{array} \right), \tag{8.13}$$

$$(k = 3, 4, 5, \cdots).$$

Both sides of the complete symmetric functional equation are multiplied by

$$\frac{\sqrt{\pi}}{\Gamma\left(1-\frac{\theta}{2}\right)} \pi^{\frac{\theta}{2}}$$

to obtain the function $\zeta(1-\theta)$.

$$\zeta(1-\theta) = \frac{\sqrt{\pi}}{\Gamma\left(1-\frac{\theta}{2}\right)} \pi^{\theta} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta), \quad \theta \in \mathbb{C}\backslash\{0\}. \tag{8.14}$$

For equation (8.7), 0 is substituted for k.

$$\begin{aligned} \zeta(0) &= \frac{2\pi^0}{(-1)\Gamma(1)\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n} \left(-3\int_0^\infty x^2 e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx + 2\int_0^\infty x^4 e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx \right) \\ &= -\frac{2}{\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n} \left(-3 \cdot \frac{(2\pi n\alpha)+1}{4} \sqrt{\pi}\, e^{-2\pi n\alpha} + 2 \cdot \frac{(2\pi n\alpha)^2 + 3(2\pi n\alpha)+3}{8} \sqrt{\pi}\, e^{-2\pi n\alpha} \right) \\ &= -2 \lim_{\alpha\to+0} (\pi\alpha)^2 \sum_{n=1}^{\infty} n e^{-2\pi n\alpha} = -2 \lim_{\alpha\to+0} (\pi\alpha)^2 \cdot \frac{1}{4} \left(\left(\frac{1}{\pi\alpha}\right)^2 - \frac{1}{3} + O(\pi\alpha)^2 \right). \end{aligned} \tag{8.15}$$

Therefore

$$\zeta(0) = -\frac{1}{2} \lim_{\alpha\to+0} \left(1 - O(\pi\alpha)^2\right) = -\frac{1}{2}. \tag{8.16}$$

For equation (8.7), 1 is substituted for k.

$$\begin{aligned} \zeta(2) &= \frac{2\pi^2}{\Gamma(2)\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n^{-1}} \left(-3\int_0^\infty x^0 e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx + 2\int_0^\infty x^2 e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx \right) \\ &= \frac{2\pi^2}{\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} n \left(-3 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi n\alpha} + 2 \cdot \frac{(2\pi n\alpha)+1}{4} \sqrt{\pi}\, e^{-2\pi n\alpha} \right) \\ &= 2\pi^2 \lim_{\alpha\to+0} \left((\pi\alpha) \sum_{n=1}^{\infty} n^2 e^{-2\pi n\alpha} - \sum_{n=1}^{\infty} n e^{-2\pi n\alpha} \right) \\ &= 2\pi^2 \lim_{\alpha\to+0} \left((\pi\alpha) \cdot \left(\left(\frac{1}{\pi\alpha}\right)^3 - \frac{(\pi\alpha)}{15} + O(\pi\alpha)^3 \right) - \frac{1}{4} \left(\left(\frac{1}{\pi\alpha}\right)^2 - \frac{1}{3} + \frac{(\pi\alpha)^2}{15} - O(\pi\alpha)^4 \right) \right) \\ &= 2\pi^2 \lim_{\alpha\to+0} \left(\left(\frac{1}{4} - \frac{1}{4}\right) \left(\frac{1}{\pi\alpha}\right)^2 - \frac{1}{4}\left(-\frac{1}{3}\right) - O(\pi\alpha)^2 \right). \end{aligned} \tag{8.17}$$

Therefore

$$\zeta(2) = 2\pi^2 \lim_{\alpha\to+0} \left(0 \times \left(\frac{1}{\pi\alpha}\right)^2 + \frac{1}{12} - O(\pi\alpha)^2 \right) = \frac{\pi^2}{6}. \tag{8.18}$$

For equation (8.14), 2 is substituted for θ.

$$\zeta(-1) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{1}{2}\right)} \pi^{-2} \Gamma(1) \zeta(2) = \frac{\sqrt{\pi}}{-2\sqrt{\pi}} \pi^{-2} \cdot 1 \cdot \frac{\pi^2}{6} = -\frac{1}{12}. \tag{8.19}$$

For equation (8.7), 2 is substituted for k.

$$\zeta(4) = \frac{2\,\pi^4}{3\,\Gamma(3)\,\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} \frac{1}{n^{-3}} \left(-3 \int_0^{\infty} x^{-2} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx + 2 \int_0^{\infty} x^0 e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx \right)$$

(8.20)
$$\begin{aligned}
&= \frac{\pi^4}{3\sqrt{\pi}} \lim_{\alpha\to+0} \sum_{n=1}^{\infty} n^3 \left(-3\cdot\frac{\sqrt{\pi}}{2\pi n\alpha} e^{-2\pi n\alpha} + 2\cdot\frac{\sqrt{\pi}}{2} e^{-2\pi n\alpha} \right) \\
&= \frac{\pi^4}{6} \lim_{\alpha\to+0} \left(-\frac{3}{\pi\alpha} \sum_{n=1}^{\infty} n^2 e^{-2\pi n\alpha} + 2\sum_{n=1}^{\infty} n^3 e^{-2\pi n\alpha} \right) \\
&= \frac{\pi^4}{6} \lim_{\alpha\to+0} \left(\begin{aligned} -\frac{3}{\pi\alpha}\cdot\frac{1}{4}\left(\left(\frac{1}{\pi\alpha}\right)^3 - \frac{(\pi\alpha)}{15} + \frac{4(\pi\alpha)^3}{189} - O(\pi\alpha)^5 \right) \\ +2\cdot\frac{3}{8}\left(\left(\frac{1}{\pi\alpha}\right)^4 + \frac{1}{45} - \frac{4(\pi\alpha)^2}{189} + O(\pi\alpha)^4 \right) \end{aligned} \right) \\
&= \frac{\pi^4}{6} \lim_{\alpha\to+0} \left(\left(-3\cdot\frac{1}{4} + 2\cdot\frac{3}{8}\right)\left(\frac{1}{\pi\alpha}\right)^4 - 3\cdot\frac{1}{4}\left(-\frac{1}{15}\right) + 2\cdot\frac{3}{8}\cdot\frac{1}{45} - O(\pi\alpha)^2 \right).
\end{aligned}$$

Therefore

(8.21)
$$\zeta(4) = \frac{\pi^4}{6} \lim_{\alpha\to+0} \left(0\times\left(\frac{1}{\pi\alpha}\right)^4 + \frac{1}{15} - O(\pi\alpha)^2 \right) = \frac{\pi^4}{90}.$$

For equation (8.14), 4 is substituted for θ.

(8.22)
$$\zeta(-3) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{3}{2}\right)} \pi^{-4} \Gamma(2)\,\zeta(4) = \frac{\sqrt{\pi}}{\frac{4\sqrt{\pi}}{3}} \pi^{-4}\cdot 1\cdot\frac{\pi^4}{90} = \frac{1}{120}.$$

For equation (8.13), 3 is substituted for k.

$$\zeta(6) = \frac{2\,\pi^6}{5\,\Gamma(4)\,\sqrt{\pi}} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^3 \sum_{n=1}^{\infty} n^2 \left(\begin{aligned} -3\int_0^{\infty} y^2 e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \\ +2(\pi n\alpha)^2 \int_0^{\infty} y^0 e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)} dy \end{aligned} \right)$$

(8.23)
$$\begin{aligned}
&= \frac{\pi^6}{15\sqrt{\pi}} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^3 \sum_{n=1}^{\infty} n^2 \left(-3\cdot\frac{(2\pi n\alpha)+1}{4}\sqrt{\pi}\, e^{-2\pi n\alpha} + \frac{(2\pi n\alpha)^2}{2}\cdot\frac{\sqrt{\pi}}{2} e^{-2\pi n\alpha} \right) \\
&= \frac{\pi^6}{60} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^3 \left(4(\pi\alpha)^2 \sum_{n=1}^{\infty} n^4 e^{-2\pi n\alpha} - 6(\pi\alpha)\sum_{n=1}^{\infty} n^3 e^{-2\pi n\alpha} - 3\sum_{n=1}^{\infty} n^2 e^{-2\pi n\alpha} \right) \\
&= \frac{\pi^6}{60} \lim_{\alpha\to+0} \left(\frac{1}{\pi\alpha}\right)^3 \left(\begin{aligned} 4(\pi\alpha)^2\cdot\frac{3}{4}\left(\left(\frac{1}{\pi\alpha}\right)^5 + \frac{2(\pi\alpha)}{189} - \frac{(\pi\alpha)^3}{135} + O(\pi\alpha)^5 \right) \\ -6(\pi\alpha)\cdot\frac{3}{8}\left(\left(\frac{1}{\pi\alpha}\right)^4 + \frac{1}{45} - \frac{4(\pi\alpha)^2}{189} + \frac{(\pi\alpha)^4}{135} - O(\pi\alpha)^6 \right) \\ -3\cdot\frac{1}{4}\left(\left(\frac{1}{\pi\alpha}\right)^3 - \frac{(\pi\alpha)}{15} + \frac{4(\pi\alpha)^3}{189} - \frac{(\pi\alpha)^5}{225} + O(\pi\alpha)^7 \right) \end{aligned} \right) \\
&= \frac{\pi^6}{60} \lim_{\alpha\to+0} \left(\begin{aligned} \left(4\cdot\frac{3}{4} - 6\cdot\frac{3}{8} - 3\cdot\frac{1}{4}\right)\left(\frac{1}{\pi\alpha}\right)^6 + \left(-6\cdot\frac{3}{8}\cdot\frac{1}{45} - 3\cdot\frac{1}{4}\left(-\frac{1}{15}\right)\right)\left(\frac{1}{\pi\alpha}\right)^2 \\ +4\cdot\frac{3}{4}\cdot\frac{2}{189} - 6\cdot\frac{3}{8}\left(-\frac{4}{189}\right) - 3\cdot\frac{1}{4}\cdot\frac{4}{189} - O(\pi\alpha)^2 \end{aligned} \right).
\end{aligned}$$

Therefore

(8.24)
$$\zeta(6) = \frac{\pi^6}{60} \lim_{\alpha\to+0} \left(0\times\left(\left(\frac{1}{\pi\alpha}\right)^6 + \left(\frac{1}{\pi\alpha}\right)^2 \right) + \frac{4}{63} - O(\pi\alpha)^2 \right) = \frac{\pi^6}{945}.$$

For equation (8.14), 6 is substituted for θ.

$$\zeta(-5) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{5}{2}\right)}\pi^{-6}\Gamma(3)\,\zeta(6) = \frac{\sqrt{\pi}}{-\frac{8\sqrt{\pi}}{15}}\pi^{-6}\cdot 2\cdot\frac{\pi^6}{945} = -\frac{1}{252}. \tag{8.25}$$

For equation (8.13), 4 is substituted for k.

$$\begin{aligned}
\zeta(8) &= \frac{2\,\pi^8}{7\,\Gamma(5)\sqrt{\pi}}\lim_{\alpha\to+0}\left(\frac{1}{\pi\alpha}\right)^5\sum_{n=1}^{\infty}n^2\left(\begin{aligned}&-3\int_0^{\infty}y^4e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)}dy\\&+2(\pi n\alpha)^2\int_0^{\infty}y^2e^{-\left(y^2+\frac{(\pi n\alpha)^2}{y^2}\right)}dy\end{aligned}\right)\\
&= \frac{\pi^8}{84\sqrt{\pi}}\lim_{\alpha\to+0}\left(\frac{1}{\pi\alpha}\right)^5\sum_{n=1}^{\infty}n^2\left(\begin{aligned}&-3\cdot\frac{(2\pi n\alpha)^2+3\,(2\pi n\alpha)+3}{8}\sqrt{\pi}\,e^{-2\pi n\alpha}\\&+\frac{(2\pi n\alpha)^2}{2}\cdot\frac{(2\pi n\alpha)+1}{4}\sqrt{\pi}\,e^{-2\pi n\alpha}\end{aligned}\right)\\
&= \frac{\pi^8}{672}\lim_{\alpha\to+0}\left(\frac{1}{\pi\alpha}\right)^5\left(\begin{aligned}&8(\pi\alpha)^3\sum_{n=1}^{\infty}n^5e^{-2\pi n\alpha}-8(\pi\alpha)^2\sum_{n=1}^{\infty}n^4e^{-2\pi n\alpha}\\&-18\,(\pi\alpha)\sum_{n=1}^{\infty}n^3e^{-2\pi n\alpha}-9\sum_{n=1}^{\infty}n^2e^{-2\pi n\alpha}\end{aligned}\right)\\
&= \frac{\pi^8}{672}\lim_{\alpha\to+0}\left(\frac{1}{\pi\alpha}\right)^5\left(\begin{aligned}&8(\pi\alpha)^3\cdot\frac{15}{8}\left(\left(\frac{1}{\pi\alpha}\right)^6-\frac{2}{945}+\frac{(\pi\alpha)^2}{225}-\frac{4(\pi\alpha)^4}{1485}+O\,(\pi\alpha)^6\right)\\&-8(\pi\alpha)^2\cdot\frac{3}{4}\left(\left(\frac{1}{\pi\alpha}\right)^5+\frac{2\,(\pi\alpha)}{189}-\frac{(\pi\alpha)^3}{135}+\frac{4(\pi\alpha)^5}{1485}-O\,(\pi\alpha)^7\right)\\&-18\,(\pi\alpha)\cdot\frac{3}{8}\left(\left(\frac{1}{\pi\alpha}\right)^4+\frac{1}{45}-\frac{4(\pi\alpha)^2}{189}+\frac{(\pi\alpha)^4}{135}-\frac{8(\pi\alpha)^6}{4455}+O\,(\pi\alpha)^8\right)\\&-9\cdot\frac{1}{4}\left(\left(\frac{1}{\pi\alpha}\right)^3-\frac{(\pi\alpha)}{15}+\frac{4(\pi\alpha)^3}{189}-\frac{(\pi\alpha)^5}{225}+\frac{8(\pi\alpha)^7}{10395}-O\,(\pi\alpha)^9\right)\end{aligned}\right)\\
&= \frac{\pi^8}{672}\lim_{\alpha\to+0}\left(\begin{aligned}&\left(8\cdot\frac{15}{8}-8\cdot\frac{3}{4}-18\cdot\frac{3}{8}-9\cdot\frac{1}{4}\right)\left(\frac{1}{\pi\alpha}\right)^8\\&+\left(-18\cdot\frac{3}{8}\cdot\frac{1}{45}-9\cdot\frac{1}{4}\left(-\frac{1}{15}\right)\right)\left(\frac{1}{\pi\alpha}\right)^4\\&+\left(8\cdot\frac{15}{8}\left(-\frac{2}{945}\right)-8\cdot\frac{3}{4}\cdot\frac{2}{189}-18\cdot\frac{3}{8}\left(-\frac{4}{189}\right)-9\cdot\frac{1}{4}\cdot\frac{4}{189}\right)\left(\frac{1}{\pi\alpha}\right)^2\\&+8\cdot\frac{15}{8}\cdot\frac{1}{225}-8\cdot\frac{3}{4}\left(-\frac{1}{135}\right)-18\cdot\frac{3}{8}\cdot\frac{1}{135}-9\cdot\frac{1}{4}\left(-\frac{1}{225}\right)-O\,(\pi\alpha)^2\end{aligned}\right).
\end{aligned} \tag{8.26}$$

Therefore

$$\zeta(8) = \frac{\pi^8}{672}\lim_{\alpha\to+0}\left(0\times\left(\left(\frac{1}{\pi\alpha}\right)^8+\left(\frac{1}{\pi\alpha}\right)^4+\left(\frac{1}{\pi\alpha}\right)^2\right)+\frac{16}{225}-O\,(\pi\alpha)^2\right) = \frac{\pi^8}{9450}. \tag{8.27}$$

For equation (8.14), 8 is substituted for θ.

$$\zeta(-7) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{7}{2}\right)}\pi^{-8}\Gamma(4)\,\zeta(8) = \frac{\sqrt{\pi}}{\frac{16\sqrt{\pi}}{105}}\pi^{-8}\cdot 6\cdot\frac{\pi^8}{9450} = \frac{1}{240}. \tag{8.28}$$

For even numbers of 2, 4, 6, and 8, the zeta function has the shape of
(Coefficient)×{0×(Divergent terms)+(Constant)−(Vanishing terms)} in the stage just before taking the limit.
Let remember equation (7.26), which is shown again.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{2k+1}} =& \frac{(k+1)\,(k-1)!\,2^k}{(2k-1)!!\,\pi}\zeta(2k+2)\\&-\frac{2}{k\,(2k-1)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left(\sum_{\mu=0}^{k}b_{k,\mu+1}(2\pi p)^{\mu}\right)(2\pi p)\,e^{-2\pi p},\quad k\in\mathbb{N}.\end{aligned} \tag{8.29}$$

The series representation of the zeta function is demonstrated for every even number of 4 or more.
The following are some calculation examples:

$$\begin{aligned}\zeta(4) &= \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^3}+\frac{\pi}{2}\sum_{p=1}^{\infty}\sigma_{-3}(p)\left((2\pi p)+1\right)(2\pi p)\,e^{-2\pi p}\\ &= \frac{7\pi^4}{720}+\frac{\pi}{2}\sum_{p=1}^{\infty}\sigma_{-3}(p)\left((2\pi p)+1\right)(2\pi p)\,e^{-2\pi p}.\end{aligned} \tag{8.30}$$

$$\zeta(6) = \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^5}+\frac{\pi}{12}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^2+3(2\pi p)+6\right)(2\pi p)\,e^{-2\pi p}. \tag{8.31}$$

$$\begin{aligned}\zeta(8) &= \frac{15\pi}{64}\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^7}+\frac{\pi}{96}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^3+6(2\pi p)^2+21(2\pi p)+45\right)(2\pi p)\,e^{-2\pi p}\\ &= \frac{19\pi^8}{241920}+\frac{\pi}{96}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^3+6(2\pi p)^2+21(2\pi p)+45\right)(2\pi p)\,e^{-2\pi p}.\end{aligned} \tag{8.32}$$

$$\zeta(10) = \frac{7\pi}{32}\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^9}+\frac{\pi}{960}\sum_{p=1}^{\infty}\sigma_{-9}(p)\begin{pmatrix}(2\pi p)^4+10(2\pi p)^3+55(2\pi p)^2\\ +195(2\pi p)+420\end{pmatrix}(2\pi p)\,e^{-2\pi p}. \tag{8.33}$$

The zeta function has the shape of
{(Constant or infinite series to be considered as constant)+(Coefficient)×(Infinite series)} for any even number of 4 or more. To understand the kind of series representation $\zeta(2)$ have, recall equation (6.9), which is shown again.

$$\begin{aligned}\frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta) &= \frac{\theta(\theta+1)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)\\ &-2\theta\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+2(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right),\quad \theta\in\mathbb{C}.\end{aligned} \tag{8.34}$$

After exchanging both sides of the equation, by moving the second term from the left to the right side,

$$\begin{aligned}\frac{\theta(\theta+1)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta) &= \frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)\\ &+2\theta\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+2(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right),\quad \theta\in\mathbb{C}.\end{aligned} \tag{8.35}$$

For the second term on the right side of equation (8.35), the recurrence formula for the modified Bessel function of the second kind is applied.

$$\begin{aligned}&2\theta\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+4(\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right)\\ &= 2\theta\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+4\left(\left(-\frac{\theta}{2}\right)\mathrm{K}_{-\frac{\theta}{2}}(2\pi p)+(\pi p)\,\mathrm{K}_{\frac{-2-\theta}{2}}(2\pi p)\right)\right)\\ &= 2\theta\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(4(\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)-(\theta+1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\right)\\ &= 2\theta\pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(4(\pi p)^{\frac{2+\theta}{2}}\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)-(\theta+1)(\pi p)^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\right)\\ &= 2\theta\pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(4\int_0^{\infty}y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy-(\theta+1)\int_0^{\infty}y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\right).\end{aligned} \tag{8.36}$$

Here, the relation of equation (19.102) is used.
Therefore

$$\frac{\theta(\theta+1)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)=\frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta) + 2\theta\pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(4\int_0^\infty y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy-(\theta+1)\int_0^\infty y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\right),\quad \theta\in\mathbb{C}. \tag{8.37}$$

Both sides of equation(8.37) are multiplied by

$$\frac{\pi^{\frac{1+\theta}{2}}}{\theta\,\Gamma\left(1+\frac{1+\theta}{2}\right)}$$

to obtain the function $\zeta(1+\theta)$.

$$\zeta(1+\theta) = \frac{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)\sqrt{\pi}}{2\Gamma\left(1+\frac{1+\theta}{2}\right)}+\frac{2\sqrt{\pi}}{\Gamma\left(1+\frac{1+\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\begin{pmatrix}4\int_0^\infty y^{\theta+1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\\ -(\theta+1)\int_0^\infty y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\end{pmatrix},\quad \theta\in\mathbb{C}\backslash\{0\}. \tag{8.38}$$

For equation (8.38), -1 is substituted for θ.

$$\begin{aligned}\zeta(0) &= \frac{-2\Gamma\left(-\frac{1}{2}\right)\zeta(-1)\sqrt{\pi}}{2\Gamma(1)}+\frac{2\sqrt{\pi}}{\Gamma(1)}\sum_{p=1}^{\infty}\sigma_1(p)\begin{pmatrix}4\int_0^\infty y^0e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\\ -0\cdot\int_0^\infty y^{-2}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\end{pmatrix}\\ &= -\left(-2\sqrt{\pi}\right)\zeta(-1)\sqrt{\pi}+2\sqrt{\pi}\sum_{p=1}^{\infty}\sigma_1(p)\,4\cdot\frac{\sqrt{\pi}}{2}e^{-2\pi p}\\ &= 2\pi\zeta(-1)+4\pi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}m\,e^{-2\pi mn}\\ &= 2\pi\zeta(-1)+\pi\sum_{n=1}^{\infty}\frac{1}{\sinh^2(\pi n)}.\end{aligned} \tag{8.39}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{\sinh^2(\pi n)}=\frac{1}{\pi}\left(\zeta(0)-2\pi\zeta(-1)\right)=\frac{1}{\pi}\left(-\frac{1}{2}-2\pi\left(-\frac{1}{12}\right)\right)=\frac{1}{6}-\frac{1}{2\pi}. \tag{8.40}$$

This result is known as "[13]."
For equation (8.38), 1 is substituted for θ.

$$\begin{aligned}\zeta(2) &= \frac{\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(2)}+\frac{2\sqrt{\pi}}{\Gamma(2)}\sum_{p=1}^{\infty}\sigma_{-1}(p)\begin{pmatrix}4\int_0^\infty y^2e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\\ -2\int_0^\infty y^0e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\end{pmatrix}\\ &= \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{2}+2\sqrt{\pi}\sum_{p=1}^{\infty}\sigma_{-1}(p)\left(4\cdot\frac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p}-2\cdot\frac{\sqrt{\pi}}{2}e^{-2\pi p}\right)\\ &= \frac{\pi}{2}+2\pi\sum_{p=1}^{\infty}\sigma_{-1}(p)(2\pi p)\,e^{-2\pi p}\\ &= \frac{\pi}{2}+4\pi^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n}(mn)\,e^{-2\pi mn}.\end{aligned} \tag{8.41}$$

Thus, the double series representation is obtained.

$$\zeta(2) = \frac{\pi}{2} + 4\pi^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m\, e^{-2\pi mn}. \tag{8.42}$$

The hyperbolic sine is used to obtain the series representation.

$$\zeta(2) = \frac{\pi}{2} + \pi^2 \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n)}. \tag{8.43}$$

The exact value of $\zeta(2)$ is finally reached:

$$\zeta(2) = \frac{\pi}{2} + \pi^2\left(\frac{1}{6} - \frac{1}{2\pi}\right) = \frac{\pi^2}{6}. \tag{8.44}$$

9 Hadamard Product Representation for the Zeta Function and Derivation of a New Chi Function

9.1 Hadamard Product Representation for the Zeta Function

The Riemann Xi function's Hadamard product is shown as follows:

$$\xi(\theta) = \xi(0)\prod_{m=1}^{\infty}\left(1 - \frac{\theta}{\frac{1}{2}+i\rho_m}\right)\left(1 - \frac{\theta}{\frac{1}{2}-i\rho_m}\right) = \frac{1}{2}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (2\theta-1)^2}{(2\rho_m)^2+1}, \quad \theta \in \mathbb{C}. \tag{9.1}$$

The definition of the Riemann Xi function is again presented.

$$\xi(\theta) := \frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta), \quad \theta \in \mathbb{C}. \tag{9.2}$$

Therefore, the zeta function's Hadamard product representation is as follows:

$$\begin{aligned}\zeta(\theta) &= \frac{2\pi^{\frac{\theta}{2}}}{\theta(\theta-1)\Gamma\left(\frac{\theta}{2}\right)}\cdot\frac{1}{2}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (2\theta-1)^2}{(2\rho_m)^2+1} \\ &= \frac{\pi^{\frac{\theta}{2}}}{2(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (2\theta-1)^2}{(2\rho_m)^2+1}, \quad \theta \in \mathbb{C}\backslash\{1\}.\end{aligned} \tag{9.3}$$

For equation (9.3), 0 is substituted for θ.

$$\zeta(0) = \frac{\pi^{\frac{\theta}{2}}}{2(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (2\theta-1)^2}{(2\rho_m)^2+1}\Bigg|_{\theta=0} = -\frac{1}{2}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2+1}{(2\rho_m)^2+1} = -\frac{1}{2}. \tag{9.4}$$

Any odd number of -1 or less is substituted for θ in equation (9.3).

$$\begin{aligned}\zeta(1-2k) &= \frac{\pi^{\frac{1-2k}{2}}}{2(1-2k-1)\Gamma\left(1+\frac{1-2k}{2}\right)}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (2(1-2k)-1)^2}{(2\rho_m)^2+1} \\ &= -\frac{1}{2k}\cdot\frac{\sqrt{\pi}}{2\left(\frac{1-2k}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\pi^k}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2+1} \\ &= -\frac{1}{2k}\cdot\frac{\sqrt{\pi}}{-(2k-1)(-1)^k\frac{2^k\sqrt{\pi}}{(2k-1)!!}\pi^k}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2+1} \\ &= -\frac{1}{2k}\cdot\frac{(-1)^{k+1}(2k-3)!!}{2^k\pi^k}\prod_{m=1}^{\infty}\frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2+1}, \quad k \in \mathbb{N}.\end{aligned} \tag{9.5}$$

Here, $(-1)!! = 1$ is agreed.
Conversely, the exact value of the zeta function for any odd number of -1 or less is well known as the formula.

$$\zeta(1-2k) = -\frac{B_{2k}}{2k}, \quad k \in \mathbb{N}. \tag{9.6}$$

By combining the equations (9.5) and (9.6),

$$\prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1} = \frac{(-1)^{k+1} 2^k \pi^k}{(2k-3)!!} B_{2k}, \quad k \in \mathbb{N}. \tag{9.7}$$

Some examples of calculation are shown.

$$\prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 3^2}{(2\rho_m)^2 + 1} = \frac{(-1)^2 2\pi}{(-1)!!} B_2 = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3}. \tag{9.8}$$

$$\prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 7^2}{(2\rho_m)^2 + 1} = \frac{(-1)^3 2^2 \pi^2}{1!!} B_4 = -4\pi^2 \left(-\frac{1}{30}\right) = \frac{2\pi^2}{15}. \tag{9.9}$$

$$\prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 11^2}{(2\rho_m)^2 + 1} = \frac{(-1)^4 2^3 \pi^3}{3!!} B_6 = \frac{8\pi^3}{3} \cdot \frac{1}{42} = \frac{4\pi^3}{63}. \tag{9.10}$$

$$\prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 15^2}{(2\rho_m)^2 + 1} = \frac{(-1)^5 2^4 \pi^4}{5!!} B_8 = -\frac{16\pi^4}{15} \left(-\frac{1}{30}\right) = \frac{8\pi^4}{225}. \tag{9.11}$$

For equation (9.3), any even number of 2 or more is substituted for θ.

$$\begin{aligned} \zeta(2k) &= \frac{\pi^k}{2(2k-1)\Gamma(1+k)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1} \\ &= \frac{\pi^k}{2(2k-1)k!} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1}, \quad k \in \mathbb{N}. \end{aligned} \tag{9.12}$$

Any odd number of 3 or more is substituted for θ in equation (9.3).

$$\begin{aligned} \zeta(2k+1) &= \frac{\pi^{\frac{2k+1}{2}}}{2(2k)\Gamma\left(1+\frac{2k+1}{2}\right)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k+1)^2}{(2\rho_m)^2 + 1} \\ &= \frac{\pi^{\frac{2k+1}{2}}}{4k\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{2k+1}{2}\right)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k+1)^2}{(2\rho_m)^2 + 1} \\ &= \frac{\pi^{\frac{2k+1}{2}}}{2k(2k+1)\frac{(2k-1)!!\sqrt{\pi}}{2^k}} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k+1)^2}{(2\rho_m)^2 + 1} \\ &= \frac{2^{k-1}\pi^k}{k(2k+1)!!} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k+1)^2}{(2\rho_m)^2 + 1}, \quad k \in \mathbb{N}. \end{aligned} \tag{9.13}$$

The following are some calculation examples:

$$\zeta(2) = \frac{\pi}{2} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 3^2}{(2\rho_m)^2 + 1} = \frac{\pi}{2} \cdot \frac{\pi}{3} = \frac{\pi^2}{6}. \tag{9.14}$$

$$\zeta(3) = \frac{\pi}{3} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 5^2}{(2\rho_m)^2 + 1}. \tag{9.15}$$

$$\zeta(4) = \frac{\pi^2}{12} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 7^2}{(2\rho_m)^2 + 1} = \frac{\pi^2}{12} \cdot \frac{2\pi^2}{15} = \frac{\pi^4}{90}. \tag{9.16}$$

$$\zeta(5) = \frac{\pi^2}{15} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 9^2}{(2\rho_m)^2 + 1}. \tag{9.17}$$

$$\zeta(6) = \frac{\pi^3}{60} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 11^2}{(2\rho_m)^2 + 1} = \frac{\pi^3}{60} \cdot \frac{4\,\pi^3}{63} = \frac{\pi^6}{945}. \tag{9.18}$$

$$\zeta(7) = \frac{4\,\pi^3}{315} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 13^2}{(2\rho_m)^2 + 1}. \tag{9.19}$$

$$\zeta(8) = \frac{\pi^4}{336} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 15^2}{(2\rho_m)^2 + 1} = \frac{\pi^4}{336} \cdot \frac{8\,\pi^4}{225} = \frac{\pi^8}{9450}. \tag{9.20}$$

$$\zeta(9) = \frac{2\,\pi^4}{945} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 17^2}{(2\rho_m)^2 + 1}. \tag{9.21}$$

9.2 Derivation of a new Chi function

By the way, the origin symmetric functional equation with correction terms is similar to the functional equation that the Riemann Xi function follows.
The new function, called the Chi function, is defined as follows:

$$\chi(\theta) := A_{[2]}(\theta) + \hat{F}_{[2]}(\theta), \quad \theta \in \mathbb{C}. \tag{9.22}$$

To generate the representation for the Chi function, equations (5.82) and (5.84) are substituted on the right side of equation (9.22).

$$\chi(\theta) = \frac{-\theta(1-\theta)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta) - 2(1-\theta)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) + (2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta \in \mathbb{C}. \tag{9.23}$$

The Chi function has clear origin symmetry based on functional equation (5.81).

$$\chi(\theta) = \chi(-\theta), \quad \theta \in \mathbb{C}. \tag{9.24}$$

For equation (9.23), 1 is substituted for θ.

$$\begin{aligned} \chi(1) &= \frac{-\theta(1-\theta)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)\Big|_{\theta=1} - 2\cdot 0\cdot\sum_{p=1}^{\infty}\sigma_{-1}(p)\,p^{\frac{1}{2}}\left(1\cdot\mathrm{K}_{\frac{1}{2}}(2\pi p) + (2\pi p)\,\mathrm{K}_{\frac{1}{2}}(2\pi p)\right) \\ &= -\theta\pi^{-\frac{1-\theta}{2}}\Gamma\left(1+\frac{1-\theta}{2}\right)\zeta(1-\theta)\Big|_{\theta=1} = \frac{1}{2}. \end{aligned} \tag{9.25}$$

Therefore

$$\chi(\pm 1) = \frac{1}{2}. \tag{9.26}$$

For equation (9.23), 0 is substituted for θ.

$$\begin{aligned} \chi(0) &= -(1-\theta)\,\pi^{-\frac{\theta}{2}}\Gamma\left(1+\frac{\theta}{2}\right)\zeta(\theta)\Big|_{\theta=0} - 2\sum_{p=1}^{\infty}\sigma_0(p)\left(0\cdot\mathrm{K}_0(2\pi p) + (2\pi p)\,\mathrm{K}_1(2\pi p)\right) \\ &= \frac{1}{2} - 2\sum_{p=1}^{\infty}\sigma_0(p)\,(2\pi p)\,\mathrm{K}_1(2\pi p) \simeq 0.4875331150002445. \end{aligned} \tag{9.27}$$

On the real axis, the Chi function has a minimum value at the point $\theta = 0$.

$$\chi(x) \geq \chi(0), \quad x \in \mathbb{R}. \tag{9.28}$$

An attempt is made to find 10 non-trivial zeros of the Chi function close to the real axis in order by interpolation. The result indicates that the real parts of 10 non-trivial zeros are all zero. Thus, the assumption that all non-trivial zeros of the Chi function lie on the imaginary axis is proved in the narrow domain. Extending the domain of consideration to the whole complex plane, I propose the generalized Riemann hypothesis for the Chi function that states that all non-trivial zeros lie on the imaginary axis. It will be proved in Section 14.
Furthermore, I assume that there are infinitely many non-trivial zeros on the imaginary axis in the Chi function. I define that the positive imaginary part of the m-th non-trivial zero of the Chi function to which the number is allocated in order to be closer to the real axis is τ_m. Under the assumptions,

$$\chi\left(\pm i\tau_m\right) = 0, \quad \left(0 \leq \tau_1 \leq \tau_2 \leq \tau_3 \cdots\right). \tag{9.29}$$

The positive imaginary parts of 10 non-trivial zeros of the Chi function close to the real axis are shown in order in a table. For comparison, the positive imaginary parts of 10 non-trivial zeros of the zeta function close to the real axis are also demonstrated in order in the same table.

Table shows the positive imaginary parts of 10 non-trivial zeros of the Chi function and their comparisons

m	τ_m	$<$, or $>$	ρ_m
1	12.041...	$<$	14.134...
2	20.487...	$<$	21.022...
3	25.976...	$>$	25.010...
4	28.269...	$<$	30.424...
5	32.685...	$<$	32.935...
6	36.583...	$<$	37.586...
7	42.044...	$>$	40.918...
8	42.901...	$<$	43.327...
9	46.556...	$<$	48.005...
10	50.021...	$>$	49.773...

Table. 9.1

Under the generalized Riemann hypothesis for the Chi function, assuming that there are infinitely many non-trivial zeros on the imaginary axis, The Hadamard product representation of the Chi function can alternatively be written down.

$$\chi\left(\theta\right) = \chi\left(0\right) \prod_{m=1}^{\infty} \left(1 - \frac{\theta}{i\tau_m}\right)\left(1 - \frac{\theta}{-i\tau_m}\right) = \chi\left(0\right) \prod_{m=1}^{\infty} \left(1 + \frac{\theta^2}{\left(\tau_m\right)^2}\right), \quad \theta \in \mathbb{C}. \tag{9.30}$$

To obtain the value of the infinite product in equation (9.30), 1/2 is substituted for θ.

$$\chi\left(\frac{1}{2}\right) = \chi\left(0\right) \prod_{m=1}^{\infty} \left(1 + \frac{\left(1/2\right)^2}{\left(\tau_m\right)^2}\right). \tag{9.31}$$

Therefore,

$$\prod_{m=1}^{\infty} \left(1 + \frac{1}{\left(2\tau_m\right)^2}\right) = \frac{\chi\left(\frac{1}{2}\right)}{\chi\left(0\right)} \simeq 1.0063381050061486. \tag{9.32}$$

1/2 is substituted for θ in equation (9.3).

$$\zeta\left(\frac{1}{2}\right) = -\frac{4\,\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)} \prod_{m=1}^{\infty} \frac{\left(2\rho_m\right)^2}{\left(2\rho_m\right)^2 + 1} = -\frac{4\,\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)} \prod_{m=1}^{\infty} \frac{1}{1 + \frac{1}{\left(2\rho_m\right)^2}}. \tag{9.33}$$

Therefore,

$$\prod_{m=1}^{\infty} \left(1 + \frac{1}{\left(2\rho_m\right)^2}\right) = -\frac{4\,\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)} \simeq 1.0057917953503750. \tag{9.34}$$

From the equations (9.32) and (9.34), the following inequality is obtained:

$$\prod_{m=1}^{\infty} \left(1 + \frac{1}{\left(2\rho_m\right)^2}\right) < \prod_{m=1}^{\infty} \left(1 + \frac{1}{\left(2\tau_m\right)^2}\right). \tag{9.35}$$

According to the inequality (9.35), the distribution of the positive imaginary parts of non-trivial zeros of the Chi function is closer to the real axis than the distribution of those of the zeta function on the whole.

10. Functional Equation Transformations, Part3

Compared to the previous sections of 4 and 5, this section will be conducted in a more straightforward manner.

10.1 Functional Equation Transformation for the third-order $\mathrm{I_r}$ type functional equation

The following are the FET operations for the third-order $\mathrm{I_r}$ type functional equation:

$$\begin{aligned}L_{[3]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty l_{[3]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx\\ &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-3n^2x^{\theta+1}+12\pi n^4x^{\theta+3}-4\pi^2n^6x^{\theta+3}\right)e^{-\pi\left(n^2+\alpha^2\right)x^2}dx.\end{aligned}\tag{10.1}$$

$$\begin{aligned}R_{[3]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty r_{[3]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx\\ &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(12n^2x^{\theta-4}-18\pi n^4x^{\theta-6}+4\pi^2n^6x^{\theta-8}\right)e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx.\end{aligned}\tag{10.2}$$

For the integral of $L_{[3]}(\theta)$, I perform the variable transformation

$$x=\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}.$$

$$L_{[3]}(\theta)=2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\begin{pmatrix}-3n^2\left(\dfrac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+1}{2}}\\ +12\pi n^4\left(\dfrac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+3}{2}}\\ -4\pi^2n^6\left(\dfrac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+5}{2}}\end{pmatrix}e^{-y}\frac{1}{2}\left(\frac{1}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}y^{-\frac{1}{2}}dy.\tag{10.3}$$

Using the left-sided limit of the positive real variable α,

$$\begin{aligned}L_{[3]}(\theta) &= \pi^{-\frac{\theta}{2}}\int_0^\infty\sum_{n=1}^\infty\frac{1}{n^\theta}\left(-3y^{\frac{\theta}{2}}+12y^{\frac{\theta}{2}+1}-4y^{\frac{\theta}{2}+2}\right)e^{-y}dy\\ &= \pi^{-\frac{\theta}{2}}\sum_{n=1}^\infty\frac{1}{n^\theta}\left(-3\int_0^\infty y^{1+\frac{\theta}{2}-1}e^{-y}dy+12\int_0^\infty y^{2+\frac{\theta}{2}-1}e^{-y}dy-4\int_0^\infty y^{3+\frac{\theta}{2}-1}e^{-y}dy\right).\end{aligned}\tag{10.4}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta)>1$ can be added to the above result because it is a convergent function under the condition.

$$L_{[3]}(\theta)=\pi^{-\frac{\theta}{2}}\sum_{n=1}^\infty\frac{1}{n^\theta}\left(-3\,\Gamma\left(1+\frac{\theta}{2}\right)+12\,\Gamma\left(2+\frac{\theta}{2}\right)-4\,\Gamma\left(3+\frac{\theta}{2}\right)\right),\quad \mathrm{Re}(\theta)>1.\tag{10.5}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$L_{[3]}(\theta)=\pi^{-\frac{\theta}{2}}\zeta(\theta)\left(-3\,\Gamma\left(1+\frac{\theta}{2}\right)+12\,\Gamma\left(2+\frac{\theta}{2}\right)-4\,\Gamma\left(3+\frac{\theta}{2}\right)\right),\quad \theta\in\mathbb{C}.\tag{10.6}$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned}L_{[3]}(\theta) &= \pi^{-\frac{\theta}{2}}\zeta(\theta)\left(-3\left(\frac{\theta}{2}\right)+12\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)-4\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)\right)\Gamma\left(\frac{\theta}{2}\right)\\ &= \pi^{-\frac{\theta}{2}}\zeta(\theta)\cdot\left(-\frac{\theta}{2}\right)\left(\theta^2-1\right)\cdot\Gamma\left(\frac{\theta}{2}\right),\quad \theta\in\mathbb{C}.\end{aligned}\tag{10.7}$$

Therefore,

(10.8) $$L_{[3]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta), \quad \theta \in \mathbb{C}.$$

The function $L_{[3]}(\theta)$ is a convergent function in the whole complex plane.
For the integral of $R_{[3]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

(10.9) $$\begin{aligned} R_{[3]}(\theta) &= 2\pi \lim_{\alpha\to+0}\int_{\infty}^{0}\sum_{n=1}^{\infty}\left(12n^2\left(\frac{1}{y}\right)^{\theta-4} - 18\pi n^4\left(\frac{1}{y}\right)^{\theta-6} + 4\pi^2 n^6\left(\frac{1}{y}\right)^{\theta-8}\right)e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}\left(-y^{-2}\right)dy \\ &= 2\pi \lim_{\alpha\to+0}\int_{0}^{\infty}\sum_{n=1}^{\infty}\left(12n^2y^{2-\theta} - 18\pi n^4y^{4-\theta} + 4\pi^2n^6y^{6-\theta}\right)e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}dy. \end{aligned}$$

For the integral of $R_{[3]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi}n},$$

one more time.

(10.10) $$\begin{aligned} R_{[3]}(\theta) &= 2\pi \lim_{\alpha\to+0}\int_{0}^{\infty}\sum_{n=1}^{\infty}\left(12n^2\left(\frac{x}{\sqrt{\pi}n}\right)^{2-\theta} - 18\pi n^4\left(\frac{x}{\sqrt{\pi}n}\right)^{4-\theta} + 4\pi^2n^6\left(\frac{x}{\sqrt{\pi}n}\right)^{6-\theta}\right)e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)}\frac{1}{\sqrt{\pi}n}dx \\ &= 2\pi^{-\frac{1-\theta}{2}}\lim_{\alpha\to+0}\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(12x^{2-\theta} - 18x^{4-\theta} + 4x^{6-\theta}\right)e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)}dx. \end{aligned}$$

Using the left-sided limit of the positive real variable α,

(10.11) $$R_{[3]}(\theta) = 2\pi^{-\frac{1-\theta}{2}}\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(12x^{2-\theta} - 18x^{4-\theta} + 4x^{6-\theta}\right)e^{-x^2}dx.$$

For the integral of $R_{[3]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.
(10.12)
$$\begin{aligned} R_{[3]}(\theta) &= 2\pi^{-\frac{1-\theta}{2}}\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(12y^{\frac{2-\theta}{2}} - 18y^{\frac{4-\theta}{2}} + 4y^{\frac{6-\theta}{2}}\right)e^{-y}\frac{1}{2}y^{-\frac{1}{2}}dy \\ &= 2\pi^{-\frac{1-\theta}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(6\int_{0}^{\infty}y^{\frac{1-\theta}{2}}e^{-y}dy - 9\int_{0}^{\infty}y^{\frac{3-\theta}{2}}e^{-y}dy + 2\int_{0}^{\infty}y^{\frac{5-\theta}{2}}e^{-y}dy\right) \\ &= 2\pi^{-\frac{1-\theta}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(6\int_{0}^{\infty}y^{1+\frac{1-\theta}{2}-1}e^{-y}dy - 9\int_{0}^{\infty}y^{2+\frac{1-\theta}{2}-1}e^{-y}dy + 2\int_{0}^{\infty}y^{3+\frac{1-\theta}{2}-1}e^{-y}dy\right). \end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta) < 0$ can be added to the above result because it is a convergent function under the condition.

(10.13) $$R_{[3]}(\theta) = 2\pi^{-\frac{1-\theta}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{1-\theta}}\left(6\,\Gamma\left(1+\frac{1-\theta}{2}\right) - 9\,\Gamma\left(2+\frac{1-\theta}{2}\right) + 2\,\Gamma\left(3+\frac{1-\theta}{2}\right)\right), \quad \mathrm{Re}(\theta) < 0.$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

(10.14) $$R_{[3]}(\theta) = 2\pi^{-\frac{1-\theta}{2}}\zeta(1-\theta)\left(6\,\Gamma\left(1+\frac{1-\theta}{2}\right) - 9\,\Gamma\left(2+\frac{1-\theta}{2}\right) + 2\,\Gamma\left(3+\frac{1-\theta}{2}\right)\right), \quad \theta \in \mathbb{C}.$$

The difference formula for the gamma function is applied to the above result.

$$
\begin{aligned}
R_{[3]}(\theta) &= 2\pi^{-\frac{1-\theta}{2}}\zeta(1-\theta)\begin{pmatrix} 6\left(\frac{1-\theta}{2}\right) - 9\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \\ +2\left(2+\frac{1-\theta}{2}\right)\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right)\end{pmatrix}\Gamma\left(\frac{1-\theta}{2}\right) \\
&= 2\pi^{-\frac{1-\theta}{2}}\zeta(1-\theta)\cdot\left(-\frac{\theta}{4}\right)\left(\theta^2-1\right)\cdot\Gamma\left(\frac{1-\theta}{2}\right), \quad \theta\in\mathbb{C}.
\end{aligned} \tag{10.15}
$$

Therefore,

$$
R_{[3]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\zeta(1-\theta), \quad \theta\in\mathbb{C}. \tag{10.16}
$$

The function $R_{[3]}(\theta)$ is also a convergent function in the whole complex plane.
Because of the convergent equations (10.8) and (10.16), the following functional equation holds true:

$$
L_{[3]}(\theta) = R_{[3]}(\theta), \quad \theta\in\mathbb{C}. \tag{10.17}
$$

Therefore

$$
-\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{1-\theta}{2}}\Gamma\left(\frac{1-\theta}{2}\right)\zeta(1-\theta), \quad \theta\in\mathbb{C}. \tag{10.18}
$$

I decide that the functional equation (10.18) is called the third-order $\mathrm{I_c}$ type functional equation.
Additionally, it is equivalent to the Riemann Xi function's functional equation.

10.2 Functional Equation Transformation for the third-order $\mathrm{II_r}$ type functional equation

As you will see later, another way to prove the absolute convergence of the correction term is chosen due to a symmetry collapse in the third-order $\mathrm{II_C}$ type functional equation.
The operations of the FET for the third-order $\mathrm{II_r}$ type functional equation are shown as follows:

$$
\begin{aligned}
A_{[3]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty a_{[3]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\
&= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-3n^2x^{\theta+1}+12\pi n^4x^{\theta+3}-4\pi^2n^6x^{\theta+5}\right)e^{-\pi\left(n^2+\alpha^2\right)x^2}dx.
\end{aligned} \tag{10.19}
$$

$$
\begin{aligned}
F_{[3]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty f_{[3]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\
&= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\left(-6n^2x^{\theta+1}+24\pi n^4x^{\theta+3}-8\pi^2n^6x^{\theta+5}\right)e^{-\pi\left(\left(n^2+\alpha^2\right)x^2+\frac{m^2}{x^2}\right)}dx.
\end{aligned} \tag{10.20}
$$

$$
\begin{aligned}
B_{[3]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty b_{[3]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\
&= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(12n^2x^{\theta-3}-18\pi n^4x^{\theta-5}+4\pi^2n^6x^{\theta-7}\right)e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx.
\end{aligned} \tag{10.21}
$$

$$
\begin{aligned}
G_{[3]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty g_{[3]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\
&= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\left(24n^2x^{\theta-3}-36\pi n^4x^{\theta-5}+8\pi^2n^6x^{\theta-7}\right)e^{-\pi\left(\left(m^2+\alpha^2\right)x^2+\frac{n^2}{x^2}\right)}dx.
\end{aligned} \tag{10.22}
$$

Immediately, the following obvious result is obtained:

$$
A_{[3]}(\theta) = L_{[3]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta), \quad \theta\in\mathbb{C}. \tag{10.23}
$$

For the integral of $B_{[3]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$B_{[3]}(\theta) = 2\pi \lim_{\alpha\to+0} \int_{\infty}^{0} \sum_{n=1}^{\infty} \begin{pmatrix} 12n^2\left(\frac{1}{y}\right)^{\theta-3} - 18\pi n^4\left(\frac{1}{y}\right)^{\theta-5} \\ + 4\pi^2 n^6\left(\frac{1}{y}\right)^{\theta-7} \end{pmatrix} e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}\left(-y^{-2}\right)dy \tag{10.24}$$

$$= 2\pi \lim_{\alpha\to+0} \int_{0}^{\infty} \sum_{n=1}^{\infty} \left(12n^2y^{1-\theta} - 18\pi n^4y^{3-\theta} + 4\pi^2n^6y^{5-\theta}\right) e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}dy.$$

For the integral of $B_{[3]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi}n},$$

one more time.

$$B_{[3]}(\theta) = 2\pi \lim_{\alpha\to+0} \int_{0}^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} 12n^2\left(\frac{x}{\sqrt{\pi}n}\right)^{1-\theta} - 18\pi n^4\left(\frac{x}{\sqrt{\pi}n}\right)^{3-\theta} \\ + 4\pi^2 n^6\left(\frac{x}{\sqrt{\pi}n}\right)^{5-\theta} \end{pmatrix} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi}n}dx \tag{10.25}$$

$$= 2\pi^{\frac{\theta}{2}} \lim_{\alpha\to+0} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(12x^{1-\theta} - 18x^{3-\theta} + 4x^{5-\theta}\right) e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)}dx.$$

Using the left-sided limit of the positive real variable α,

$$B_{[3]}(\theta) = 2\pi^{\frac{\theta}{2}} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(12x^{1-\theta} - 18x^{3-\theta} + 4x^{5-\theta}\right) e^{-x^2}dx. \tag{10.26}$$

For the integral of $B_{[3]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$B_{[3]}(\theta) = 2\pi^{\frac{\theta}{2}} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(12y^{\frac{1-\theta}{2}} - 18y^{\frac{3-\theta}{2}} + 4y^{\frac{5-\theta}{2}}\right) e^{-y}\frac{1}{2}y^{-\frac{1}{2}}dy \tag{10.27}$$

$$= 2\pi^{\frac{\theta}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(6\int_{0}^{\infty} y^{\frac{-\theta}{2}}e^{-y}dy - 9\int_{0}^{\infty} y^{\frac{2-\theta}{2}}e^{-y}dy + 2\int_{0}^{\infty} y^{\frac{4-\theta}{2}}e^{-y}dy\right)$$

$$= 2\pi^{\frac{\theta}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(6\int_{0}^{\infty} y^{1+\frac{-\theta}{2}-1}e^{-y}dy - 9\int_{0}^{\infty} y^{2+\frac{-\theta}{2}-1}e^{-y}dy + 2\int_{0}^{\infty} y^{3+\frac{-\theta}{2}-1}e^{-y}dy\right).$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta) < -1$ can be added to the above result because it is a convergent function under the condition.

$$B_{[3]}(\theta) = 2\pi^{\frac{\theta}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(6\,\Gamma\left(1+\frac{-\theta}{2}\right) - 9\,\Gamma\left(2+\frac{-\theta}{2}\right) + 2\,\Gamma\left(3+\frac{-\theta}{2}\right)\right), \quad \mathrm{Re}(\theta) < -1. \tag{10.28}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$B_{[3]}(\theta) = 2\pi^{\frac{\theta}{2}}\zeta(-\theta) \left(6\,\Gamma\left(1+\frac{-\theta}{2}\right) - 9\,\Gamma\left(2+\frac{-\theta}{2}\right) + 2\,\Gamma\left(3+\frac{-\theta}{2}\right)\right), \quad \theta \in \mathbb{C}. \tag{10.29}$$

The difference formula for the gamma function is applied to the above result.

$$B_{[3]}(\theta) = 2\pi^{\frac{\theta}{2}}\zeta(-\theta) \begin{pmatrix} 6\left(\frac{-\theta}{2}\right) - 9\left(1+\frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \\ + 2\left(2+\frac{-\theta}{2}\right)\left(1+\frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \end{pmatrix} \Gamma\left(\frac{-\theta}{2}\right) \tag{10.30}$$

$$= 2\pi^{\frac{\theta}{2}}\zeta(-\theta) \cdot \left(-\frac{\theta}{4}\right)(\theta+1)(\theta+2) \cdot \Gamma\left(-\frac{\theta}{2}\right), \quad \theta \in \mathbb{C}.$$

Combining the relation of the complete symmetric functional equation to the result,

$$B_{[3]}(\theta) = -\frac{\theta(\theta+1)(\theta+2)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta) = -\frac{\theta(\theta+1)(\theta+2)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta),\ \theta\in\mathbb{C}. \tag{10.31}$$

The function $B_{[3]}(\theta)$ is also a convergent function in the whole complex plane.
The left-sided limit of the positive real variable α, is used for the defining equation (10.20),

$$F_{[3]}(\theta) = 2\pi\int_0^\infty \sum_{n=1}^\infty\sum_{m=1}^\infty\left(-6n^2x^{\theta+1}+24\pi n^4x^{\theta+3}-8\pi^2n^6x^{\theta+5}\right)e^{-\pi\left(n^2x^2+\frac{m^2}{x^2}\right)}dx. \tag{10.32}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $F_{[3]}(\theta)$, I perform the variable transformation.

$$x = \left(\frac{my}{n}\right)^{\frac{1}{2}}.$$

$$\begin{aligned}F_{[3]}(\theta) &= 2\pi\sum_{n=1}^\infty\sum_{m=1}^\infty\int_0^\infty\left(\begin{array}{r}-6n^2\left(\frac{my}{n}\right)^{\frac{\theta+1}{2}}+24\pi n^4\left(\frac{my}{n}\right)^{\frac{\theta+3}{2}}\\ -8\pi^2n^6\left(\frac{my}{n}\right)^{\frac{\theta+5}{2}}\end{array}\right)e^{-\pi mn\left(y+\frac{1}{y}\right)}\frac{1}{2}\left(\frac{m}{n}\right)^{\frac{1}{2}}y^{-\frac{1}{2}}dy\\ &= 4\sum_{n=1}^\infty\sum_{m=1}^\infty(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{array}{l}-3\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+2}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy\\ +12(\pi mn)\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+4}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy\\ -4(\pi mn)^2\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+6}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy\end{array}\right).\end{aligned} \tag{10.33}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$F_{[3]}(\theta) = 4\sum_{n=1}^\infty\sum_{m=1}^\infty(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{array}{r}-3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)+12(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)\\ -4(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn)\end{array}\right). \tag{10.34}$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.
The recurrence formula for the modified Bessel function of the second kind is applied three times.

$$\begin{aligned}&F_{[3]}(\theta)\\ &= 4\sum_{n=1}^\infty\sum_{m=1}^\infty(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{array}{l}-3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)+12(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)\\ \quad -4(\pi mn)\left(\frac{\theta+4}{2}\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)\right)\end{array}\right)\\ &= 4\sum_{n=1}^\infty\sum_{m=1}^\infty(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(-\left(3+4(\pi mn)^2\right)\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)-2(\theta-2)(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)\right)\\ &= -4\sum_{n=1}^\infty\sum_{m=1}^\infty(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{array}{l}\left(3+4(\pi mn)^2\right)\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)\\ \quad +2(\theta-2)\left(\frac{\theta+2}{2}\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right)\end{array}\right)\\ &= -4\sum_{n=1}^\infty\sum_{m=1}^\infty\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(2(\theta-2)(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)+\left(\theta^2-1+4(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)\right)\\ &= -4\sum_{n=1}^\infty\sum_{m=1}^\infty\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{array}{l}2(\theta-2)(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\\ \quad +\left(\theta^2-1+4(\pi mn)^2\right)\left(\frac{\theta}{2}\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn)\right)\end{array}\right)\\ &= -2\sum_{n=1}^\infty\sum_{m=1}^\infty\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta\left(\theta^2-1\right)+2(\theta-1)(2\pi mn)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\\ \quad +\left(\theta^2-1+(2\pi mn)^2\right)(2\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn)\end{array}\right).\end{aligned} \tag{10.35}$$

From equation(5.37) and the result of equation (10.35),

$$F_{[3]}(\theta) = -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta(\theta-1)(\theta+1)+2(\theta-1)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ +\left((\theta-1)(\theta+1)+(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{pmatrix}. \tag{10.36}$$

For the defining equation (10.22), using the left-sided limit of the positive real variable α,

$$G_{[3]}(\theta) = 2\pi\int_0^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(24n^2x^{\theta-3}-36\pi n^4x^{\theta-5}+8\pi^2n^6x^{\theta-7}\right)e^{-\pi\left(m^2x^2+\frac{n^2}{x^2}\right)}dx. \tag{10.37}$$

For the integral of $G_{[3]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

(10.38)

$$\begin{aligned}G_{[3]}(\theta) &= 2\pi\int_{\infty}^{0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(24n^2\left(\frac{1}{y}\right)^{\theta-3}-36\pi n^4\left(\frac{1}{y}\right)^{\theta-5}+8\pi^2n^6\left(\frac{1}{y}\right)^{\theta-7}\right)e^{-\pi\left(\frac{m^2}{y^2}+n^2y^2\right)}\left(-y^{-2}\right)dy\\ &= 2\pi\int_0^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(24n^2y^{1-\theta}-36\pi n^4y^{3-\theta}+8\pi^2n^6y^{5-\theta}\right)e^{-\pi\left(n^2y^2+\frac{m^2}{y^2}\right)}dy.\end{aligned}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $G_{[3]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{mx}{n}\right)^{\frac{1}{2}}.$$

(10.39)

$$\begin{aligned}G_{[3]}(\theta) &= 2\pi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\begin{pmatrix}24n^2\left(\frac{mx}{n}\right)^{\frac{1-\theta}{2}}-36\pi n^4\left(\frac{mx}{n}\right)^{\frac{3-\theta}{2}}\\ +8\pi^2n^6\left(\frac{mx}{n}\right)^{\frac{5-\theta}{2}}\end{pmatrix}e^{-\pi mn\left(x+\frac{1}{x}\right)}\frac{1}{2}\left(\frac{m}{n}\right)^{\frac{1}{2}}x^{-\frac{1}{2}}dx\\ &= 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}6\cdot\frac{1}{2}\int_0^{\infty}x^{\frac{2-\theta}{2}-1}e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)}dx\\ -9(\pi mn)\cdot\frac{1}{2}\int_0^{\infty}x^{\frac{4-\theta}{2}-1}e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)}dx\\ +2(\pi mn)^2\cdot\frac{1}{2}\int_0^{\infty}x^{\frac{6-\theta}{2}-1}e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)}dx\end{pmatrix}.\end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$G_{[3]}(\theta) = 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}6\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)-9(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\\ +2(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn)\end{pmatrix}. \tag{10.40}$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied twice.

(10.41)

$$\begin{aligned}&G_{[3]}(\theta)\\ &= 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}6\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)-9(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\\ +2(\pi mn)\left(\frac{4-\theta}{2}\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\right)\end{pmatrix}\\ &= -8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\left(-\left(6+2(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)+(\theta+5)(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\right)\\ &= -8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}-\left(6+2(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\\ +(\theta+5)\left(\frac{2-\theta}{2}\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{-\frac{\theta}{2}}(2\pi mn)\right)\end{pmatrix}\\ &= -4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}2(\theta+5)(\pi mn)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\\ -\left((\theta+1)(\theta+2)+4(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\end{pmatrix}\\ &= -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}(\theta+5)(2\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\\ -\left((\theta+1)(\theta+2)+(2\pi mn)^2\right)(2\pi mn)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\end{pmatrix}.\end{aligned}$$

Based on the result of (10.41), from equations (3.30) and (5.38),

$$G_{[3]}(\theta) = -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((\theta+5)(2\pi p)^2 \mathrm{K}_{\frac{\theta}{2}}(2\pi p) - \left((\theta+1)(\theta+2)+(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\right). \tag{10.42}$$

I define the function $H_{[3]}(\theta)$ as follows:

$$H_{[3]}(\theta) := G_{[3]}(\theta) - F_{[3]}(\theta). \tag{10.43}$$

From (the right side of equation (10.42)) − (the right side of equation (10.36)),

$$H_{[3]}(\theta) = 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta(\theta-1)(\theta+1)+(\theta-7)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ +\left((2\theta+1)(\theta+1)+2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{pmatrix}. \tag{10.44}$$

The modified Bessel functions of the second kind are written using the integral respectively.

$$\begin{aligned}&H_{[3]}(\theta)\\&= 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta(\theta-1)(\theta+1)+(\theta-7)(2\pi p)^2\right)\cdot\frac{1}{2}\int_0^{\infty}x^{\frac{\theta}{2}-1}e^{-\frac{2\pi p}{2}\left(x+\frac{1}{x}\right)}dx\\ +\left((2\theta+1)(\theta+1)+2(2\pi p)^2\right)(2\pi p)\cdot\frac{1}{2}\int_0^{\infty}x^{\frac{\theta-2}{2}-1}e^{-\frac{2\pi p}{2}\left(x+\frac{1}{x}\right)}dx\end{pmatrix}.\end{aligned} \tag{10.45}$$

For the integrals of $H_{[3]}(\theta)$, I perform the variable transformation

$$x = \frac{y^2}{\pi p}.$$

$$\begin{aligned}&H_{[3]}(\theta)\\&= 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta(\theta-1)(\theta+1)+(\theta-7)(2\pi p)^2\right)\cdot\frac{1}{2}\int_0^{\infty}\left(\frac{y^2}{\pi p}\right)^{\frac{\theta-2}{2}}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}\frac{2y}{\pi p}dy\\ +\left((2\theta+1)(\theta+1)+2(2\pi p)^2\right)(2\pi p)\cdot\frac{1}{2}\int_0^{\infty}\left(\frac{y^2}{\pi p}\right)^{\frac{\theta-4}{2}}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}\frac{2y}{\pi p}dy\end{pmatrix}\\&= \pi^{-\frac{\theta}{2}}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\begin{pmatrix}2\left(\theta(\theta-1)(\theta+1)+(\theta-7)(2\pi p)^2\right)\int_0^{\infty}y^{\theta-1}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\\ +\left((2\theta+1)(\theta+1)+2(2\pi p)^2\right)(2\pi p)^2\int_0^{\infty}y^{\theta-3}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy\end{pmatrix}.\end{aligned} \tag{10.46}$$

For convenience, the above result's absolutely convergent integrals in the whole complex plane are defined as follows:

$$I_p(\theta) := \int_0^{\infty}y^{-1+\mathrm{Re}(\theta)}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy, \quad (p\in\mathbb{N},\theta\in\mathbb{C}). \tag{10.47}$$

$$J_p(\theta) := \int_0^{\infty}y^{-3+\mathrm{Re}(\theta)}e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)}dy, \quad (p\in\mathbb{N},\theta\in\mathbb{C}). \tag{10.48}$$

Using the expressions $I_p(\theta)$ and $J_p(\theta)$,

$$\left|H_{[3]}(\theta)\right| \le \pi^{-\frac{\mathrm{Re}(\theta)}{2}}\sum_{p=1}^{\infty}\sigma_{-\mathrm{Re}(\theta)}(p)\begin{pmatrix}2\left|\theta(\theta-1)(\theta+1)\right|I_p(\theta)+2\left|(\theta-7)\right|(2\pi p)^2 I_p(\theta)\\ \left|(2\theta+1)(\theta+1)\right|(2\pi p)^2 J_p(\theta)+2(2\pi p)^4 J_p(\theta)\end{pmatrix}. \tag{10.49}$$

Now, the following obvious fact is introduced into the inequality (10.49):

$$\sigma_{-\mathrm{Re}(\theta)}(p) < p^{2k}\zeta(2k+\mathrm{Re}(\theta)), \quad (k,p\in\mathbb{N},\mathrm{Re}(\theta)>1-2k). \tag{10.50}$$

Under the condition of $\mathrm{Re}(\theta) > 1-2k$,

(10.51)
$$\left|H_{[3]}(\theta)\right| < \pi^{-\frac{\mathrm{Re}(\theta)}{2}} \sum_{p=1}^{\infty} p^{2k}\zeta(2k+\mathrm{Re}(\theta)) \begin{pmatrix} 2\left|\theta(\theta-1)(\theta+1)\right| I_p(\theta) + 2\left|(\theta-7)\right|(2\pi p)^2 I_p(\theta) \\ \left|(2\theta+1)(\theta+1)\right|(2\pi p)^2 J_p(\theta) + 2(2\pi p)^4 J_p(\theta) \end{pmatrix}, \quad (k\in\mathbb{N}, \mathrm{Re}(\theta) > 1-2k).$$

Because the variable of the zeta function is irrelevant to the variable p, the zeta function of the sum can move outside.

(10.52)
$$\left|H_{[3]}(\theta)\right| < 2\pi^{-\frac{\mathrm{Re}(\theta)}{2}} \zeta(2k+\mathrm{Re}(\theta)) \sum_{p=1}^{\infty} \begin{pmatrix} \left|\theta(\theta-1)(\theta+1)\right| p^{2k} I_p(\theta) + (2\pi)^2\left|(\theta-7)\right| p^{2(k+1)} I_p(\theta) \\ +2\pi^2\left|(2\theta+1)(\theta+1)\right| p^{2(k+1)} J_p(\theta) + (2\pi)^4 p^{2(k+2)} J_p(\theta) \end{pmatrix}, \quad (k\in\mathbb{N}, \mathrm{Re}(\theta) > 1-2k).$$

Because the variable of the zeta function is irrelevant to the variable p, the zeta function of the sum can move outside.
To estimate the absolute convergence of the function $H_{[3]}(\theta)$, the following equations with the condition are defined:

(10.53)
$$S_k(x) := \sum_{p=1}^{\infty} p^{2k} \int_0^{\infty} y^{x-1} e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)} dy, \quad (k\in\mathbb{N}, x > -2k).$$

(10.54)
$$T_k(x) := \sum_{p=1}^{\infty} p^{2k} \int_0^{\infty} y^{x-3} e^{-\left(y^2+\frac{(\pi p)^2}{y^2}\right)} dy, \quad (k\in\mathbb{N}, x > 2-2k).$$

For the function $S_k(x)$, because the integral is an absolutely convergent integral, I can change the order of relation between the sum and the integral.

(10.55)
$$\begin{aligned} S_k(x) &= \int_0^{\infty} y^{x-1} e^{-y^2} \lim_{L\to+\infty} \sum_{p=1}^{L} p^{2k} e^{-\frac{(\pi p)^2}{y^2}} dy \\ &< \int_0^{\infty} y^{x-1} e^{-y^2} \lim_{L\to+\infty} \sum_{p=1}^{L^2} p^{k} e^{-\frac{\pi^2 p}{y^2}} dy \\ &= \int_0^{\infty} y^{x-1} e^{-y^2} \sum_{p=1}^{\infty} p^{k} e^{-\frac{\pi^2 p}{y^2}} dy, \quad (k\in\mathbb{N}, x > -2k). \end{aligned}$$

To evaluate the inside sum of the integral, I define the function $f_k(t,y)$ of double positive variables with the parameter k as follows:

(10.56)
$$f_k(t,y) := t^k e^{-\frac{\pi^2 t}{y^2}}, \quad (t,y>0, k\in\mathbb{N}).$$

Taking the limits yields function values on both sides of the open interval of the positive variable t.

(10.57)
$$\lim_{t\to+0} f_k(t,y) = 0, \quad (y>0, k\in\mathbb{N}).$$

(10.58)
$$\lim_{t\to\infty} f_k(t,y) = 0, \quad (y>0, k\in\mathbb{N}).$$

In order to look into the behavior of the function, it is partially differentiated.

(10.59)
$$\frac{\partial}{\partial t} f_k(t,y) = \left(k - \frac{\pi^2}{y^2}t\right) t^{k-1} e^{-\frac{\pi^2 t}{y^2}}, \quad (t,y>0, k\in\mathbb{N}).$$

I assume that the zero of the right side of above equation is α_2.

(10.60)
$$\left.\frac{\partial}{\partial t} f_k(t,y)\right|_{t=\alpha_2} = 0, \quad (t,y>0, k\in\mathbb{N}).$$

The solution is easily obtained since it has a positive value depending on both the positive variable y and parameter k.

$$\alpha_2 = \frac{ky^2}{\pi^2}, \quad (y > 0, k \in \mathbb{N}). \tag{10.61}$$

Table of the increase and decrease for $f_k(t, y)$ of the positive variable t

t	$+0$	$\cdots$	α_2	$\cdots$	∞
$\frac{\partial}{\partial t} f_k(t, y)$		$+$	0	$-$	
$f_k(t, y)$	0	$\nearrow$	maximum	$\searrow$	0

Table. 10.1

By considering mainly the rectangle consisting of a base $(\alpha_2 + 1)$ and a height $f_k(\alpha_2, y)$ in area, I obtain the upper limit of the infinite sum.

$$\begin{aligned} \sum_{p=1}^{\infty} p^k e^{-\frac{\pi^2 p}{y^2}} &< (\alpha_2 + 1) \cdot f_k(\alpha_2, y) + \int_0^{\infty} t^k e^{-\frac{\pi^2 t}{y^2}} dt \\ &= \left(\frac{ky^2}{\pi^2} + 1\right) \cdot \left(\frac{ky^2}{\pi^2}\right)^k e^{-k} + \frac{y^{2(k+1)}}{\pi^{2(k+1)}} \Gamma(k+1) \\ &= \frac{k^{k+1} e^{-k} + k!}{\pi^{2(k+1)}} y^{2(k+1)} + \frac{k^k e^{-k}}{\pi^{2k}} y^{2k}, \quad (y > 0, k \in \mathbb{N}). \end{aligned} \tag{10.62}$$

This result is introduced into the right side of inequality (10.55).

$$\begin{aligned} S_k(x) &< \int_0^{\infty} y^{x-1} e^{-y^2} \left(\frac{k^{k+1} e^{-k} + k!}{\pi^{2(k+1)}} y^{2(k+1)} + \frac{k^k e^{-k}}{\pi^{2k}} y^{2k}\right) dy \\ &= \frac{k^{k+1} e^{-k} + k!}{\pi^{2(k+1)}} \int_0^{\infty} y^{x+2k+1} e^{-y^2} dy + \frac{k^k e^{-k}}{\pi^{2k}} \int_0^{\infty} y^{x+2k-1} e^{-y^2} dy, \quad (k \in \mathbb{N}, x > -2k). \end{aligned} \tag{10.63}$$

For the above integrals, I perform the variable transformation $y = u^{1/2}$.

$$\begin{aligned} S_k(x) &< \frac{k^{k+1} e^{-k} + k!}{\pi^{2(k+1)}} \int_0^{\infty} u^{\frac{x+2k+1}{2}} e^{-u} \frac{1}{2} u^{-\frac{1}{2}} du + \frac{k^k e^{-k}}{\pi^{2k}} \int_0^{\infty} u^{\frac{x+2k-1}{2}} e^{-u} \frac{1}{2} u^{-\frac{1}{2}} du \\ &= \frac{k^{k+1} e^{-k} + k!}{2\pi^{2(k+1)}} \int_0^{\infty} u^{\frac{x+2k+2}{2}-1} e^{-u} du + \frac{k^k e^{-k}}{2\pi^{2k}} \int_0^{\infty} u^{\frac{x+2k}{2}-1} e^{-u} du \\ &= \frac{k^{k+1} e^{-k} + k!}{2\pi^{2(k+1)}} \Gamma\left(\frac{x+2k+2}{2}\right) + \frac{k^k e^{-k}}{2\pi^{2k}} \Gamma\left(\frac{x+2k}{2}\right), \quad (k \in \mathbb{N}, x > -2k). \end{aligned} \tag{10.64}$$

Therefore,

$$S_k(x) < \left(\frac{k^{k+1} e^{-k} + k!}{4\pi^{2(k+1)}} (x+2k) + \frac{k^k e^{-k}}{2\pi^{2k}}\right) \Gamma\left(\frac{x+2k}{2}\right), \quad (k \in \mathbb{N}, x > -2k). \tag{10.65}$$

By the same method,

$$T_k(x) < \left(\frac{k^{k+1} e^{-k} + k!}{4\pi^{2(k+1)}} (x+2k-2) + \frac{k^k e^{-k}}{2\pi^{2k}}\right) \Gamma\left(\frac{x+2k-2}{2}\right), \quad (k \in \mathbb{N}, x > 2-2k). \tag{10.66}$$

The upper limit function of the function $S_k(x)$ is defined by the following equation:

$$\hat{S}_k(x) := \left(\frac{k^{k+1} e^{-k} + k!}{4\pi^{2(k+1)}} (x+2k) + \frac{k^k e^{-k}}{2\pi^{2k}}\right) \Gamma\left(\frac{x+2k}{2}\right), \quad (k \in \mathbb{N}, x > -2k). \tag{10.67}$$

The upper limit function of the function $T_k(x)$ is defined by the following equation:

$$\hat{T}_k(x) := \left(\frac{k^{k+1} e^{-k} + k!}{4\pi^{2(k+1)}} (x+2k-2) + \frac{k^k e^{-k}}{2\pi^{2k}}\right) \Gamma\left(\frac{x+2k-2}{2}\right), \quad (k \in \mathbb{N}, x > 2-2k). \tag{10.68}$$

Using the upper limit functions,

$$\begin{aligned}
&\left|H_{[3]}(\theta)\right| \\
(10.69)\quad &< 2\pi^{-\frac{\mathrm{Re}(\theta)}{2}}\zeta(2k+\mathrm{Re}(\theta))\sum_{p=1}^{\infty}\left(\begin{aligned}&|\theta(\theta-1)(\theta+1)|\,\hat{S}_k(\mathrm{Re}(\theta))+(2\pi)^2|(\theta-7)|\,\hat{S}_{k+1}(\mathrm{Re}(\theta))\\&+2\pi^2|(2\theta+1)(\theta+1)|\,\hat{T}_{k+1}(\mathrm{Re}(\theta))+(2\pi)^4\hat{T}_{k+2}(\mathrm{Re}(\theta))\end{aligned}\right), \\
&(k\in\mathbb{N},\mathrm{Re}(\theta)>1-2k).
\end{aligned}$$

When any positive integer k is given, absolute value of the function $H_{[3]}(\theta)$ is a convergent function in the right half-plane of $\mathrm{Re}(\theta)>1-2k$.
Because any positive integer for the variable k can be taken, the domain of definition can be expanded to the whole complex plane.
Thus, the function $H_{[3]}(\theta)$ is an absolutely convergent function in the whole complex plane.
Combining the above results yield the following functional equation, which holds true:

$$A_{[3]}(\theta)=B_{[3]}(\theta)+H_{[3]}(\theta),\quad \theta\in\mathbb{C}. \tag{10.70}$$

Therefore

$$\begin{aligned}
&-\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)=-\frac{\theta(\theta+1)(\theta+2)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta)+H_{[3]}(\theta)\\
&=-\frac{\theta(\theta+1)(\theta+2)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)+H_{[3]}(\theta),\quad\theta\in\mathbb{C}.
\end{aligned} \tag{10.71}$$

Where

$$H_{[3]}(\theta)=2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{aligned}&\left(\theta(\theta-1)(\theta+1)+(\theta-7)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\&+\left((2\theta+1)(\theta+1)+2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{aligned}\right). \tag{10.72}$$

I decide that the functional equation (10.71) is called the third-order $\mathrm{II_c}$ type functional equation.

10.3 Functional Equation Transformation for the fourth-order $\mathrm{I_r}$ type functional equation

The following are the FET operations for the fourth-order $\mathrm{I_r}$ type functional equation:

$$\begin{aligned}
L_{[4]}(\theta)&:=\lim_{\alpha\to+0}\int_0^{\infty}l_{[4]}(x)\,x^{\theta}e^{-\pi\alpha^2x^2}dx\\
&=2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\left(-3n^2x^{\theta+1}+42\pi n^4x^{\theta+3}-44\pi^2n^6x^{\theta+3}+8\pi^3n^8x^{\theta+5}\right)e^{-\pi\left(n^2+\alpha^2\right)x^2}dx.
\end{aligned} \tag{10.73}$$

(10.74)

$$\begin{aligned}
R_{[4]}(\theta)&:=\lim_{\alpha\to+0}\int_0^{\infty}r_{[4]}(x)\,x^{\theta}e^{-\pi\alpha^2x^2}dx\\
&=2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\left(-48n^2x^{\theta-4}+132\pi n^4x^{\theta-6}-68\pi^2n^6x^{\theta-8}+8\pi^3n^8x^{\theta-10}\right)e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx.
\end{aligned}$$

For the integral of $L_{[4]}(\theta)$, I perform the variable transformation

$$x=\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}.$$

$$L_{[4]}(\theta)=2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\left(\begin{aligned}&-3n^2\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+1}{2}}\\&+42\pi n^4\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+3}{2}}\\&-44\pi^2n^6\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+5}{2}}\\&+8\pi^3n^8\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+7}{2}}\end{aligned}\right)e^{-y}\frac{1}{2}\left(\frac{1}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}y^{-\frac{1}{2}}dy. \tag{10.75}$$

Using the left-sided limit of the positive real variable α,

$$\begin{aligned} L_{[4]}(\theta) &= \pi^{-\frac{\theta}{2}} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^\theta} \left(-3y^{\frac{\theta}{2}} + 42y^{\frac{\theta}{2}+1} - 44y^{\frac{\theta}{2}+2} + 8y^{\frac{\theta}{2}+3}\right) e^{-y} dy \\ &= \pi^{-\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^\theta} \begin{pmatrix} -3\int_0^\infty y^{1+\frac{\theta}{2}-1} e^{-y} dy + 42\int_0^\infty y^{2+\frac{\theta}{2}-1} e^{-y} dy \\ -44\int_0^\infty y^{3+\frac{\theta}{2}-1} e^{-y} dy + 8\int_0^\infty y^{4+\frac{\theta}{2}-1} e^{-y} dy \end{pmatrix}. \end{aligned} \tag{10.76}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta) > 1$ can be added to the above result because it is a convergent function under the condition.

$$L_{[4]}(\theta) = \pi^{-\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^\theta} \left(-3\Gamma\left(1+\frac{\theta}{2}\right) + 42\Gamma\left(2+\frac{\theta}{2}\right) - 44\Gamma\left(3+\frac{\theta}{2}\right) + 8\Gamma\left(4+\frac{\theta}{2}\right)\right), \quad \mathrm{Re}(\theta) > 1. \tag{10.77}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$L_{[4]}(\theta) = \pi^{-\frac{\theta}{2}} \zeta(\theta) \left(-3\Gamma\left(1+\frac{\theta}{2}\right) + 42\Gamma\left(2+\frac{\theta}{2}\right) - 44\Gamma\left(3+\frac{\theta}{2}\right) + 8\Gamma\left(4+\frac{\theta}{2}\right)\right), \quad \theta \in \mathbb{C}. \tag{10.78}$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned} L_{[4]}(\theta) &= \pi^{-\frac{\theta}{2}} \zeta(\theta) \begin{pmatrix} -3\left(\frac{\theta}{2}\right) + 42\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) \\ -44\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) + 8\left(3+\frac{\theta}{2}\right)\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) \end{pmatrix} \Gamma\left(\frac{\theta}{2}\right) \\ &= \pi^{-\frac{\theta}{2}} \zeta(\theta) \cdot \left(\frac{\theta}{2}\right)(\theta-1)(\theta+1)^2 \cdot \Gamma\left(\frac{\theta}{2}\right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{10.79}$$

Therefore,

$$L_{[4]}(\theta) = \frac{\theta(\theta-1)(\theta+1)^2}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta), \quad \theta \in \mathbb{C}. \tag{10.80}$$

The function $L_{[4]}(\theta)$ is a convergent function in the whole complex plane.
For the integral of $R_{[4]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned} R_{[4]}(\theta) &= 2\pi \lim_{\alpha\to+0} \int_\infty^0 \sum_{n=1}^\infty \begin{pmatrix} -48n^2\left(\frac{1}{y}\right)^{\theta-4} + 132\pi n^4\left(\frac{1}{y}\right)^{\theta-6} \\ -68\pi^2 n^6\left(\frac{1}{y}\right)^{\theta-8} + 8\pi^3 n^8\left(\frac{1}{y}\right)^{\theta-10} \end{pmatrix} e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)} \left(-y^{-2}\right) dy \\ &= 2\pi \lim_{\alpha\to+0} \int_0^\infty \sum_{n=1}^\infty \left(-48n^2y^{2-\theta} + 132\pi n^4y^{4-\theta} - 68\pi^2n^6y^{6-\theta} + 8\pi^3n^8y^{8-\theta}\right) e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)} dy. \end{aligned} \tag{10.81}$$

For the integral of $R_{[4]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi}n},$$

one more time.

$$\begin{aligned} R_{[4]}(\theta) &= 2\pi \lim_{\alpha\to+0} \int_0^\infty \sum_{n=1}^\infty \begin{pmatrix} -48n^2\left(\frac{x}{\sqrt{\pi}n}\right)^{2-\theta} + 132\pi n^4\left(\frac{x}{\sqrt{\pi}n}\right)^{4-\theta} \\ -68\pi^2 n^6\left(\frac{x}{\sqrt{\pi}n}\right)^{6-\theta} + 8\pi^3 n^8\left(\frac{x}{\sqrt{\pi}n}\right)^{8-\theta} \end{pmatrix} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi}n} dx \\ &= 2\pi^{-\frac{1-\theta}{2}} \lim_{\alpha\to+0} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(-48x^{2-\theta} + 132x^{4-\theta} - 68x^{6-\theta} + 8x^{8-\theta}\right) e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx. \end{aligned} \tag{10.82}$$

Using the left-sided limit of the positive real variable α,

$$R_{[4]}(\theta) = 2\pi^{-\frac{1-\theta}{2}} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(-48x^{2-\theta} + 132x^{4-\theta} - 68x^{6-\theta} + 8x^{8-\theta}\right) e^{-x^2} dx. \tag{10.83}$$

For the integral of $R_{[4]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$\begin{aligned} R_{[4]}(\theta) &= 2\pi^{-\frac{1-\theta}{2}} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(-48y^{\frac{2-\theta}{2}} + 132y^{\frac{4-\theta}{2}} - 68y^{\frac{6-\theta}{2}} + 8y^{\frac{8-\theta}{2}}\right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\ &= 2\pi^{-\frac{1-\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \begin{pmatrix} -24\int_0^\infty y^{\frac{1-\theta}{2}} e^{-y} dy + 66\int_0^\infty y^{\frac{3-\theta}{2}} e^{-y} dy \\ -34\int_0^\infty y^{\frac{5-\theta}{2}} e^{-y} dy + 4\int_0^\infty y^{\frac{7-\theta}{2}} e^{-y} dy \end{pmatrix} \\ &= 2\pi^{-\frac{1-\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \begin{pmatrix} -24\int_0^\infty y^{1+\frac{1-\theta}{2}-1} e^{-y} dy + 66\int_0^\infty y^{2+\frac{1-\theta}{2}-1} e^{-y} dy \\ -34\int_0^\infty y^{3+\frac{1-\theta}{2}-1} e^{-y} dy + 4\int_0^\infty y^{4+\frac{1-\theta}{2}-1} e^{-y} dy \end{pmatrix}. \end{aligned} \tag{10.84}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta) < 0$ can be added to the above result because it is a convergent function under the condition.

$$R_{[4]}(\theta) = 2\pi^{-\frac{1-\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \begin{pmatrix} -24\Gamma\left(1+\frac{1-\theta}{2}\right) + 66\Gamma\left(2+\frac{1-\theta}{2}\right) \\ -34\Gamma\left(3+\frac{1-\theta}{2}\right) + 4\Gamma\left(4+\frac{1-\theta}{2}\right) \end{pmatrix}, \quad \mathrm{Re}(\theta) < 0. \tag{10.85}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$R_{[4]}(\theta) = 2\pi^{-\frac{1-\theta}{2}} \zeta(1-\theta) \begin{pmatrix} -24\Gamma\left(1+\frac{1-\theta}{2}\right) + 66\Gamma\left(2+\frac{1-\theta}{2}\right) \\ -34\Gamma\left(3+\frac{1-\theta}{2}\right) + 4\Gamma\left(4+\frac{1-\theta}{2}\right) \end{pmatrix}, \quad \theta \in \mathbb{C}. \tag{10.86}$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned} R_{[4]}(\theta) &= 2\pi^{-\frac{1-\theta}{2}} \zeta(1-\theta) \begin{pmatrix} -24\left(\frac{1-\theta}{2}\right) + 66\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \\ -34\left(2+\frac{1-\theta}{2}\right)\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \\ +4\left(3+\frac{1-\theta}{2}\right)\left(2+\frac{1-\theta}{2}\right)\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \end{pmatrix} \Gamma\left(\frac{1-\theta}{2}\right) \\ &= 2\pi^{-\frac{1-\theta}{2}} \zeta(1-\theta) \cdot \left(\frac{\theta}{4}\right)(\theta-1)(\theta+1)^2 \cdot \Gamma\left(\frac{1-\theta}{2}\right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{10.87}$$

Therefore,

$$R_{[4]}(\theta) = \frac{\theta(\theta-1)(\theta+1)^2}{2} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta), \quad \theta \in \mathbb{C}. \tag{10.88}$$

The function $R_{[4]}(\theta)$ is also a convergent function in the whole complex plane.
Because of the convergent equations (10.80) and (10.88), the following functional equation holds true:

$$L_{[4]}(\theta) = R_{[4]}(\theta), \quad \theta \in \mathbb{C}. \tag{10.89}$$

Therefore

$$\frac{\theta(\theta-1)(\theta+1)^2}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\theta(\theta-1)(\theta+1)^2}{2} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta), \quad \theta \in \mathbb{C}. \tag{10.90}$$

I decide that the functional equation (10.90) is called the fourth-order $\mathrm{I_c}$ type functional equation.
Additionally, it is equivalent to the Riemann Xi function's functional equation.

10.4 Functional Equation Transformation for the fourth-order $\mathrm{II_r}$ type functional equation

The operations of the FET for the fourth-order $\mathrm{II_r}$ type functional equation are shown as follows:

$$\begin{aligned} A_{[4]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty a_{[4]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\ &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-3n^2x^{\theta+1}+42\pi n^4x^{\theta+3}-44\pi^2n^6x^{\theta+5}+8\pi^3n^8x^{\theta+7}\right)e^{-\pi\left(n^2+\alpha^2\right)x^2}dx. \end{aligned} \tag{10.91}$$

$$\begin{aligned} F_{[4]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty f_{[4]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\ &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\begin{pmatrix}-6n^2x^{\theta+1}+84\pi n^4x^{\theta+3}\\ -88\pi^2n^6x^{\theta+5}+16\pi^3n^8x^{\theta+7}\end{pmatrix}e^{-\pi\left(\left(n^2+\alpha^2\right)x^2+\frac{m^2}{x^2}\right)}dx. \end{aligned} \tag{10.92}$$

$$\begin{aligned} B_{[4]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty b_{[4]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\ &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-48n^2x^{\theta-3}+132\pi n^4x^{\theta-5}-68\pi^2n^6x^{\theta-7}+8\pi^3n^8x^{\theta-9}\right)e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx. \end{aligned} \tag{10.93}$$

$$\begin{aligned} G_{[4]}(\theta) &:= \lim_{\alpha\to+0}\int_0^\infty g_{[4]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx \\ &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\begin{pmatrix}-96n^2x^{\theta-3}+264\pi n^4x^{\theta-5}\\ -136\pi^2n^6x^{\theta-7}+16\pi^3n^8x^{\theta-9}\end{pmatrix}e^{-\pi\left(\left(m^2+\alpha^2\right)x^2+\frac{n^2}{x^2}\right)}dx. \end{aligned} \tag{10.94}$$

Immediately, the following obvious result is obtained:

$$A_{[4]}(\theta)=L_{[4]}(\theta)=\frac{\theta(\theta-1)(\theta+1)^2}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta),\quad \theta\in\mathbb{C}. \tag{10.95}$$

For the integral of $B_{[4]}(\theta)$, I perform the variable transformation $x=y^{-1}$.

$$\begin{aligned} B_{[4]}(\theta) &= 2\pi\lim_{\alpha\to+0}\int_\infty^0\sum_{n=1}^\infty\begin{pmatrix}-48n^2\left(\frac{1}{y}\right)^{\theta-3}+132\pi n^4\left(\frac{1}{y}\right)^{\theta-5}\\ -68\pi^2n^6\left(\frac{1}{y}\right)^{\theta-7}+8\pi^3n^8\left(\frac{1}{y}\right)^{\theta-9}\end{pmatrix}e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}\left(-y^{-2}\right)dy \\ &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\left(-48n^2y^{1-\theta}+132\pi n^4y^{3-\theta}-68\pi^2n^6y^{5-\theta}+8\pi^3n^8y^{7-\theta}\right)e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}dy. \end{aligned} \tag{10.96}$$

For the integral of $B_{[4]}(\theta)$, I perform the variable transformation

$$y=\frac{x}{\sqrt{\pi}n},$$

one more time.

$$\begin{aligned} B_{[4]}(\theta) &= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\begin{pmatrix}-48n^2\left(\frac{x}{\sqrt{\pi}n}\right)^{1-\theta}+132\pi n^4\left(\frac{x}{\sqrt{\pi}n}\right)^{3-\theta}\\ -68\pi^2n^6\left(\frac{x}{\sqrt{\pi}n}\right)^{5-\theta}+8\pi^3n^8\left(\frac{x}{\sqrt{\pi}n}\right)^{7-\theta}\end{pmatrix}e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)}\frac{1}{\sqrt{\pi}n}dx \\ &= 2\pi^{\frac{\theta}{2}}\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\frac{1}{n^{-\theta}}\left(-48x^{1-\theta}+132x^{3-\theta}-68x^{5-\theta}+8x^{7-\theta}\right)e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)}dx. \end{aligned} \tag{10.97}$$

Using the left-sided limit of the positive real variable α,

$$B_{[4]}(\theta)=2\pi^{\frac{\theta}{2}}\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\frac{1}{n^{-\theta}}\left(-48x^{1-\theta}+132x^{3-\theta}-68x^{5-\theta}+8x^{7-\theta}\right)e^{-x^2}dx. \tag{10.98}$$

For the integral of $B_{[4]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$(10.99)\quad \begin{aligned} B_{[4]}(\theta) &= 2\pi^{\frac{\theta}{2}} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(-48y^{\frac{1-\theta}{2}} + 132y^{\frac{3-\theta}{2}} - 68y^{\frac{5-\theta}{2}} + 8y^{\frac{7-\theta}{2}}\right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\ &= 2\pi^{\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(\begin{array}{c} -24\int_0^\infty y^{\frac{-\theta}{2}} e^{-y} dy + 66\int_0^\infty y^{\frac{2-\theta}{2}} e^{-y} dy \\ -34\int_0^\infty y^{\frac{4-\theta}{2}} e^{-y} dy + 4\int_0^\infty y^{\frac{6-\theta}{2}} e^{-y} dy \end{array} \right) \\ &= 2\pi^{\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(\begin{array}{c} -24\int_0^\infty y^{1+\frac{-\theta}{2}-1} e^{-y} dy + 66\int_0^\infty y^{2+\frac{-\theta}{2}-1} e^{-y} dy \\ -34\int_0^\infty y^{3+\frac{-\theta}{2}-1} e^{-y} dy + 4\int_0^\infty y^{4+\frac{-\theta}{2}-1} e^{-y} dy \end{array} \right). \end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta) < -1$ can be added to the above result because it is a convergent function under the condition.

$$(10.100)\quad B_{[4]}(\theta) = 2\pi^{\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(\begin{array}{c} -24\Gamma\left(1+\frac{-\theta}{2}\right) + 66\Gamma\left(2+\frac{-\theta}{2}\right) \\ -34\Gamma\left(3+\frac{-\theta}{2}\right) + 4\Gamma\left(4+\frac{-\theta}{2}\right) \end{array} \right), \quad \mathrm{Re}(\theta) < -1.$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10.101)\quad B_{[4]}(\theta) = 2\pi^{\frac{\theta}{2}} \zeta(-\theta) \left(\begin{array}{c} -24\Gamma\left(1+\frac{-\theta}{2}\right) + 66\Gamma\left(2+\frac{-\theta}{2}\right) \\ -34\Gamma\left(3+\frac{-\theta}{2}\right) + 4\Gamma\left(4+\frac{-\theta}{2}\right) \end{array} \right), \quad \theta \in \mathbb{C}.$$

The difference formula for the gamma function is applied to the above result.

$$(10.102)\quad \begin{aligned} B_{[4]}(\theta) &= 2\pi^{\frac{\theta}{2}} \zeta(-\theta) \left(\begin{array}{c} -24\left(\frac{-\theta}{2}\right) + 66\left(1+\frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \\ -34\left(2+\frac{-\theta}{2}\right)\left(1+\frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \\ +4\left(3+\frac{-\theta}{2}\right)\left(2+\frac{-\theta}{2}\right)\left(1+\frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \end{array} \right) \Gamma\left(\frac{-\theta}{2}\right) \\ &= 2\pi^{\frac{\theta}{2}} \zeta(-\theta) \cdot \left(\frac{\theta}{4}\right)(\theta+1)(\theta+2)^2 \cdot \Gamma\left(-\frac{\theta}{2}\right), \quad \theta \in \mathbb{C}. \end{aligned}$$

Combining the relation of the complete symmetric functional equation to the result,

$$(10.103)\quad \begin{aligned} B_{[4]}(\theta) &= \frac{\theta(\theta+1)(\theta+2)^2}{2} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) \\ &= \frac{\theta(\theta+1)(\theta+2)^2}{2} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta), \ \theta \in \mathbb{C}. \end{aligned}$$

The function $B_{[4]}(\theta)$ is also a convergent function in the whole complex plane.
The left-sided limit of the positive real variable α, is used for the defining equation (10.92),

$$(10.104)\quad F_{[4]}(\theta) = 2\pi \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty \left(-6n^2 x^{\theta+1} + 84\pi n^4 x^{\theta+3} - 88\pi^2 n^6 x^{\theta+5} + 16\pi^3 n^8 x^{\theta+7}\right) e^{-\pi\left(n^2 x^2 + \frac{m^2}{x^2}\right)} dx.$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $F_{[4]}(\theta)$, I perform the variable transformation,

$$x = \left(\frac{my}{n}\right)^{\frac{1}{2}}.$$

$$F_{[4]}(\theta) = 2\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty} \begin{pmatrix} -6n^2\left(\frac{my}{n}\right)^{\frac{\theta+1}{2}} + 84\pi n^4\left(\frac{my}{n}\right)^{\frac{\theta+3}{2}} \\ -88\pi^2 n^6\left(\frac{my}{n}\right)^{\frac{\theta+5}{2}} + 16\pi^2 n^6\left(\frac{my}{n}\right)^{\frac{\theta+5}{2}} \end{pmatrix} e^{-\pi mn\left(y+\frac{1}{y}\right)}\frac{1}{2}\left(\frac{m}{n}\right)^{\frac{1}{2}} y^{-\frac{1}{2}}dy$$

(10.105)

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\cdot\frac{1}{2}\int_0^{\infty} y^{\frac{\theta+2}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy \\ +42(\pi mn)\cdot\frac{1}{2}\int_0^{\infty} y^{\frac{\theta+4}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy \\ -44(\pi mn)^2\cdot\frac{1}{2}\int_0^{\infty} y^{\frac{\theta+6}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy \\ +8(\pi mn)^3\cdot\frac{1}{2}\int_0^{\infty} y^{\frac{\theta+8}{2}-1}e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)}dy \end{pmatrix}.$$

The integrals can be written using the modified Bessel functions of the second kind.

(10.106)
$$F_{[4]}(\theta) = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 42(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ -44(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) + 8(\pi mn)^3\,\mathrm{K}_{\frac{\theta+8}{2}}(2\pi mn) \end{pmatrix}.$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.
The recurrence formula for the modified Bessel function of the second kind is applied four times.
(10.107)

$$F_{[4]}(\theta)$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 42(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) - 44(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) \\ +8(\pi mn)^2\left(\frac{\theta+6}{2}\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)\right) \end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 2\left(21+4(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ +4(\theta-5)(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) \end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 2\left(21+4(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ +4(\theta-5)(\pi mn)\left(\frac{\theta+4}{2}\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)\right) \end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} \left(-3+4(\theta-5)(\pi mn)^2\right)\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) \\ +2\left(\theta^2-\theta+1+4(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} \left(-3+4(\theta-5)(\pi mn)^2\right)\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) \\ +2\left(\theta^2-\theta+1+4(\pi mn)^2\right)\left(\frac{\theta+2}{2}\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right) \end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} 2\left(\theta^2-\theta+1+4(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ +\left((\theta-1)(\theta+1)^2+4(2\theta-3)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) \end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} 2\left(\theta^2-\theta+1+4(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ +\left((\theta-1)(\theta+1)^2+4(2\theta-3)(\pi mn)^2\right)\left(\frac{\theta}{2}\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn)\right) \end{pmatrix}$$

$$= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} \left(\theta(\theta-1)(\theta+1)^2+4\left((\theta-1)(3\theta-1)+4(\pi mn)^2\right)(\pi mn)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ +2\left((\theta-1)(\theta+1)^2+4(2\theta-3)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn) \end{pmatrix}.$$

Based on the result of equation (10.107), from equations (3.30) and (5.37),

(10.108)
$$F_{[4]}(\theta) = 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}} \begin{pmatrix} \left(\theta(\theta-1)(\theta+1)^2+\left((\theta-1)(3\theta-1)+(2\pi p)^2\right)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ +\left((\theta-1)(\theta+1)^2+(2\theta-3)(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \end{pmatrix}.$$

The left-sided limit of the positive real variable α, is used for the defining equation (10.94),

(10.109)
$$G_{[4]}(\theta) = 2\pi \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty \begin{pmatrix} -96n^2x^{\theta-3} + 264\pi n^4 x^{\theta-5} \\ -136\pi^2 n^6 x^{\theta-7} + 16\pi^3 n^8 x^{\theta-9} \end{pmatrix} e^{-\pi\left(m^2x^2+\frac{n^2}{x^2}\right)} dx.$$

For the integrals of $G_{[4]}(\theta)$, I perform the variable transformation $x = y^{-1}$.
(10.110)
$$G_{[4]}(\theta) = 2\pi \int_\infty^0 \sum_{n=1}^\infty \sum_{m=1}^\infty \begin{pmatrix} -96n^2\left(\frac{1}{y}\right)^{\theta-3} + 264\pi n^4 \left(\frac{1}{y}\right)^{\theta-5} \\ -136\pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-7} + 16\pi^3 n^8 \left(\frac{1}{y}\right)^{\theta-9} \end{pmatrix} e^{-\pi\left(\frac{m^2}{y^2}+n^2y^2\right)} \left(-y^{-2}\right) dy$$
$$= 2\pi \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty \left(-96n^2y^{1-\theta} + 264\pi n^4 y^{3-\theta} - 136\pi^2 n^6 y^{5-\theta} + 16\pi^3 n^8 y^{7-\theta}\right) e^{-\pi\left(n^2y^2+\frac{m^2}{y^2}\right)} dy.$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $G_{[4]}(\theta)$, I perform the variable transformation
$$y = \left(\frac{mx}{n}\right)^{\frac{1}{2}}.$$

(10.111)
$$G_{[4]}(\theta) = 2\pi \sum_{n=1}^\infty \sum_{m=1}^\infty \int_0^\infty \begin{pmatrix} -96n^2\left(\frac{mx}{n}\right)^{\frac{1-\theta}{2}} + 264\pi n^4 \left(\frac{mx}{n}\right)^{\frac{3-\theta}{2}} \\ -136\pi^2 n^6 \left(\frac{mx}{n}\right)^{\frac{5-\theta}{2}} + 16\pi^3 n^8 \left(\frac{mx}{n}\right)^{\frac{7-\theta}{2}} \end{pmatrix} e^{-\pi mn\left(x+\frac{1}{x}\right)} \frac{1}{2}\left(\frac{m}{n}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx$$
$$= 16 \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} -12 \cdot \frac{1}{2}\int_0^\infty x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \\ +33(\pi mn)\cdot \frac{1}{2}\int_0^\infty x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \\ -17(\pi mn)^2\cdot \frac{1}{2}\int_0^\infty x^{\frac{6-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \\ +2(\pi mn)^3\cdot \frac{1}{2}\int_0^\infty x^{\frac{8-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \end{pmatrix}.$$

The integrals can be written using the modified Bessel functions of the second kind.

(10.112)
$$G_{[4]}(\theta) = 16 \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} -12\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) + 33(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) \\ -17(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn) + 2(\pi mn)^3\,\mathrm{K}_{\frac{8-\theta}{2}}(2\pi mn) \end{pmatrix}.$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied three times.
(10.113)
$$G_{[4]}(\theta)$$
$$= 16 \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} -12\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) + 33(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) - 17(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn) \\ +2(\pi mn)^2\left(\frac{6-\theta}{2}\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\right) \end{pmatrix}$$
$$= 16 \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} -12\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) + \left(33 + 2(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) \\ -(\theta+11)(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn) \end{pmatrix}$$
$$= 16 \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} -12\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) + \left(33 + 2(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) \\ -(\theta+11)(\pi mn)\left(\frac{4-\theta}{2}\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\right) \end{pmatrix}$$

$$= 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix} -2\left(12+(\theta+11)(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) \\ +\left(\theta^2+7\theta+22+4(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\end{pmatrix}$$

$$= 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix} -2\left(12+(\theta+11)(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) \\ +\left(\theta^2+7\theta+22+4(\pi mn)^2\right)\left(\frac{2-\theta}{2}\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{-\frac{\theta}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix} 2\left(\theta^2+7\theta+22+4(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ -\left((\theta+1)(\theta+2)^2+4(2\theta+9)(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\end{pmatrix}$$

$$= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix} \left(\theta^2+7\theta+22+(2\pi mn)^2\right)(2\pi mn)^2\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ -\left((\theta+1)(\theta+2)^2+(2\theta+9)(2\pi mn)^2\right)(2\pi mn)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\end{pmatrix}.$$

Based on the result of equation (10.113), from equation (5.38),

$$G_{[4]}(\theta) = 2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix} \left(\theta^2+7\theta+22+(2\pi p)^2\right)(2\pi p)^2\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ -\left((\theta+1)(\theta+2)^2+(2\theta+9)(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{pmatrix}. \tag{10.114}$$

I define the function $H_{[4]}(\theta)$ as follows:

$$H_{[4]}(\theta) := G_{[4]}(\theta) - F_{[4]}(\theta). \tag{10.115}$$

From (the right side of equation (10.114)) − (the right side of equation (10.108)), I obtain the above result, which will be shown soon. Additionally, since the procedure is the same as the one used in the previous subsection, I may confirm the absolute convergence of the function $H_{[4]}(\theta)$ without a proof. Combining the above results, the following functional equation holds true:

$$A_{[4]}(\theta) = B_{[4]}(\theta) + H_{[4]}(\theta), \quad \theta\in\mathbb{C}. \tag{10.116}$$

Therefore

$$\begin{aligned}\frac{\theta(\theta-1)(\theta+1)^2}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta) &= \frac{\theta(\theta+1)(\theta+2)^2}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta)+H_{[4]}(\theta)\\ &= \frac{\theta(\theta+1)(\theta+2)^2}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)+H_{[4]}(\theta), \quad \theta\in\mathbb{C}.\end{aligned} \tag{10.117}$$

Where

$$H_{[4]}(\theta) = -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix} \left(\theta(\theta-1)(\theta+1)^2+(2\theta+3)(\theta-7)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ \left((\theta+1)\left(2\theta^2+4\theta+3\right)+2(2\theta+3)(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{pmatrix}. \tag{10.118}$$

The functional equation (10.117) is called the fourth-order $\mathrm{II_c}$ type functional equation

10.5 Functional Equation Transformation for the Fifth-Order $\mathrm{I_r}$ Type Functional Equation

The following are the FET operations for the fifth-order $\mathrm{I_r}$ type functional equation:

$$\begin{aligned}L_{[5]}(\theta) &:= \lim_{\alpha\to+0}\int_0^{\infty} l_{[5]}(x)\,x^{\theta}e^{-\pi\alpha^2x^2}dx\\ &= 2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} -3n^2x^{\theta+1}+72\pi n^4x^{\theta+3}-224\pi^2n^6x^{\theta+5} \\ +128\pi^3n^8x^{\theta+7}-16\pi^4n^{10}x^{\theta+9}\end{pmatrix}e^{-\pi\left(n^2+\alpha^2\right)x^2}dx.\end{aligned} \tag{10.119}$$

$$\begin{aligned}R_{[5]}(\theta) &:= \lim_{\alpha\to+0}\int_0^{\infty} r_{[5]}(x)\,x^{\theta}e^{-\pi\alpha^2x^2}dx\\ &= 2\pi\lim_{\alpha\to+0}\int_0^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} 312n^2x^{\theta-4}-1188\pi n^4x^{\theta-6}+952\pi^2n^6x^{\theta-8} \\ -232\pi^3n^8x^{\theta-10}+16\pi^4n^{10}x^{\theta-12}\end{pmatrix}e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx.\end{aligned} \tag{10.120}$$

For the integral of $L_{[5]}(\theta)$, I perform the variable transformation

$$x=\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}.$$

$$\text{(10.121)}\qquad L_{[5]}(\theta)=2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\begin{pmatrix}-3n^2\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+1}{2}}\\+72\pi n^4\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+3}{2}}\\-224\pi^2n^6\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+5}{2}}\\+128\pi^3n^8\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+7}{2}}\\-16\pi^4n^{10}\left(\frac{y}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{\theta+9}{2}}\end{pmatrix}e^{-y}\frac{1}{2}\left(\frac{1}{\pi\left(n^2+\alpha^2\right)}\right)^{\frac{1}{2}}y^{-\frac{1}{2}}dy.$$

Using the left-sided limit of the positive real variable α,

$$\text{(10.122)}\qquad \begin{aligned}L_{[5]}(\theta)&=\pi^{-\frac{\theta}{2}}\int_0^\infty\sum_{n=1}^\infty\frac{1}{n^\theta}\left(-3y^{\frac{\theta}{2}}+72y^{\frac{\theta}{2}+1}-224y^{\frac{\theta}{2}+2}+128y^{\frac{\theta}{2}+3}-16y^{\frac{\theta}{2}+4}\right)e^{-y}dy\\&=\pi^{-\frac{\theta}{2}}\sum_{n=1}^\infty\frac{1}{n^\theta}\begin{pmatrix}-3\int_0^\infty y^{1+\frac{\theta}{2}-1}e^{-y}dy+72\int_0^\infty y^{2+\frac{\theta}{2}-1}e^{-y}dy-224\int_0^\infty y^{3+\frac{\theta}{2}-1}e^{-y}dy\\+128\int_0^\infty y^{4+\frac{\theta}{2}-1}e^{-y}dy-16\int_0^\infty y^{5+\frac{\theta}{2}-1}e^{-y}dy\end{pmatrix}.\end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta)>1$ can be added to the above result because it is a convergent function under the condition.

$$\text{(10.123)}\qquad L_{[5]}(\theta)=\pi^{-\frac{\theta}{2}}\sum_{n=1}^\infty\frac{1}{n^\theta}\begin{pmatrix}-3\Gamma\left(1+\frac{\theta}{2}\right)+72\Gamma\left(2+\frac{\theta}{2}\right)-224\Gamma\left(3+\frac{\theta}{2}\right)\\+128\Gamma\left(4+\frac{\theta}{2}\right)-16\Gamma\left(5+\frac{\theta}{2}\right)\end{pmatrix},\quad \mathrm{Re}(\theta)>1.$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$\text{(10.124)}\qquad L_{[5]}(\theta)=\pi^{-\frac{\theta}{2}}\zeta(\theta)\begin{pmatrix}-3\Gamma\left(1+\frac{\theta}{2}\right)+72\Gamma\left(2+\frac{\theta}{2}\right)-224\Gamma\left(3+\frac{\theta}{2}\right)\\+128\Gamma\left(4+\frac{\theta}{2}\right)-16\Gamma\left(5+\frac{\theta}{2}\right)\end{pmatrix},\quad \theta\in\mathbb{C}.$$

The difference formula for the gamma function is applied to the above result.

$$\text{(10.125)}\qquad \begin{aligned}L_{[5]}(\theta)&=\pi^{-\frac{\theta}{2}}\zeta(\theta)\begin{pmatrix}-3\left(\frac{\theta}{2}\right)+72\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)-224\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)\\+128\left(3+\frac{\theta}{2}\right)\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)\\-16\left(4+\frac{\theta}{2}\right)\left(3+\frac{\theta}{2}\right)\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)\end{pmatrix}\Gamma\left(\frac{\theta}{2}\right)\\&=\pi^{-\frac{\theta}{2}}\zeta(\theta)\cdot\left(-\frac{\theta}{2}\right)\left(\theta^2-1\right)\left(\theta^2+4\theta+5\right)\cdot\Gamma\left(\frac{\theta}{2}\right),\quad\theta\in\mathbb{C}.\end{aligned}$$

Therefore,

$$\text{(10.126)}\qquad L_{[5]}(\theta)=-\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+4\theta+5\right)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta),\quad\theta\in\mathbb{C}.$$

The function $L_{[5]}(\theta)$ is a convergent function in the whole complex plane.
For the integral of $R_{[5]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

(10.127)
$$R_{[5]}(\theta) = 2\pi \lim_{\alpha\to+0} \int_{\infty}^{0} \sum_{n=1}^{\infty} \begin{pmatrix} 312n^2\left(\frac{1}{y}\right)^{\theta-4} - 1188\pi n^4\left(\frac{1}{y}\right)^{\theta-6} \\ +952\pi^2 n^6\left(\frac{1}{y}\right)^{\theta-8} - 232\pi^3 n^8\left(\frac{1}{y}\right)^{\theta-10} \\ +16\pi^4 n^{10}\left(\frac{1}{y}\right)^{\theta-12} \end{pmatrix} e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}\left(-y^{-2}\right) dy$$
$$= 2\pi \lim_{\alpha\to+0} \int_{0}^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} 312n^2y^{2-\theta} - 1188\pi n^4 y^{4-\theta} + 952\pi^2 n^6 y^{6-\theta} \\ -232\pi^3 n^8 y^{8-\theta} + 16\pi^4 n^{10} y^{10-\theta} \end{pmatrix} e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)} dy.$$

For the integral of $R_{[5]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi}n},$$

one more time.
(10.128)
$$R_{[5]}(\theta) = 2\pi \lim_{\alpha\to+0} \int_{0}^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} 312n^2\left(\frac{x}{\sqrt{\pi}n}\right)^{2-\theta} - 1188\pi n^4\left(\frac{x}{\sqrt{\pi}n}\right)^{4-\theta} \\ +952\pi^2 n^6\left(\frac{x}{\sqrt{\pi}n}\right)^{6-\theta} - 232\pi^3 n^8\left(\frac{x}{\sqrt{\pi}n}\right)^{8-\theta} \\ +16\pi^4 n^{10}\left(\frac{x}{\sqrt{\pi}n}\right)^{10-\theta} \end{pmatrix} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi}n} dx$$
$$= 2\pi^{-\frac{1-\theta}{2}} \lim_{\alpha\to+0} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \begin{pmatrix} 312x^{2-\theta} - 1188x^{4-\theta} + 952x^{6-\theta} \\ -232x^{8-\theta} + 16x^{10-\theta} \end{pmatrix} e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)} dx.$$

Using the left-sided limit of the positive real variable α,

(10.129)
$$R_{[5]}(\theta) = 2\pi^{-\frac{1-\theta}{2}} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(312x^{2-\theta} - 1188x^{4-\theta} + 952x^{6-\theta} - 232x^{8-\theta} + 16x^{10-\theta}\right) e^{-x^2} dx.$$

For the integral of $R_{[5]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

(10.130)
$$R_{[5]}(\theta) = 2\pi^{-\frac{1-\theta}{2}} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \begin{pmatrix} 312y^{\frac{2-\theta}{2}} - 1188y^{\frac{4-\theta}{2}} + 952y^{\frac{6-\theta}{2}} \\ -232y^{\frac{8-\theta}{2}} + 16y^{\frac{10-\theta}{2}} \end{pmatrix} e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy$$
$$= 2\pi^{-\frac{1-\theta}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \begin{pmatrix} 156\int_0^\infty y^{\frac{1-\theta}{2}} e^{-y} dy - 594\int_0^\infty y^{\frac{3-\theta}{2}} e^{-y} dy \\ +476\int_0^\infty y^{\frac{5-\theta}{2}} e^{-y} dy - 116\int_0^\infty y^{\frac{7-\theta}{2}} e^{-y} dy \\ +8\int_0^\infty y^{\frac{9-\theta}{2}} e^{-y} dy \end{pmatrix}$$
$$= 2\pi^{-\frac{1-\theta}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \begin{pmatrix} 156\int_0^\infty y^{1+\frac{1-\theta}{2}-1} e^{-y} dy - 594\int_0^\infty y^{2+\frac{1-\theta}{2}-1} e^{-y} dy \\ +476\int_0^\infty y^{3+\frac{1-\theta}{2}-1} e^{-y} dy - 116\int_0^\infty y^{4+\frac{1-\theta}{2}-1} e^{-y} dy \\ +8\int_0^\infty y^{5+\frac{1-\theta}{2}-1} e^{-y} dy \end{pmatrix}.$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta) < 0$ can be added to the above result because it is a convergent function under the condition.

$$R_{[5]}(\theta) = 2\pi^{-\frac{1-\theta}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \begin{pmatrix} 156\,\Gamma\left(1+\frac{1-\theta}{2}\right) - 594\,\Gamma\left(2+\frac{1-\theta}{2}\right) \\ +476\,\Gamma\left(3+\frac{1-\theta}{2}\right) - 116\,\Gamma\left(4+\frac{1-\theta}{2}\right) \\ +8\,\Gamma\left(5+\frac{1-\theta}{2}\right) \end{pmatrix}, \quad \mathrm{Re}(\theta) < 0. \tag{10.131}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$R_{[5]}(\theta) = 2\pi^{-\frac{1-\theta}{2}} \zeta(1-\theta) \begin{pmatrix} 156\,\Gamma\left(1+\frac{1-\theta}{2}\right) - 594\,\Gamma\left(2+\frac{1-\theta}{2}\right) \\ +476\,\Gamma\left(3+\frac{1-\theta}{2}\right) - 116\,\Gamma\left(4+\frac{1-\theta}{2}\right) \\ +8\,\Gamma\left(5+\frac{1-\theta}{2}\right) \end{pmatrix}, \quad \theta \in \mathbb{C}. \tag{10.132}$$

The difference formula for the gamma function is applied to the above result.

(10.133)

$$\begin{aligned} R_{[5]}(\theta) &= 2\pi^{-\frac{1-\theta}{2}} \zeta(1-\theta) \begin{pmatrix} 156\left(\frac{1-\theta}{2}\right) - 594\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \\ +476\left(2+\frac{1-\theta}{2}\right)\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \\ -116\left(3+\frac{1-\theta}{2}\right)\left(2+\frac{1-\theta}{2}\right)\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \\ +8\left(4+\frac{1-\theta}{2}\right)\left(3+\frac{1-\theta}{2}\right)\left(2+\frac{1-\theta}{2}\right)\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \end{pmatrix} \Gamma\left(\frac{1-\theta}{2}\right) \\ &= 2\pi^{-\frac{1-\theta}{2}} \zeta(1-\theta) \cdot \left(-\frac{\theta}{4}\right)\left(\theta^2-1\right)\left(\theta^2+4\theta+5\right) \cdot \Gamma\left(\frac{1-\theta}{2}\right), \quad \theta \in \mathbb{C}. \end{aligned}$$

Therefore,

$$R_{[5]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+4\theta+5\right)}{2} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta), \quad \theta \in \mathbb{C}. \tag{10.134}$$

The function $R_{[5]}(\theta)$ is also a convergent function in the whole complex plane.
Because of the convergent equations (10.126) and (10.134), the following functional equation holds true:

$$L_{[5]}(\theta) = R_{[5]}(\theta), \quad \theta \in \mathbb{C}. \tag{10.135}$$

Therefore

$$\begin{aligned} &-\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+4\theta+5\right)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \\ &= -\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+4\theta+5\right)}{2} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta), \quad \theta \in \mathbb{C}. \end{aligned} \tag{10.136}$$

I decide that the functional equation (10.136) is called the fifth-order $\mathrm{I_c}$ type functional equation. Additionally, it is equivalent to the Riemann Xi function's functional equation.

10.6 Functional Equation Transformation for the fifth-order $\mathrm{II_r}$ type functional equation

The operations of the FET for the fifth-order $\mathrm{II_r}$ type functional equation are shown as follows:

$$\begin{aligned} A_{[5]}(\theta) &:= \lim_{\alpha \to +0} \int_0^{\infty} a_{[5]}(x)\, x^{\theta} e^{-\pi\alpha^2 x^2} dx \\ &= 2\pi \lim_{\alpha \to +0} \int_0^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} -3n^2 x^{\theta+1} + 72\pi n^4 x^{\theta+3} - 224\pi^2 n^6 x^{\theta+5} \\ +128\pi^3 n^8 x^{\theta+7} - 16\pi^4 n^{10} x^{\theta+9} \end{pmatrix} e^{-\pi\left(n^2+\alpha^2\right)x^2} dx. \end{aligned} \tag{10.137}$$

(10.138)
$$F_{[5]}(\theta) := \lim_{\alpha\to+0}\int_0^\infty f_{[5]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\begin{pmatrix}-6n^2x^{\theta+1}+144\pi n^4x^{\theta+3}\\ -448\pi^2n^6x^{\theta+5}+256\pi^3n^8x^{\theta+7}\\ -32\pi^4n^{10}x^{\theta+9}\end{pmatrix}e^{-\pi\left(\left(n^2+\alpha^2\right)x^2+\frac{m^2}{x^2}\right)}dx.$$

(10.139)
$$B_{[5]}(\theta) := \lim_{\alpha\to+0}\int_0^\infty b_{[5]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\begin{pmatrix}312n^2x^{\theta-3}-1188\pi n^4x^{\theta-5}+952\pi^2n^6x^{\theta-7}\\ -232\pi^3n^8x^{\theta-9}+16\pi^4n^{10}x^{\theta-11}\end{pmatrix}e^{-\pi\left(\frac{n^2}{x^2}+\alpha^2x^2\right)}dx.$$

(10.140)
$$G_{[5]}(\theta) := \lim_{\alpha\to+0}\int_0^\infty g_{[5]}(x)\,x^\theta e^{-\pi\alpha^2x^2}dx$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\sum_{m=1}^\infty\begin{pmatrix}624n^2x^{\theta-3}-2376\pi n^4x^{\theta-5}\\ +1904\pi^2n^6x^{\theta-7}-464\pi^3n^8x^{\theta-9}\\ +32\pi^4n^{10}x^{\theta-11}\end{pmatrix}e^{-\pi\left(\left(m^2+\alpha^2\right)x^2+\frac{n^2}{x^2}\right)}dx.$$

Immediately, the following obvious result is obtained:

(10.141)
$$A_{[5]}(\theta) = L_{[5]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+4\theta+5\right)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta),\quad \theta\in\mathbb{C}.$$

For the integral of $B_{[5]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

(10.142)
$$B_{[5]}(\theta) = 2\pi\lim_{\alpha\to+0}\int_\infty^0\sum_{n=1}^\infty\begin{pmatrix}312n^2\left(\frac{1}{y}\right)^{\theta-3}-1188\pi n^4\left(\frac{1}{y}\right)^{\theta-5}\\ +952\pi^2n^6\left(\frac{1}{y}\right)^{\theta-7}-232\pi^3n^8\left(\frac{1}{y}\right)^{\theta-9}\\ +16\pi^4n^{10}\left(\frac{1}{y}\right)^{\theta-11}\end{pmatrix}e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}\left(-y^{-2}\right)dy$$
$$= 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\begin{pmatrix}312n^2y^{1-\theta}-1188\pi n^4y^{3-\theta}+952\pi^2n^6y^{5-\theta}\\ -232\pi^3n^8y^{7-\theta}+16\pi^4n^{10}y^{9-\theta}\end{pmatrix}e^{-\pi\left(n^2y^2+\frac{\alpha^2}{y^2}\right)}dy.$$

For the integral of $B_{[5]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi}n},$$

one more time.

(10.143)
$$B_{[5]}(\theta) = 2\pi\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\begin{pmatrix}312n^2\left(\frac{x}{\sqrt{\pi}n}\right)^{1-\theta}-1188\pi n^4\left(\frac{x}{\sqrt{\pi}n}\right)^{3-\theta}\\ +952\pi^2n^6\left(\frac{x}{\sqrt{\pi}n}\right)^{5-\theta}-232\pi^3n^8\left(\frac{x}{\sqrt{\pi}n}\right)^{7-\theta}\\ +16\pi^4n^{10}\left(\frac{x}{\sqrt{\pi}n}\right)^{9-\theta}\end{pmatrix}e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)}\frac{1}{\sqrt{\pi}n}dx$$
$$= 2\pi^{\frac{\theta}{2}}\lim_{\alpha\to+0}\int_0^\infty\sum_{n=1}^\infty\frac{1}{n^{-\theta}}\begin{pmatrix}312x^{1-\theta}-1188x^{3-\theta}+952x^{5-\theta}\\ -232x^{7-\theta}+16x^{9-\theta}\end{pmatrix}e^{-\left(x^2+\frac{(\pi n\alpha)^2}{x^2}\right)}dx.$$

Using the left-sided limit of the positive real variable α,

$$B_{[5]}(\theta) = 2\pi^{\frac{\theta}{2}} \lim_{\alpha\to+0} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \begin{pmatrix} 312x^{1-\theta} - 1188x^{3-\theta} + 952x^{5-\theta} \\ - 232x^{7-\theta} + 16x^{9-\theta} \end{pmatrix} e^{-x^2} dx. \tag{10.144}$$

For the integral of $B_{[5]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$\begin{aligned} B_{[5]}(\theta) &= 2\pi^{\frac{\theta}{2}} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \begin{pmatrix} 312x^{1-\theta} - 1188x^{3-\theta} + 952x^{5-\theta} \\ - 232x^{7-\theta} + 16x^{9-\theta} \end{pmatrix} e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\ &= 2\pi^{\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \begin{pmatrix} 312 \int_0^\infty y^{\frac{-\theta}{2}} e^{-y} dy - 1188 \int_0^\infty y^{\frac{2-\theta}{2}} e^{-y} dy \\ + 952 \int_0^\infty y^{\frac{4-\theta}{2}} e^{-y} dy - 232 \int_0^\infty y^{\frac{6-\theta}{2}} e^{-y} dy \\ + 16 \int_0^\infty y^{\frac{8-\theta}{2}} e^{-y} dy \end{pmatrix} \\ &= 2\pi^{\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \begin{pmatrix} 312 \int_0^\infty y^{1+\frac{-\theta}{2}-1} e^{-y} dy - 1188 \int_0^\infty y^{2+\frac{-\theta}{2}-1} e^{-y} dy \\ + 952 \int_0^\infty y^{3+\frac{-\theta}{2}-1} e^{-y} dy - 232 \int_0^\infty y^{4+\frac{-\theta}{2}-1} e^{-y} dy \\ + 16 \int_0^\infty y^{5+\frac{-\theta}{2}-1} e^{-y} dy \end{pmatrix}. \end{aligned} \tag{10.145}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.
Now, the condition $\mathrm{Re}(\theta) < -1$ can be added to the above result because it is a convergent function under the condition.

$$B_{[5]}(\theta) = 2\pi^{\frac{\theta}{2}} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \begin{pmatrix} 312\,\Gamma\left(1 + \frac{-\theta}{2}\right) - 1188\,\Gamma\left(2 + \frac{-\theta}{2}\right) \\ + 952\,\Gamma\left(3 + \frac{-\theta}{2}\right) - 232\,\Gamma\left(4 + \frac{-\theta}{2}\right) \\ + 16\,\Gamma\left(5 + \frac{-\theta}{2}\right) \end{pmatrix}, \quad \mathrm{Re}(\theta) < -1. \tag{10.146}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$B_{[5]}(\theta) = 2\pi^{\frac{\theta}{2}} \zeta(-\theta) \begin{pmatrix} 312\,\Gamma\left(1 + \frac{-\theta}{2}\right) - 1188\,\Gamma\left(2 + \frac{-\theta}{2}\right) + 952\,\Gamma\left(3 + \frac{-\theta}{2}\right) \\ - 232\,\Gamma\left(4 + \frac{-\theta}{2}\right) + 16\,\Gamma\left(5 + \frac{-\theta}{2}\right) \end{pmatrix}, \quad \theta \in \mathbb{C}. \tag{10.147}$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned} B_{[5]}(\theta) &= 2\pi^{\frac{\theta}{2}} \zeta(-\theta) \begin{pmatrix} 312\left(\frac{-\theta}{2}\right) - 1188\left(1 + \frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \\ + 952\left(2 + \frac{-\theta}{2}\right)\left(1 + \frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \\ - 232\left(3 + \frac{-\theta}{2}\right)\left(2 + \frac{-\theta}{2}\right)\left(1 + \frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \\ + 16\left(4 + \frac{-\theta}{2}\right)\left(3 + \frac{-\theta}{2}\right)\left(2 + \frac{-\theta}{2}\right)\left(1 + \frac{-\theta}{2}\right)\left(\frac{-\theta}{2}\right) \end{pmatrix} \Gamma\left(\frac{-\theta}{2}\right) \\ &= 2\pi^{\frac{\theta}{2}} \zeta(-\theta) \cdot \left(-\frac{\theta}{4}\right)(\theta+1)(\theta+2)\left(\theta^2 + 6\theta + 10\right) \cdot \Gamma\left(-\frac{\theta}{2}\right), \quad \theta \in \mathbb{C}. \end{aligned} \tag{10.148}$$

Combining the relation of the complete symmetric functional equation to the result,

$$B_{[5]}(\theta) = -\frac{\theta(\theta+1)(\theta+2)\left(\theta^2+6\theta+10\right)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta)$$

(10.149)

$$= -\frac{\theta(\theta+1)(\theta+2)\left(\theta^2+6\theta+10\right)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta), \quad \theta \in \mathbb{C}.$$

The function $B_{[5]}(\theta)$ is also a convergent function in the whole complex plane.
The left-sided limit of the positive real variable α, is used for the defining equation (10.138),

(10.150)
$$F_{[5]}(\theta) = 2\pi \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty \begin{pmatrix} -6n^2x^{\theta+1} + 144\pi n^4 x^{\theta+3} - 448\pi^2 n^6 x^{\theta+5} \\ + 256\pi^3 n^8 x^{\theta+7} - 32\pi^4 n^{10} x^{\theta+9} \end{pmatrix} e^{-\pi\left(n^2x^2+\frac{m^2}{x^2}\right)} dx.$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $F_{[5]}(\theta)$, I perform the variable transformation,

$$x = \left(\frac{my}{n}\right)^{\frac{1}{2}}.$$

$$F_{[5]}(\theta) = 2\pi \sum_{n=1}^\infty \sum_{m=1}^\infty \int_0^\infty \begin{pmatrix} -6n^2\left(\frac{my}{n}\right)^{\frac{\theta+1}{2}} + 144\pi n^4\left(\frac{my}{n}\right)^{\frac{\theta+3}{2}} \\ -448\pi^2 n^6\left(\frac{my}{n}\right)^{\frac{\theta+5}{2}} + 256\pi^3 n^8\left(\frac{my}{n}\right)^{\frac{\theta+7}{2}} \\ -32\pi^4 n^{10}\left(\frac{my}{n}\right)^{\frac{\theta+9}{2}} \end{pmatrix} e^{-\pi mn\left(y+\frac{1}{y}\right)} \frac{1}{2}\left(\frac{m}{n}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy$$

(10.151)

$$= 4\sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+2}{2}-1} e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)} dy \\ +72(\pi mn)\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+4}{2}-1} e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)} dy \\ -224(\pi mn)^2\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+6}{2}-1} e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)} dy \\ +128(\pi mn)^3\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+8}{2}-1} e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)} dy \\ -16(\pi mn)^4\cdot\frac{1}{2}\int_0^\infty y^{\frac{\theta+10}{2}-1} e^{-\frac{2\pi mn}{2}\left(y+\frac{1}{y}\right)} dy \end{pmatrix}.$$

The integrals can be written using the modified Bessel functions of the second kind.

(10.152)
$$F_{[5]}(\theta) = 4\sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 72(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ -224(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) + 128(\pi mn)^3\,\mathrm{K}_{\frac{\theta+8}{2}}(2\pi mn) \\ -16(\pi mn)^4\,\mathrm{K}_{\frac{\theta+10}{2}}(2\pi mn) \end{pmatrix}.$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied five times.
(10.153)

$$F_{[5]}(\theta) = 4\sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 72(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ -224(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) + 128(\pi mn)^3\,\mathrm{K}_{\frac{\theta+8}{2}}(2\pi mn) \\ -16(\pi mn)^3\left(\left(\frac{\theta+8}{2}\right)\mathrm{K}_{\frac{\theta+8}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn)\right) \end{pmatrix}$$

$$= 4\sum_{n=1}^\infty \sum_{m=1}^\infty (\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}} \begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 72(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ -16\left(14+(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) \\ -8(\theta-8)(\pi mn)^3\,\mathrm{K}_{\frac{\theta+8}{2}}(2\pi mn) \end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + 72(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ -16\left(14+(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) \\ -8(\theta-8)(\pi mn)^2\left(\left(\frac{\theta+6}{2}\right)\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) - 8\left(-9+(\theta-8)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ -4\left(\theta^2-2\theta+8+4(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{\theta+6}{2}}(2\pi mn)\end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} -3\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) - 8\left(-9+(\theta-8)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) \\ -4\left(\theta^2-2\theta+8+4(\pi mn)^2\right)(\pi mn) \\ \times\left(\left(\frac{\theta+4}{2}\right)\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= -4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} \left(3+4\left(\theta^2-2\theta+8+4(\pi mn)^2\right)(\pi mn)^2\right)\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) \\ +2\left(\theta^3+2\theta^2-4+8(\theta-2)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta+4}{2}}(2\pi mn)\end{pmatrix}$$

$$= -4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} \left(3+4\left(\theta^2-2\theta+8+4(\pi mn)^2\right)(\pi mn)^2\right)\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) \\ +2\left(\theta^3+2\theta^2-4+8(\theta-2)(\pi mn)^2\right) \\ \times\left(\left(\frac{\theta+2}{2}\right)\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} 4\left(\theta^3+2\theta^2-4+8(\theta-2)(\pi mn)^2\right)(\pi mn)^2\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ +2\left(\theta^4+4\theta^3+4\theta^2-4\theta-5+4\left(3\theta^2-2\theta+4(\pi mn)^2\right)(\pi mn)^2\right) \\ \times(\pi mn)\,\mathrm{K}_{\frac{\theta+2}{2}}(2\pi mn)\end{pmatrix}$$

$$= -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} 4\left(\theta^3+2\theta^2-4+8(\theta-2)(\pi mn)^2\right)(\pi mn)^2\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ +2\left(\theta^4+4\theta^3+4\theta^2-4\theta-5+4\left(3\theta^2-2\theta+4(\pi mn)^2\right)(\pi mn)^2\right) \\ \times\left(\frac{\theta}{2}\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\theta}{2}}\begin{pmatrix} \left(\theta^5+4\theta^4+4\theta^3-4\theta^2-5\theta+4\left(4\theta^3-4+4(3\theta-4)(\pi mn)^2\right)(\pi mn)^2\right) \\ \times\mathrm{K}_{\frac{\theta}{2}}(2\pi mn) \\ +2\left(\theta^4+4\theta^3+4\theta^2-4\theta-5+4\left(3\theta^2-2\theta+4(\pi mn)^2\right)(\pi mn)^2\right) \\ \times(\pi mn)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi mn)\end{pmatrix}.$$

Based on the result of equation (10.153), from equations (3.30) and (5.37),
(10.154)

$$F_{[5]}(\theta) = -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix} \left(\theta^5+4\theta^4+4\theta^3-4\theta^2-5\theta+\left(4\theta^3-4+(3\theta-4)(2\pi p)^2\right)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ +\left(\theta^4+4\theta^3+4\theta^2-4\theta-5+\left(3\theta^2-2\theta+(2\pi p)^2\right)(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{pmatrix}.$$

The left-sided limit of the positive real variable α, is used for the defining equation (10.140),

$$G_{[5]}(\theta) = 2\pi\int_0^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\begin{pmatrix} 624n^2x^{\theta-3} - 2376\pi n^4x^{\theta-5} \\ +1904\pi^2n^6x^{\theta-7} - 464\pi^3n^8x^{\theta-9} \\ +32\pi^4n^{10}x^{\theta-11}\end{pmatrix} e^{-\pi\left(m^2x^2+\frac{n^2}{x^2}\right)}dx. \tag{10.155}$$

For the integral of $G_{[5]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

(10.156)

$$G_{[5]}(\theta) = 2\pi \int_{\infty}^{0} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{pmatrix} 624n^2\left(\frac{1}{y}\right)^{\theta-3} - 2376\pi n^4\left(\frac{1}{y}\right)^{\theta-5} \\ + 1904\pi^2 n^6\left(\frac{1}{y}\right)^{\theta-7} - 464\pi^3 n^8\left(\frac{1}{y}\right)^{\theta-9} \\ + 32\pi^4 n^{10}\left(\frac{1}{y}\right)^{\theta-11} \end{pmatrix} e^{-\pi\left(\frac{m^2}{y^2}+n^2y^2\right)} \left(-y^{-2}\right) dy$$

$$= 2\pi \int_{0}^{\infty} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{pmatrix} 624n^2y^{1-\theta} - 2376\pi n^4 y^{3-\theta} + 1904\pi^2 n^6 y^{5-\theta} \\ - 464\pi^3 n^8 y^{7-\theta} + 32\pi^4 n^{10} y^{9-\theta} \end{pmatrix} e^{-\pi\left(n^2y^2+\frac{m^2}{y^2}\right)} dy.$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $G_{[5]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{mx}{n}\right)^{\frac{1}{2}}.$$

(10.157)

$$G_{[5]}(\theta) = 2\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \int_{0}^{\infty} \begin{pmatrix} 624n^2\left(\frac{mx}{n}\right)^{\frac{1-\theta}{2}} - 2376\pi n^4\left(\frac{mx}{n}\right)^{\frac{3-\theta}{2}} \\ + 1904\pi^2 n^6\left(\frac{mx}{n}\right)^{\frac{5-\theta}{2}} - 464\pi^3 n^8\left(\frac{mx}{n}\right)^{\frac{7-\theta}{2}} \\ + 32\pi^4 n^{10}\left(\frac{mx}{n}\right)^{\frac{9-\theta}{2}} \end{pmatrix} e^{-\pi mn\left(x+\frac{1}{x}\right)} \frac{1}{2}\left(\frac{m}{n}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx$$

$$= 16 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} 78\cdot\frac{1}{2}\int_0^\infty x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \\ - 297(\pi mn)\cdot\frac{1}{2}\int_0^\infty x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \\ + 238(\pi mn)^2\cdot\frac{1}{2}\int_0^\infty x^{\frac{6-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \\ - 58(\pi mn)^3\cdot\frac{1}{2}\int_0^\infty x^{\frac{8-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \\ + 4(\pi mn)^4\cdot\frac{1}{2}\int_0^\infty x^{\frac{10-\theta}{2}-1} e^{-\frac{2\pi mn}{2}\left(x+\frac{1}{x}\right)} dx \end{pmatrix}.$$

The integrals can be written using the modified Bessel functions of the second kind. The integrals can be written using the modified Bessel functions of the second kind.

(10.158)

$$G_{[5]}(\theta) = 16 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} 78\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) - 297(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) \\ + 238(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn) - 58(\pi mn)^3\,\mathrm{K}_{\frac{8-\theta}{2}}(2\pi mn) \\ + 4(\pi mn)^4\,\mathrm{K}_{\frac{10-\theta}{2}}(2\pi mn) \end{pmatrix}.$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied four times.

(10.159)

$$G_{[5]}(\theta) = 16 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} 78\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) - 297(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) \\ + 238(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn) - 58(\pi mn)^3\,\mathrm{K}_{\frac{8-\theta}{2}}(2\pi mn) \\ + 4(\pi mn)^3\left(\left(\frac{8-\theta}{2}\right)\mathrm{K}_{\frac{8-\theta}{2}}(2\pi mn) + (\pi mn)\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn)\right) \end{pmatrix}$$

$$= 16 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \begin{pmatrix} 78\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) - 297(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn) \\ + \left(238 + 4(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn) \\ - 2(\theta+21)(\pi mn)^3\,\mathrm{K}_{\frac{8-\theta}{2}}(2\pi mn) \end{pmatrix}$$

$$= 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}78\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) - 297(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\\ +\left(238+4(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn)\\ -2(\theta+21)(\pi mn)^2\left(\left(\dfrac{6-\theta}{2}\right)\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}78\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) - \left(297+2(\theta+21)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\\ +\left(\theta^2+15\theta+112+4(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{6-\theta}{2}}(2\pi mn)\end{pmatrix}$$

$$= 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}78\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn) - \left(297+2(\theta+21)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\\ +\left(\theta^2+15\theta+112+4(\pi mn)^2\right)(\pi mn)\\ \times\left(\left(\dfrac{4-\theta}{2}\right)\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}\left(156+2\left(\theta^2+15\theta+112+4(\pi mn)^2\right)(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\\ -\left(\theta^3+11\theta^2+52\theta+146+4(2\theta+17)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{4-\theta}{2}}(2\pi mn)\end{pmatrix}$$

$$= 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}\left(156+2\left(\theta^2+15\theta+112+4(\pi mn)^2\right)(\pi mn)^2\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\\ -\left(\theta^3+11\theta^2+52\theta+146+4(2\theta+17)(\pi mn)^2\right)\\ \times\left(\left(\dfrac{2-\theta}{2}\right)\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)+(\pi mn)\,\mathrm{K}_{-\frac{\theta}{2}}(2\pi mn)\right)\end{pmatrix}$$

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\pi mn)\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}-2\left(\theta^3+11\theta^2+52\theta+146+4(2\theta+17)(\pi mn)^2\right)(\pi mn)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\\ +\left(\theta^4+9\theta^3+30\theta^2+42\theta+20+4\left(3\theta^2+28\theta+78+4(\pi mn)^2\right)(\pi mn)^2\right)\\ \times\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\end{pmatrix}$$

$$= -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{n}\right)^{-\frac{\theta}{2}}\begin{pmatrix}4\left(\theta^3+11\theta^2+52\theta+146+4(2\theta+17)(\pi mn)^2\right)(\pi mn)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi mn)\\ -2\left(\theta^4+9\theta^3+30\theta^2+42\theta+20+4\left(3\theta^2+28\theta+78+4(\pi mn)^2\right)(\pi mn)^2\right)\\ \times(\pi mn)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi mn)\end{pmatrix}.$$

Based on the result of equation (10.159), from equation (5.38),
(10.160)

$$G_{[5]}(\theta) = -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta^3+11\theta^2+52\theta+146+(2\theta+17)(2\pi p)^2\right)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ -\left(\theta^4+9\theta^3+30\theta^2+42\theta+20+\left(3\theta^2+28\theta+78+(2\pi p)^2\right)(2\pi p)^2\right)\\ \times(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{pmatrix}.$$

The function $H_{[5]}(\theta)$ is defined as follows:

(10.161) $$H_{[5]}(\theta) := G_{[5]}(\theta) - F_{[5]}(\theta).$$

From (the right side of equation (10.160)) − (the right side of equation (10.154)), I obtain the above result, which will be shown soon. And I may also omit a proof for the absolute convergence of the function $H_{[5]}(\theta)$. Combining the above results, the following functional equation holds true:

(10.162) $$A_{[5]}(\theta) = B_{[5]}(\theta) + H_{[5]}(\theta), \quad \theta\in\mathbb{C}.$$

Therefore

$$\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+4\theta+5\right)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)$$

(10.163) $$= \frac{\theta(\theta+1)(\theta+2)\left(\theta^2+6\theta+10\right)}{2}\pi^{\frac{\theta}{2}}\Gamma\left(-\frac{\theta}{2}\right)\zeta(-\theta) + H_{[5]}(\theta)$$

$$= \frac{\theta(\theta+1)(\theta+2)\left(\theta^2+6\theta+10\right)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)+H_{[5]}(\theta), \quad \theta\in\mathbb{C}.$$

Where
(10.164)

$$H_{[5]}(\theta)=2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(\theta^5+4\theta^4+4\theta^3-4\theta^2-5\theta \right. \\ \left. +\left(3\theta^3-11\theta^2-52\theta-150+(\theta-21)(2\pi p)^2\right)(2\pi p)^2\right) \\ \times \mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ +\left(2\theta^4+13\theta^3+34\theta^2+38\theta+15+2\left(3\theta^2+13\theta+39+(2\pi p)^2\right)(2\pi p)^2\right) \\ \times(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right).$$

The functional equation (10.163) is called the fifth-order II_c type functional equation.

11. The Second Proof of the Riemann Hypothesis

11.1. Derivation of a New Explicit Formula for the Zeta Function

The fourth-order II_c type functional equation is shown again.

(11.1) $$\frac{\theta(\theta-1)(\theta+1)^2}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)=\frac{\theta(\theta+1)(\theta+2)^2}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)+H_{[4]}(\theta), \quad \theta\in\mathbb{C}.$$

The third-order II_c type functional equation is also presented.

(11.2) $$-\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)=-\frac{\theta(\theta+1)(\theta+2)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)+H_{[3]}(\theta), \quad \theta\in\mathbb{C}.$$

From (fourth-order II_c type functional equation)+(third-order II_c type functional equation)$\times(\theta+2)$,

(11.3) $$-\frac{\theta(\theta-1)(\theta+1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)=(\theta+2)H_{[3]}(\theta)+H_{[4]}(\theta), \quad \theta\in\mathbb{C}.$$

Where

(11.4) $$H_{[3]}(\theta)=2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta(\theta-1)(\theta+1)+(\theta-7)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ +\left((2\theta+1)(\theta+1)+2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{array}\right).$$

(11.5) $$H_{[4]}(\theta)=-2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta(\theta-1)(\theta+1)^2+(2\theta+3)(\theta-7)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ \left((\theta+1)\left(2\theta^2+4\theta+3\right)+2(2\theta+3)(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right).$$

The right side of equation (11.3) is calculated to obtain the following equation:

(11.6) $$\begin{aligned}&(\theta+2)H_{[3]}(\theta)+H_{[4]}(\theta) \\ &=2(\theta+1)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ +\left((\theta-1)-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right), \quad \theta\in\mathbb{C}.\end{aligned}$$

The common factor $(\theta+1)$ is simplified to obtain the following equation:

(11.7) $$-\frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)=2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ +\left((\theta-1)-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right), \quad \theta\in\mathbb{C}.$$

Both sides of the above equation are multiplied by

$$\frac{\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)}, \quad \theta\in\mathbb{C}\backslash\{1\}$$

to obtain the new explicit formula for the zeta function.

$$\zeta(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \begin{pmatrix} \left(\theta(\theta-1) + (7-\theta)(2\pi p)^2\right) \mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ + \left((\theta-1) - 2(2\pi p)^2\right)(2\pi p)\, \mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \end{pmatrix}, \quad \theta \in \mathbb{C}\backslash\{1\}. \tag{11.8}$$

11.2. The Second Proof of the Riemann Hypothesis

Both x and y are assumed to be real numbers, and $(x + iy)$ is assumed to be a non-trivial zeta function zero. From the point symmetry of the Riemann Xi function, $(1 - x - iy)$ is also non-trivial zero of the zeta function. The real axis must be excluded from the target region because there are trivial zeros of the zeta function on it. I consider that the non-trivial zeros are determined as the solutions of the simultaneous equations comprising the following equations:

$$\begin{cases} \zeta(x+iy) = 0 & (x \in \mathbb{R}, y \in \mathbb{R}\backslash\{0\}), \\ \zeta(1-x-iy) = 0 & (x \in \mathbb{R}, y \in \mathbb{R}\backslash\{0\}). \end{cases} \tag{11.9}$$

For the explicit formula for the zeta function, $(x + iy)$ is substituted for θ.

$$\begin{aligned} \zeta(x+iy) =& \frac{2\pi^{\frac{x+iy}{2}}}{(1-x-iy)\Gamma\left(1+\frac{x+iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-(x+iy)}(p)\, p^{\frac{x+iy}{2}} \\ & \times \begin{pmatrix} \left((x+iy)(x-1+iy) + (7-x-iy)(2\pi p)^2\right) \mathrm{K}_{\frac{x+iy}{2}}(2\pi p) \\ + \left((x-1+iy) - 2(2\pi p)^2\right)(2\pi p)\, \mathrm{K}_{\frac{2-x-iy}{2}}(2\pi p) \end{pmatrix} \\ =& \, 0, \quad (x \in \mathbb{R}, y \in \mathbb{R}\backslash\{0\}). \end{aligned} \tag{11.10}$$

Similarly, for the explicit formula for the zeta function, $(1 - x - iy)$ is substituted for θ.

$$\begin{aligned} \zeta(1-x-iy) =& \frac{2\pi^{\frac{1-x-iy}{2}}}{(x+iy)\Gamma\left(1+\frac{1-x-iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-(1-x-iy)}(p)\, p^{\frac{1-x-iy}{2}} \\ & \times \begin{pmatrix} \left((1-x-iy)(-x+iy) + (6+x+iy)(2\pi p)^2\right) \mathrm{K}_{\frac{1-x-iy}{2}}(2\pi p) \\ + \left((-x-iy) - 2(2\pi p)^2\right)(2\pi p)\, \mathrm{K}_{\frac{1+x+iy}{2}}(2\pi p) \end{pmatrix} \\ =& \, 0, \quad (x \in \mathbb{R}, y \in \mathbb{R}\backslash\{0\}). \end{aligned} \tag{11.11}$$

The non-trivial zeros are determined as the solutions of the simultaneous equations comprising the equations (11.10) and (11.11). Because of the symmetric pair of the equations,

$$x = \frac{1}{2} \tag{11.12}$$

is immediately determined.

For equation (11.10), 1/2 is substituted for x.

$$\zeta\left(\frac{1}{2}+iy\right)=\frac{2\pi^{\frac{1+2iy}{4}}}{\left(\frac{1}{2}-iy\right)\Gamma\left(\frac{5+2iy}{4}\right)}\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-iy}(p)\,p^{\frac{1+2iy}{4}}$$

$$\times\left(\begin{array}{r}\left(\left(\frac{1}{2}+iy\right)\left(-\frac{1}{2}+iy\right)+\left(\frac{13}{2}-iy\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)\\ +\left(-\frac{1}{2}+iy-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{3-2iy}{4}}(2\pi p)\end{array}\right)$$

$$=0,\quad y\in\mathbb{R}\backslash\{0\}. \tag{11.13}$$

For equation (11.11), $1/2$ is substituted for x.

$$\zeta\left(\frac{1}{2}-iy\right)=\frac{2\pi^{\frac{1-2iy}{4}}}{\left(\frac{1}{2}+iy\right)\Gamma\left(\frac{5-2iy}{4}\right)}\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}+iy}(p)\,p^{\frac{1-2iy}{4}}$$

$$\times\left(\begin{array}{r}\left(\left(\frac{1}{2}-iy\right)\left(-\frac{1}{2}-iy\right)+\left(\frac{13}{2}+iy\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1-2iy}{4}}(2\pi p)\\ +\left(-\frac{1}{2}-iy-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{3+2iy}{4}}(2\pi p)\end{array}\right)$$

$$=0,\quad y\in\mathbb{R}\backslash\{0\}. \tag{11.14}$$

The equations (11.13) and (11.14) are complex conjugates of one another. And I can choose either equation (11.13) or (11.14) as the determining equation of the real variable y. On the critical line, the gamma factor of the zeta function does not take the value zero. Furthermore, the condition of $y=0$ can be excepted because there is no non-trivial zero of the zeta function on the real axis.
Therefore, the following equation gives the determining equation of the real variable y:

$$\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-iy}(p)\,p^{\frac{1+2iy}{4}}\left(\begin{array}{r}\left(-y^2-\frac{1}{4}+\left(\frac{13}{2}-iy\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)\\ +\left(-\frac{1}{2}+iy-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{3-2iy}{4}}(2\pi p)\end{array}\right)=0,\quad y\in\mathbb{R}. \tag{11.15}$$

The explicit formula for the Riemann Xi function is also calculated.

$$\xi(\theta)=\frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)$$

$$=-2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ +\left((\theta-1)-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad\theta\in\mathbb{C}. \tag{11.16}$$

The function value of the Riemann Xi function on the critical line is,

$$\xi\left(\frac{1}{2}+iy\right)=-2\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-iy}(p)\,p^{\frac{1+2iy}{4}}\left(\begin{array}{r}\left(-y^2-\frac{1}{4}+\left(\frac{13}{2}-iy\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)\\ +\left(-\frac{1}{2}+iy-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{3-2iy}{4}}(2\pi p)\end{array}\right),\quad y\in\mathbb{R}. \tag{11.17}$$

This result shows that the Riemann Xi function on the critical line is equivalent to the left side of the determining equation with difference constant multiplication.
Applying the recurrence formula and the origin symmetry with respect to the index of the modified Bessel function of the second kind,

(11.18)

$$\xi\left(\frac{1}{2}+iy\right)$$

$$=-2\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-iy}(p)\,p^{\frac{1+2iy}{4}}\left(\begin{array}{r}\left(-y^2-\frac{1}{4}+\left(\frac{13}{2}-iy\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)\\ +\left(-\frac{1}{2}+iy-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{-3+2iy}{4}}(2\pi p)\end{array}\right)$$

$$= -2\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-iy}(p)\,p^{\frac{1+2iy}{4}}\left(\begin{array}{l}\left(-y^2-\frac{1}{4}+\left(\frac{13}{2}-iy\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)\\ \quad+\left(-\frac{1}{2}+iy-2(2\pi p)^2\right)\left(-\frac{1+2iy}{2}\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)+(2\pi p)\mathrm{K}_{\frac{5+2iy}{4}}(2\pi p)\right)\end{array}\right)$$

$$= -2\sum_{p=1}^{\infty}(2\pi p)\,\sigma_{-\frac{1}{2}-iy}(p)\,p^{\frac{1+2iy}{4}}\left(\begin{array}{l}\left(\frac{15}{2}+iy\right)(2\pi p)\,\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)\\ \quad-\left(\frac{1}{2}-iy+2(2\pi p)^2\right)\mathrm{K}_{\frac{5+2iy}{4}}(2\pi p)\end{array}\right),\quad y\in\mathbb{R}.$$

Using the complex conjugate on both sides of the above equation,

$$\xi^*\left(\frac{1}{2}+iy\right) = -2\sum_{p=1}^{\infty}(2\pi p)\,\sigma_{-\frac{1}{2}+iy}(p)\,p^{\frac{1-2iy}{4}}\left(\begin{array}{l}\left(\frac{15}{2}-iy\right)(2\pi p)\,\mathrm{K}_{\frac{1-2iy}{4}}(2\pi p)\\ \quad-\left(\frac{1}{2}+iy+2(2\pi p)^2\right)\mathrm{K}_{\frac{5-2iy}{4}}(2\pi p)\end{array}\right),\quad y\in\mathbb{R}. \tag{11.19}$$

Therefore, it is shown that the Riemann Xi function takes real value on the critical line as follows:

$$\mathrm{Im}\left(\xi\left(\frac{1}{2}+iy\right)\right)=\frac{1}{2i}\left(\xi\left(\frac{1}{2}+iy\right)-\xi^*\left(\frac{1}{2}+iy\right)\right)=0,\quad y\in\mathbb{R}. \tag{11.20}$$

The Hadamard product representation for the Riemann Xi function on the critical line is

$$\xi\left(\frac{1}{2}+iy\right)=\frac{1}{2}\prod_{m=1}^{\infty}\left(1-\frac{4y^2+1}{(2\rho_m)^2+1}\right),\quad y\in\mathbb{R}. \tag{11.21}$$

From the equations (11.17) and (11.21),

$$\begin{aligned}&\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-iy}(p)\,p^{\frac{1+2iy}{4}}\left(\begin{array}{l}\left(-y^2-\frac{1}{4}+\left(\frac{13}{2}-iy\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2iy}{4}}(2\pi p)\\ \quad+\left(-\frac{1}{2}+iy-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{3-2iy}{4}}(2\pi p)\end{array}\right)\\ &=-\frac{1}{4}\prod_{m=1}^{\infty}\left(1-\frac{4y^2+1}{(2\rho_m)^2+1}\right)\in\mathbb{R},\quad y\in\mathbb{R}.\end{aligned} \tag{11.22}$$

The equation (11.22) insists on two points. One is that the left side of the determining equation has a real value on the critical line, and the other is that there are an unlimited number of non-trivial zeros. It is possible to assert that there is no non-trivial zero off the critical line.
The second proof of the Riemann hypothesis for the zeta function is thus completed.

It may appear strange because the left side of the determining equation takes a real value on the critical line, although the imaginary unit i is described as explicit in equation (11.15).
Then see the following analogy as a reference:
Consider the cubic equations with three different real solutions. Additionally, the imaginary unit i appeared when the formula for solving cubic equations was applied. When you actually proceed with the calculation according to the formula, the imaginary unit has disappeared and the real solutions have been obtained.

It is possible to design a zeta function variant that takes real value on the critical line. The method is to take the absolute value for the gamma factor in the explicit formula for the zeta function.

I decide to write the zeta function variation as zeta-tilde by adding the symbol ˜ (tilde) just above the Greek letter ζ.
(11.23)

$$\tilde{\zeta}(\theta):=\left|\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\right|\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad+\left((\theta-1)-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}\backslash\{1\}.$$

11.3 Behavior of the Number-Theoretic Function $\sigma_{-\theta}(n)\, n^{\theta/2}$, Part1

From now, I consider the behavior of the number-theoretic function $\sigma_{-\theta}(n)\, n^{\theta/2}$ on the critical line by presenting specific examples.
As its preparation, I begin with the classification of the whole positive integers into four subsets. Assuming k as a positive integer, I consider the first subset: $\{1\}$, the second subset of the $(2k-1)$ power of any prime number: $\{p^{2k-1}\}$, the third subset of the $2k$ power of any prime number: $\{p^{2k}\}$, and the fourth subset of composite numbers that does not belong to the aforementioned three subsets.
In the following discussions, a, b and c are assumed to be ttree different arbitrary prime numbers.
The magnitude correlation of a, b and c does not matter.
When $n = 1$,

$$n^{\frac{1+2iy}{4}} \sigma_{-\frac{1+2iy}{2}}(n)\Big|_{n=1} = 1. \quad y \in \mathbb{R}. \tag{11.24}$$

This is a direct current component when y is considered as the time variable and applied in the analogy of an electric circuit's current waveform.
When n is a prime number a,

$$\begin{aligned} a^{\frac{1+2iy}{4}} \sigma_{-\frac{1+2iy}{2}}(a) &= \left(1 + a^{-\frac{1+2iy}{4}}\right) a^{\frac{1+2iy}{4}} = 2\cosh\left(\frac{1+2iy}{4}\log(a)\right) \\ &= 2\cosh\left(\frac{\log(a)}{4}\right)\cos\left(\frac{y\log(a)}{2}\right) + 2i\sinh\left(\frac{\log(a)}{4}\right)\sin\left(\frac{y\log(a)}{2}\right), \quad y \in \mathbb{R}. \end{aligned} \tag{11.25}$$

This corresponds to an alternate current waveform with a single spectrum.
When n is the cubic of a prime number a,

$$\begin{aligned} \left(a^3\right)^{\frac{1+2iy}{4}} \sigma_{-\frac{1+2iy}{2}}\left(a^3\right) &= \left(1 + a^{-\frac{1+2iy}{2}} + \left(a^2\right)^{-\frac{1+2iy}{2}} + \left(a^3\right)^{-\frac{1+2iy}{2}}\right)\left(a^3\right)^{\frac{1+2iy}{4}} \\ &= 2\cosh\left(\frac{1+2iy}{4}\log(a)\right) + 2\cosh\left(\frac{3+6iy}{4}\log(a)\right), \quad y \in \mathbb{R}. \end{aligned} \tag{11.26}$$

When n is the $(2k-1)$ power of a prime number a,

$$\begin{aligned} \left(a^{2k-1}\right)^{\frac{1+2iy}{4}} \sigma_{-\frac{1+2iy}{2}}\left(a^{2k-1}\right) &= \left(1 + a^{-\frac{1+2iy}{2}} + a^{2^{-\frac{1+2iy}{2}}} + \cdots + a^{2k-1^{-\frac{1+2iy}{2}}}\right)\left(a^{2k-1}\right)^{\frac{1+2iy}{4}} \\ &= 2\sum_{m=1}^{k} \cosh\left(\frac{2m-1+2(2m-1)iy}{4}\log(a)\right), \quad (m \in \mathbb{N}, y \in \mathbb{R}). \end{aligned} \tag{11.27}$$

This corresponds to an alternate current waveform with multiple spectra.
When n is the square of a prime number a,

$$\begin{aligned} \left(a^2\right)^{\frac{1+2iy}{4}} \sigma_{-\frac{1+2iy}{2}}\left(a^2\right) &= \left(1 + a^{-\frac{1+2iy}{2}} + \left(a^2\right)^{-\frac{1+2iy}{2}}\right)\left(a^2\right)^{\frac{1+2iy}{4}} \\ &= 1 + 2\cosh\left(\frac{1+2iy}{2}\log(a)\right), \quad y \in \mathbb{R}. \end{aligned} \tag{11.28}$$

When n is the fourth power of a prime number a,

$$\begin{aligned} \left(a^4\right)^{\frac{1+2iy}{4}} \sigma_{-\frac{1+2iy}{2}}\left(a^4\right) &= \left(1 + a^{-\frac{1+2iy}{2}} + \left(a^2\right)^{-\frac{1+2iy}{2}} + \left(a^3\right)^{-\frac{1+2iy}{2}} + \left(a^4\right)^{-\frac{1+2iy}{2}}\right)\left(a^4\right)^{\frac{1+2iy}{4}} \\ &= 1 + 2\cosh\left(\frac{1+2iy}{2}\log(a)\right) + 2\cosh\left((1+2iy)\log(a)\right), \quad y \in \mathbb{R}. \end{aligned} \tag{11.29}$$

When n is the $2k$ power of a prime number a,

$$\begin{aligned} \left(a^{2k}\right)^{\frac{1+2iy}{4}} \sigma_{-\frac{1+2iy}{2}}\left(a^{2k}\right) &= \left(1 + a^{-\frac{1+2iy}{2}} + \left(a^2\right)^{-\frac{1+2iy}{2}} + \cdots + \left(a^{2k}\right)^{-\frac{1+2iy}{2}}\right)\left(a^{2k}\right)^{\frac{1+2iy}{4}} \\ &= 1 + 2\sum_{m=1}^{k} \cosh\left(\frac{m+2miy}{2}\log(a)\right), \quad (m \in \mathbb{N}, y \in \mathbb{R}). \end{aligned} \tag{11.30}$$

This corresponds to an alternate current waveform with multiple spectra and superimposed direct current components.
When n is a power of any prime number, the functional type can be determined additively.
When n is a composite number $a\cdot b$,

$$\begin{aligned}(a\cdot b)^{\frac{1+2iy}{4}}\sigma_{-\frac{1+2iy}{2}}(a\cdot b)&=\left(1+a^{-\frac{1+2iy}{2}}+b^{-\frac{1+2iy}{2}}+(a\cdot b)^{-\frac{1+2iy}{2}}\right)(a\cdot b)^{\frac{1+2iy}{4}}\\&=4\cosh\left(\frac{1+2iy}{4}\log(a)\right)\cosh\left(\frac{1+2iy}{4}\log(b)\right),\quad y\in\mathbb{R}.\end{aligned}\tag{11.31}$$

When n is a composite number $a\cdot b\cdot c$,

$$\begin{aligned}(a\cdot b\cdot c)^{\frac{1+2iy}{4}}\sigma_{-\frac{1+2iy}{2}}(a\cdot b\cdot c)&=\begin{pmatrix}1+a^{-\frac{1+2iy}{2}}+b^{-\frac{1+2iy}{2}}+(a\cdot b)^{-\frac{1+2iy}{2}}\\+(b\cdot c)^{-\frac{1+2iy}{2}}+(c\cdot a)^{-\frac{1+2iy}{2}}+(a\cdot b\cdot c)^{-\frac{1+2iy}{2}}\end{pmatrix}(a\cdot b\cdot c)^{\frac{1+2iy}{4}}\\&=8\cosh\left(\frac{1+2iy}{4}\log(a)\right)\cosh\left(\frac{1+2iy}{4}\log(b)\right)\\&\quad\times\cosh\left(\frac{1+2iy}{4}\log(c)\right),\quad y\in\mathbb{R}.\end{aligned}\tag{11.32}$$

When n is a composite number $a\cdot b^2$,

$$\begin{aligned}\left(a\cdot b^2\right)^{\frac{1+2iy}{4}}\sigma_{-\frac{1+2iy}{2}}\left(a\cdot b^2\right)&=\begin{pmatrix}1+a^{-\frac{1+2iy}{2}}+b^{-\frac{1+2iy}{2}}+(a\cdot b)^{-\frac{1+2iy}{2}}\\+\left(b^2\right)^{-\frac{1+2iy}{2}}+\left(a\cdot b^2\right)^{-\frac{1+2iy}{2}}\end{pmatrix}\left(a\cdot b^2\right)^{\frac{1+2iy}{4}}\\&=2\cosh\left(\frac{1+2iy}{4}\log(a)\right)\left(1+2\cosh\left(\frac{1+2iy}{4}\log(b)\right)\right),\quad y\in\mathbb{R}.\end{aligned}\tag{11.33}$$

When n is a composite number excluding powers of any prime number, the functional type can be mainly determined using multiplicative analysis.
The number-theoretic function $\sigma_{-\theta}(n)\,n^{\theta/2}$ is described by the complex hyperbolic cosine functions excepting $n=1$ in the explicit formula for the zeta function on the critical line.

12 Some Representations for the Zeta Function for any Odd Number Except 1

I presented some representations for the zeta function for any odd number except 1 and relational mentions in the first half of this section. In the second half, I will show another two types of general representations for the zeta function for any odd number of either 3 or 7, or more.

12.1 Some Representations for the Zeta Function for any Odd Number Except 1 and Relational Mentions

The explicit formula for the zeta function is shown again.

$$\zeta(\theta)=\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+\left((\theta-1)-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{pmatrix},\quad\theta\in\mathbb{C}\backslash\{1\}.\tag{12.1}$$

For rewriting the modified Bessel function of the second kind into the quadrable integral representation, the following equation is prepared in subsection 19.7:

$$\alpha^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\alpha)=\int_0^{\infty}x^{\theta-1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx,\quad(\alpha>0,\theta\in\mathbb{C}).\tag{12.2}$$

Here, putting as

$$\alpha=\pi p,\quad p\in\mathbb{N}.$$

the following two relations are obtained:

$$(\pi p)^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)=\int_0^{\infty}x^{\theta-1}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx,\quad(p\in\mathbb{N},\theta\in\mathbb{C}).\tag{12.3}$$

$$(12.4)\qquad (\pi p)^{\frac{2-\theta}{2}}\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)=(\pi p)^{\frac{2-\theta}{2}}\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)=\int_0^\infty x^{\theta-3}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx,\quad (p\in\mathbb{N},\theta\in\mathbb{C}).$$

To apply those transformation formulae, the right side of equation (12.1) is modified as follows:

$$(12.5)\qquad \zeta(\theta)=\frac{1}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\begin{pmatrix}2\left(\theta(1-\theta)+(\theta-7)(2\pi p)^2\right)(\pi p)^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ +\left(1-\theta+2(2\pi p)^2\right)(2\pi p)^2(\pi p)^{\frac{2-\theta}{2}}\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{pmatrix},$$
$$\theta\in\mathbb{C}\backslash\{1\}.$$

Therefore,

$$(12.6)\qquad \zeta(\theta)=\frac{1}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\begin{pmatrix}2\left(\theta(1-\theta)+(\theta-7)(2\pi p)^2\right)\int_0^\infty x^{\theta-1}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ +\left(1-\theta+2(2\pi p)^2\right)(2\pi p)^2\int_0^\infty x^{\theta-3}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix},$$
$$\theta\in\mathbb{C}\backslash\{1\}.$$

The equation (12.6) can be used to calculate each zeta function representation for any odd number of 3 or more. Conversely, for calculating each zeta function representation for any odd number of −1 or less, for equation (12.1), $-\theta$ is substituted for θ to obtain the following equation:

$$(12.7)\qquad \zeta(-\theta)=\frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{\theta}(p)\,p^{-\frac{\theta}{2}}\begin{pmatrix}2\left(\theta(\theta+1)+(7+\theta)(2\pi p)^2\right)\mathrm{K}_{-\frac{\theta}{2}}(2\pi p)\\ -4\left(\theta+1+2(2\pi p)^2\right)(\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\end{pmatrix},$$
$$\theta\in\mathbb{C}\backslash\{-1\}.$$

The relations of equations (3.30) and (5.39) are applied to the equation (12.7).

$$(12.8)\qquad \zeta(-\theta)=\frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}2\left(\theta(\theta+1)+(7+\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ -4\left(\theta+1+2(2\pi p)^2\right)(\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\end{pmatrix},$$
$$\theta\in\mathbb{C}\backslash\{-1\}.$$

The recurrence formula for the modified Bessel function of the second kind is applied.

$$(12.9)\qquad \begin{aligned}\zeta(-\theta)&=\frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}2\left(\theta(\theta+1)+(7+\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ -4\left(\theta+1+2(2\pi p)^2\right)\\ \times\left(\frac{\theta}{2}\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+(\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\right)\end{pmatrix}\\
&=\frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\begin{pmatrix}2(7-\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ -2\left(\theta+1+2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{pmatrix}\\
&=\frac{\pi^{-\theta}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\begin{pmatrix}2(7-\theta)(2\pi p)^2(\pi p)^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ -\left(\theta+1+2(2\pi p)^2\right)(2\pi p)^2(\pi p)^{\frac{\theta-2}{2}}\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{pmatrix},\end{aligned}$$
$$\theta\in\mathbb{C}\backslash\{-1\}.$$

Therefore

$$
\zeta(-\theta) = \frac{\pi^{-\theta}}{(1+\theta)\,\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(\begin{array}{l} 2\,(7-\theta)\,(2\pi p)^2 \displaystyle\int_0^{\infty} x^{\theta-1} e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \\ \quad -\left(\theta+1+2(2\pi p)^2\right)(2\pi p)^2 \displaystyle\int_0^{\infty} x^{\theta-3} e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right), \quad \theta \in \mathbb{C}\backslash\{-1\}. \tag{12.10}
$$

Thus, the equation (12.10) can be used to calculate each zeta function representation for any odd number of -1 or less. Some calculation examples and the results will be shown soon.
Moreover, because the values are well known, I will post exact value for each of the zeta function for any odd number of -1 or less together.
For equation (12.10), 1 is substituted for θ,

$$
\begin{aligned}
\zeta(-1) &= \frac{\pi^{-1}}{2\,\Gamma\left(1-\frac{1}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(\begin{array}{l} 12(2\pi p)^2 \displaystyle\int_0^{\infty} x^0 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \\ \quad -2\left(1+(2\pi p)^2\right)(2\pi p)^2 \displaystyle\int_0^{\infty} x^{-2} e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right) \\
&= \frac{1}{\pi\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(6(2\pi p)^2 \cdot \frac{\sqrt{\pi}}{2}\, e^{-2\pi p} - \left(1+(2\pi p)^2\right)(2\pi p)^2 \cdot \frac{\sqrt{\pi}}{2\pi p}\, e^{-2\pi p} \right).
\end{aligned} \tag{12.11}
$$

Therefore,

$$
\zeta(-1) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left((2\pi p)^2 - 3\,(2\pi p) + 1 \right)(2\pi p)\, e^{-2\pi p} = -\frac{1}{12}. \tag{12.12}
$$

For equation (12.10), 3 is substituted for θ,

$$
\begin{aligned}
\zeta(-3) &= \frac{\pi^{-3}}{4\,\Gamma\left(1-\frac{3}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(\begin{array}{l} 8(2\pi p)^2 \displaystyle\int_0^{\infty} x^2 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \\ \quad -2\left(2+(2\pi p)^2\right)(2\pi p)^2 \displaystyle\int_0^{\infty} x^0 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right) \\
&= -\frac{1}{8\pi^3\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(\begin{array}{l} 8(2\pi p)^2 \cdot \dfrac{(2\pi p)+1}{4} \sqrt{\pi}\, e^{-2\pi p} \\ \quad -2\left(2+(2\pi p)^2\right)(2\pi p)^2 \cdot \dfrac{\sqrt{\pi}}{2}\, e^{-2\pi p} \end{array} \right).
\end{aligned} \tag{12.13}
$$

Therefore,

$$
\zeta(-3) = \frac{1}{8\pi^3} \sum_{p=1}^{\infty} \sigma_{-3}(p)\,((2\pi p)-2)\,(2\pi p)^3\, e^{-2\pi p} = \frac{1}{120}. \tag{12.14}
$$

For equation (12.10), 5 is substituted for θ,

$$
\begin{aligned}
\zeta(-5) &= \frac{\pi^{-5}}{6\,\Gamma\left(1-\frac{5}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(\begin{array}{l} 4(2\pi p)^2 \displaystyle\int_0^{\infty} x^4 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \\ \quad -2\left(3+(2\pi p)^2\right)(2\pi p)^2 \displaystyle\int_0^{\infty} x^2 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right) \\
&= -\frac{1}{8\pi^3\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(\begin{array}{l} 4(2\pi p)^2 \cdot \dfrac{(2\pi p)^2+3\,(2\pi p)+3}{8} \sqrt{\pi}\, e^{-2\pi p} \\ \quad -2\left(3+(2\pi p)^2\right)(2\pi p)^2 \cdot \dfrac{(2\pi p)+1}{4} \sqrt{\pi}\, e^{-2\pi p} \end{array} \right).
\end{aligned} \tag{12.15}
$$

Therefore,

$$
\zeta(-5) = -\frac{1}{16\pi^5} \sum_{p=1}^{\infty} \sigma_{-5}(p)\,(2\pi p)^5\, e^{-2\pi p} = -\frac{1}{252}. \tag{12.16}
$$

For equation (12.10), 7 and 9 are substituted for θ, only the results are shown.

$$\zeta\left(-7\right)=\frac{15}{256\pi^5}\sum_{p=1}^{\infty}\sigma_{-7}\left(p\right)\left((2\pi p)^4+3(2\pi p)^3+7(2\pi p)^2+12\left(2\pi p\right)+12\right)\left(2\pi p\right)^2e^{-2\pi p}=\frac{1}{240}. \tag{12.17}$$

$$\zeta\left(-9\right)=-\frac{21}{256\pi^5}\sum_{p=1}^{\infty}\sigma_{-9}\left(p\right)\begin{pmatrix}(2\pi p)^5+7(2\pi p)^4+30(2\pi p)^3\\ +90(2\pi p)^2+180\left(2\pi p\right)+180\end{pmatrix}\left(2\pi p\right)^2e^{-2\pi p}=-\frac{1}{132}. \tag{12.18}$$

For equation (12.6), 3 is substituted for θ,

$$\begin{aligned}\zeta\left(3\right)&=\frac{1}{2\,\Gamma\left(1+\frac{3}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\begin{pmatrix}2\left(-6-4(2\pi p)^2\right)\int_0^{\infty}x^2e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ +\left(-2+2(2\pi p)^2\right)\left(2\pi p\right)^2\int_0^{\infty}x^0e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix}\\ &=\frac{2}{3\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\begin{pmatrix}-4\left(3+2(2\pi p)^2\right)\cdot\frac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p}\\ +2\left(-1+(2\pi p)^2\right)\left(2\pi p\right)^2\cdot\frac{\sqrt{\pi}}{2}\,e^{-2\pi p}\end{pmatrix}.\end{aligned} \tag{12.19}$$

Therefore,

$$\zeta\left(3\right)=\frac{2}{3}\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\left((2\pi p)^4-2(2\pi p)^3-3(2\pi p)^2-3\left(2\pi p\right)-3\right)e^{-2\pi p}. \tag{12.20}$$

The right side of equation (12.20) can be expanded as follows:

$$\zeta\left(3\right)=\frac{2}{3}\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\left((2\pi p)^3-2(2\pi p)^2-3\left(2\pi p\right)-3\right)\left(2\pi p\right)e^{-2\pi p}-\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^3}+\zeta\left(3\right). \tag{12.21}$$

Thus, I can obtain the representation for the series

$$\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^3},$$

and also its exact value is known.

$$\sum_{n=1}^{\infty}\frac{\coth\left(\pi n\right)}{n^3}=\frac{2}{3}\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\left((2\pi p)^3-2(2\pi p)^2-3\left(2\pi p\right)-3\right)\left(2\pi p\right)e^{-2\pi p}=\frac{7\,\pi^3}{180}. \tag{12.22}$$

Further, the right side of equation (12.20) can be also expanded as follows:

$$\zeta\left(3\right)=\frac{2}{3}\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\left((2\pi p)-2\right)\left(2\pi p\right)^3e^{-2\pi p}-2\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\left((2\pi p)^2+(2\pi p)+1\right)e^{-2\pi p}. \tag{12.23}$$

Because the first term on the right side of equation (12.23) has already been determined, I obtain the representation that gives the transcendental number a fixed value.

$$\sum_{p=1}^{\infty}\sigma_{-3}\left(p\right)\left((2\pi p)-2\right)\left(2\pi p\right)^3e^{-2\pi p}=\frac{3}{2}\cdot\frac{2\,\pi^3}{45}=\frac{\pi^3}{15}. \tag{12.24}$$

For equation (12.6), 5 is substituted for θ,

$$\begin{aligned}\zeta\left(5\right)&=\frac{1}{4\,\Gamma\left(1+\frac{5}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-5}\left(p\right)\begin{pmatrix}2\left(-20-2(2\pi p)^2\right)\int_0^{\infty}x^4e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ +\left(-4+2(2\pi p)^2\right)\left(2\pi p\right)^2\int_0^{\infty}x^2e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix}\\ &=\frac{2}{15\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-5}\left(p\right)\begin{pmatrix}-4\left(10+(2\pi p)^2\right)\cdot\frac{(2\pi p)^2+3\left(2\pi p\right)+3}{8}\sqrt{\pi}\,e^{-2\pi p}\\ +2\left(-2+(2\pi p)^2\right)\left(2\pi p\right)^2\cdot\frac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p}\end{pmatrix}.\end{aligned} \tag{12.25}$$

Therefore,

$$\zeta(5) = \frac{1}{15}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^5 - 5(2\pi p)^3 - 15(2\pi p)^2 - 30(2\pi p) - 30\right)e^{-2\pi p}. \tag{12.26}$$

The right side of equation (12.26) can be expanded as follows:

$$\zeta(5) = \frac{1}{15}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^4 - 5(2\pi p)^2 - 15(2\pi p) - 30\right)(2\pi p)e^{-2\pi p} - \sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^5} + \zeta(5). \tag{12.27}$$

Thus, I can obtain the representation for the series

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^5},$$

but its exact value is unknown.

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^5} = \frac{1}{15}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^4 - 5(2\pi p)^2 - 15(2\pi p) - 30\right)(2\pi p)e^{-2\pi p}. \tag{12.28}$$

Furthermore, the right side of equation (12.26) can also be expanded as follows:

$$\zeta(5) = \frac{1}{15}\sum_{p=1}^{\infty}\sigma_{-5}(p)(2\pi p)^5 e^{-2\pi p} - \frac{1}{3}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^3 + 3(2\pi p)^2 + 6(2\pi p) + 6\right)e^{-2\pi p}. \tag{12.29}$$

Because the first term on the right side of equation (12.29) has already been determined, I obtain the representation that gives the transcendental number a fixed value.

$$\sum_{p=1}^{\infty}\sigma_{-5}(p)(2\pi p)^5 e^{-2\pi p} = 15\cdot\frac{4\pi^5}{945} = \frac{4\pi^5}{63}. \tag{12.30}$$

For equation (12.6), 7 is substituted for θ, and the principal results are shown.

$$\zeta(7) = \frac{2}{315}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^6 + 3(2\pi p)^5 - 30(2\pi p)^3 - 135(2\pi p)^2 - 315(2\pi p) - 315\right)e^{-2\pi p}. \tag{12.31}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^7} &= \frac{2}{315}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^5 + 3(2\pi p)^4 - 30(2\pi p)^2 - 135(2\pi p) - 315\right)(2\pi p)e^{-2\pi p}\\ &= \frac{19\pi^7}{56700}.\end{aligned} \tag{12.32}$$

$$\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^4 + 3(2\pi p)^3 + 7(2\pi p)^2 + 12(2\pi p) + 12\right)(2\pi p)^2 e^{-2\pi p} = \frac{16\pi^7}{225}. \tag{12.33}$$

For equation (12.6), 9 is substituted for θ, and the principal results are shown.

$$\zeta(9) = \frac{1}{1890}\sum_{p=1}^{\infty}\sigma_{-9}(p)\begin{pmatrix}(2\pi p)^7 + 7(2\pi p)^6 + 21(2\pi p)^5 - 315(2\pi p)^3\\ - 1575(2\pi p)^2 - 3780(2\pi p) - 3780\end{pmatrix}e^{-2\pi p}. \tag{12.34}$$

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^9} = \frac{1}{1890}\sum_{p=1}^{\infty}\sigma_{-9}(p)\begin{pmatrix}(2\pi p)^6 + 7(2\pi p)^5 + 21(2\pi p)^4\\ - 315(2\pi p)^2 - 1575(2\pi p) - 3780\end{pmatrix}(2\pi p)e^{-2\pi p}. \tag{12.35}$$

$$\sum_{p=1}^{\infty}\sigma_{-9}(p)\left((2\pi p)^5 + 7(2\pi p)^4 + 30(2\pi p)^3 + 90(2\pi p)^2 + 180(2\pi p) + 180\right)(2\pi p)^2 e^{-2\pi p} = \frac{16\pi^9}{693}. \tag{12.36}$$

For equation (12.6), 11 is substituted for θ, and the principal results are shown.

(12.37)
$$\zeta(11) = \frac{2}{51975}\sum_{p=1}^{\infty}\sigma_{-11}(p)\begin{pmatrix}(2\pi p)^8 + 12(2\pi p)^7 + 70(2\pi p)^6 + 210(2\pi p)^5 - 105(2\pi p)^4 \\ - 4410(2\pi p)^3 - 21735(2\pi p)^2 - 51975\,(2\pi p) - 51975\end{pmatrix} e^{-2\pi p}.$$

(12.38)
$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{11}} = \frac{2}{51975}\sum_{p=1}^{\infty}\sigma_{-11}(p)\begin{pmatrix}(2\pi p)^7 + 12(2\pi p)^6 + 70(2\pi p)^5 \\ + 210(2\pi p)^4 - 105(2\pi p)^3 - 4410(2\pi p)^2 \\ - 21735\,(2\pi p) - 51975\end{pmatrix}(2\pi p)\, e^{-2\pi p} = \frac{1453\,\pi^{11}}{425675250}.$$

(12.39)
$$\sum_{p=1}^{\infty}\sigma_{-11}(p)\begin{pmatrix}(2\pi p)^6 + 12(2\pi p)^5 + 81(2\pi p)^4 + 375(2\pi p)^3 \\ + 1215(2\pi p)^2 + 2520\,(2\pi p) + 2520\end{pmatrix}(2\pi p)^2 e^{-2\pi p} = \frac{176896\,\pi^{11}}{1289925}.$$

For equation (12.6), 13 is substituted for θ, and the principal results are shown.

(12.40)
$$\zeta(13) = \frac{1}{405405}\sum_{p=1}^{\infty}\sigma_{-13}(p)\begin{pmatrix}(2\pi p)^9 + 18(2\pi p)^8 + 162(2\pi p)^7 + 882(2\pi p)^6 \\ + 2457(2\pi p)^5 - 3780(2\pi p)^4 - 72765(2\pi p)^3 \\ - 343035(2\pi p)^2 - 810810\,(2\pi p) - 810810\end{pmatrix} e^{-2\pi p}.$$

(12.41)
$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{13}} = \frac{1}{405405}\sum_{p=1}^{\infty}\sigma_{-13}(p)\begin{pmatrix}(2\pi p)^8 + 18(2\pi p)^7 + 162(2\pi p)^6 \\ + 882(2\pi p)^5 + 2457(2\pi p)^4 - 3780(2\pi p)^3 \\ - 72765(2\pi p)^2 - 343035\,(2\pi p) - 810810\end{pmatrix}(2\pi p)\, e^{-2\pi p}.$$

(12.42)
$$\sum_{p=1}^{\infty}\sigma_{-13}(p)\begin{pmatrix}(2\pi p)^7 + 18(2\pi p)^6 + 175(2\pi p)^5 + 1155(2\pi p)^4 \\ + 5460(2\pi p)^3 + 18060(2\pi p)^2 + 37800\,(2\pi p) + 37800\end{pmatrix}(2\pi p)^2 e^{-2\pi p} = \frac{1024\,\pi^{13}}{4455}.$$

For equation (12.6), 15 is substituted for θ, and the principal results are shown.

(12.43)
$$\zeta(15) = \frac{2}{14189175}\sum_{p=1}^{\infty}\sigma_{-15}(p)\begin{pmatrix}(2\pi p)^{10} + 25(2\pi p)^9 + 315(2\pi p)^8 \\ + 2520(2\pi p)^7 + 12915(2\pi p)^6 + 31185(2\pi p)^5 \\ - 103950(2\pi p)^4 - 1351350(2\pi p)^3 - 6081075(2\pi p)^2 \\ - 14189175\,(2\pi p) - 14189175\end{pmatrix} e^{-2\pi p}.$$

(12.44)
$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{15}} = \frac{2}{14189175}\sum_{p=1}^{\infty}\sigma_{-15}(p)\begin{pmatrix}(2\pi p)^9 + 25(2\pi p)^8 + 315(2\pi p)^7 \\ + 2520(2\pi p)^6 + 12915(2\pi p)^5 + 31185(2\pi p)^4 \\ - 103950(2\pi p)^3 - 1351350(2\pi p)^2 \\ - 6081075\,(2\pi p) - 14189175\end{pmatrix}(2\pi p)\, e^{-2\pi p} = \frac{13687\,\pi^{15}}{390769879500}.$$

(12.45)
$$\sum_{p=1}^{\infty}\sigma_{-15}(p)\begin{pmatrix}(2\pi p)^8 + 25(2\pi p)^7 + 330(2\pi p)^6 \\ + 12915(2\pi p)^5 + 31185(2\pi p)^4 + 89775(2\pi p)^3 \\ + 297675(2\pi p)^2 + 623700\,(2\pi p) + 623700\end{pmatrix}(2\pi p)^2 e^{-2\pi p} = \frac{14815232\,\pi^{15}}{34459425}.$$

For equation (12.6), 17 is substituted for θ, and the principal results are shown.

(12.46)
$$\zeta(17) = \frac{1}{137837700}\sum_{p=1}^{\infty}\sigma_{-17}(p)\begin{pmatrix}(2\pi p)^{11} + 33(2\pi p)^{10} + 550(2\pi p)^9 + 5940(2\pi p)^8 \\ + 44055(2\pi p)^7 + 211365(2\pi p)^6 + 405405(2\pi p)^5 \\ - 2702700(2\pi p)^4 - 27702675(2\pi p)^3 - 119594475(2\pi p)^2 \\ - 275675400\,(2\pi p) - 275675400\end{pmatrix} e^{-2\pi p}.$$

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{17}}=\frac{1}{137837700}\sum_{p=1}^{\infty}\sigma_{-17}(p)\begin{pmatrix}(2\pi p)^{10}+33(2\pi p)^{9}+550(2\pi p)^{8}\\+5940(2\pi p)^{7}+44055(2\pi p)^{6}\\+211365(2\pi p)^{5}+405405(2\pi p)^{4}\\-2702700(2\pi p)^{3}-27702675(2\pi p)^{2}\\-119594475(2\pi p)-275675400\end{pmatrix}(2\pi p)\,e^{-2\pi p}. \tag{12.47}$$

$$\sum_{p=1}^{\infty}\sigma_{-17}(p)\begin{pmatrix}(2\pi p)^{9}+33(2\pi p)^{8}+567(2\pi p)^{7}\\+6552(2\pi p)^{6}+55377(2\pi p)^{5}+350595(2\pi p)^{4}\\+1642410(2\pi p)^{3}+5426190(2\pi p)^{2}\\+11351340(2\pi p)+11351340\end{pmatrix}(2\pi p)^{2}\,e^{-2\pi p}=\frac{718716928\,\pi^{17}}{808782975}. \tag{12.48}$$

12.2 Another Two Types of General Representations for the Zeta Function for any Odd Number of Either 3 or 7, or More

For equation (12.6), any odd number of 3 or more is substituted for θ.

$$\begin{aligned}\zeta(2k+1)&=\frac{1}{2k\,\Gamma\left(1+\frac{2k+1}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\begin{pmatrix}2\left((2k+1)(-2k)+(2k-6)(2\pi p)^{2}\right)\int_0^{\infty}x^{2k}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\+\left(-2k+2(2\pi p)^{2}\right)(2\pi p)^{2}\int_0^{\infty}x^{2(k-1)}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix}\\&=\frac{2^{k+1}}{k\,(2k+1)!!\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\begin{pmatrix}2\left(-k(2k+1)+(k-3)(2\pi p)^{2}\right)\int_0^{\infty}x^{2k}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\+\left(-k+(2\pi p)^{2}\right)(2\pi p)^{2}\int_0^{\infty}x^{2(k-1)}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix},\quad k\in\mathbb{N}.\end{aligned} \tag{12.49}$$

The general representation for the two integrals of the result of equation (12.49) is given by the description in the subsection 19-4 i.e.,

$$\int_0^{\infty}x^{2k}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx=\frac{1}{2^{k+1}}\sum_{\mu=0}^{k}a_{k,\mu}(2\pi p)^{\mu}\sqrt{\pi}\,e^{-2\pi p},\quad (k,p\in\mathbb{N}). \tag{12.50}$$

Where

$$a_{k,\mu}=\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!},\quad \left(k\in\mathbb{N},\mu=\{0,1,2,\cdots,k\}\right). \tag{12.51}$$

And the coefficients deserving special mention are as follows:

$$a_{k,k}=1,\quad k\in\mathbb{N}. \tag{12.52}$$

$$a_{k,1}=a_{k,0}=(2k-1)!!,\quad k\in\mathbb{N}. \tag{12.53}$$

For equation (12.51), 0 is substituted directly for k and μ.

$$a_{0,0}=\left.\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}\right|_{k=\mu=0}=1. \tag{12.54}$$

This result indicates that equation (12.50) holds even if $k = 0$.

The above general representation for the two integrals is introduced into the result of equation (12.49).

$$\zeta(2k+1)$$

$$= \frac{2^{k+1}}{k(2k+1)!!\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} 2\left(-k(2k+1)+(k-3)(2\pi p)^2\right) \\ \times \frac{1}{2^{k+1}} \sum_{\mu=0}^{k} a_{k,\mu}(2\pi p)^{\mu} \sqrt{\pi}\, e^{-2\pi p} \\ + \left(-k+(2\pi p)^2\right)(2\pi p)^2 \\ \times \frac{1}{2^{k}} \sum_{\mu=0}^{k-1} a_{k-1,\mu}(2\pi p)^{\mu} \sqrt{\pi}\, e^{-2\pi p} \end{pmatrix}$$

(12.55)
$$= \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} \left(-k(2k+1)+(k-3)(2\pi p)^2\right) \sum_{\mu=0}^{k} a_{k,\mu}(2\pi p)^{\mu} \\ + \left(-k+(2\pi p)^2\right)(2\pi p)^2 \sum_{\mu=0}^{k-1} a_{k-1,\mu}(2\pi p)^{\mu} \end{pmatrix} e^{-2\pi p}$$

$$= \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} a_{k-1,k-1}(2\pi p)^{k+3} + \left(a_{k-1,k-2}+(k-3)\,a_{k,k}\right)(2\pi p)^{k+2} \\ + \sum_{\mu=4}^{k+1} a_{k-1,\mu-4}(2\pi p)^{\mu} + (k-3)\sum_{\mu=2}^{k+1} a_{k,\mu-2}(2\pi p)^{\mu} \\ - k\sum_{\mu=2}^{k+1} a_{k-1,\mu-2}(2\pi p)^{\mu} - k(2k+1)\sum_{\mu=0}^{k} a_{k,\mu}(2\pi p)^{\mu} \end{pmatrix} e^{-2\pi p},$$

$k \in \mathbb{N}$.

From the equations (12.52) and (12.54), the coefficient of $(2\pi p)^{k+3}$ is immediately determined as follows:

(12.56)
$$a_{k-1,k-1} = 1, \quad k \in \mathbb{N}.$$

The coefficient of $(2\pi p)^{k+2}$ is determined under the condition of $k > 1$ as follows:

(12.57)
$$\begin{aligned} a_{k-1,k-2} + (k-3)\,a_{k,k} &= \left.\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}\right|_{k=k-1,\mu=k-2} + (k-3)\cdot 1 \\ &= \frac{k!}{2^1\,(k-2)!\,1!} + (k-3) = \frac{(k-2)(k+3)}{2}, \quad k \in \mathbb{N}\backslash\{1\}. \end{aligned}$$

For the result of equation (12.57), 1 is substituted directly for k.

(12.58)
$$\left.\frac{(k-2)(k+3)}{2}\right|_{k=1} = -2.$$

This result is the same as the coefficient of $(2\pi p)^3$ of the representation for $\zeta(3)$. Thus, the right side of equation (12.57) holds even for the case when $k = 1$.
The above results are introduced into the result of equation (12.55).

$$\zeta(2k+1)$$

(12.59)
$$= \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} (2\pi p)^{k+3} + \frac{(k-2)(k+3)}{2}(2\pi p)^{k+2} \\ + \sum_{\mu=4}^{k+1} a_{k-1,\mu-4}(2\pi p)^{\mu} + (k-3)\sum_{\mu=2}^{k+1} a_{k,\mu-2}(2\pi p)^{\mu} \\ - k\sum_{\mu=2}^{k+1} a_{k-1,\mu-2}(2\pi p)^{\mu} - k(2k+1)\sum_{\mu=0}^{k} a_{k,\mu}(2\pi p)^{\mu} \end{pmatrix} e^{-2\pi p},$$

$k \in \mathbb{N}$.

Here, new coefficients $c_{k,\mu}$ are used for the general representation for the zeta function for any odd number of 3 or more.

The coefficients $c_{k,\mu}$ are determined by the following equation of definition:

$$\zeta(2k+1) =: \frac{2}{k(2k+1)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left((2\pi p)^{k+3} + \frac{(k-2)(k+3)}{2}(2\pi p)^{k+2} + \sum_{\mu=0}^{k+1} c_{k,\mu}(2\pi p)^{\mu}\right)e^{-2\pi p}, \quad k \in \mathbb{N}. \tag{12.60}$$

The coefficients $c_{k,k+3}$ and $c_{k,k+2}$ are immediately determined as follows:

$$c_{k,k+3} = 1, \quad k \in \mathbb{N}. \tag{12.61}$$

$$c_{k,k+2} = \frac{(k-2)(k+3)}{2}, \quad k \in \mathbb{N}. \tag{12.62}$$

And the coefficients $c_{k,\mu}$ are determined with the condition as follows:

$$c_{k,\mu} = a_{k-1,\mu-4} + (k-3)\,a_{k,\mu-2} - k\,a_{k-1,\mu-2} - k(2k+1)\,a_{k,\mu}, \quad (k \in \mathbb{N}\backslash\{1,2\}, \mu = 4,5,6,\cdots,k+1). \tag{12.63}$$

The relation of equation (12.51) is introduced into the above equation.

$$\begin{aligned} c_{k,\mu} = & \left.\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}\right|_{k=k-1,\mu=\mu-4} + (k-3)\cdot\left.\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}\right|_{k=k,\mu=\mu-2} \\ & - k\cdot\left.\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}\right|_{k=k-1,\mu=\mu-2} - k(2k+1)\cdot\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}, \\ & (k \in \mathbb{N}\backslash\{1,2\}, \mu = 4,5,6,\cdots,k+1). \end{aligned} \tag{12.64}$$

I demonstrate the calculation steps..

$$\begin{aligned} c_{k,\mu} = & \frac{(2k+2-\mu)!}{2^{k+3-\mu}(\mu-4)!\,(k+3-\mu)!} + (k-3)\cdot\frac{(2k+2-\mu)!}{2^{k+2-\mu}(\mu-2)!\,(k+2-\mu)!} \\ & - k\cdot\frac{(2k-\mu)!}{2^{k+1-\mu}(\mu-2)!\,(k+1-\mu)!} - k(2k+1)\cdot\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!} \\ = & \left(\frac{(2k+2-\mu)(2k+1-\mu)}{2^{k+3-\mu}(\mu-4)!\,(k+3-\mu)!} + \frac{(k-3)(2k+2-\mu)(2k+1-\mu)}{2^{k+2-\mu}(\mu-2)!\,(k+2-\mu)!} \right. \\ & \left. - \frac{k}{2^{k+1-\mu}(\mu-2)!\,(k+1-\mu)!} - \frac{k(2k+1)}{2^{k-\mu}\mu!\,(k-\mu)!}\right)(2k-\mu)! \\ = & \left(\frac{\mu(\mu-1)(\mu-2)(\mu-3)(2k+2-\mu)(2k+1-\mu)}{2^{k+3-\mu}\mu!\,(k+3-\mu)!} \right. \\ & + \frac{2\mu(\mu-1)(k-3)(k+3-\mu)(2k+2-\mu)(2k+1-\mu)}{2^{k+3-\mu}\mu!\,(k+3-\mu)!} \\ & - \frac{4\mu(\mu-1)k(k+3-\mu)(k+2-\mu)}{2^{k+3-\mu}\mu!\,(k+3-\mu)!} \\ & \left. - \frac{8k(2k+1)(k+3-\mu)(k+2-\mu)(k+1-\mu)}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\right)(2k-\mu)! \end{aligned} \tag{12.65}$$

The result of the above calculation is shown.

$$\begin{aligned} = & \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\left(\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k\left(8k^4 - 4k^3\left(\mu^2+5\mu-13\right) + 4k^2\left(2\mu^3+3\mu^2-26\mu+28\right) \right.\right. \\ & \left.\left. - k\left(7\mu^4-2\mu^3-89\mu^2+168\mu-92\right) + (\mu-1)\left(3\mu^4-2\mu^3-39\mu^2+68\mu-24\right)\right)\right), \\ & (k \in \mathbb{N}\backslash\{1,2\}, \mu = 4,5,6,\cdots,k+1). \end{aligned}$$

For equation (12.65), 0 is substituted for μ.

$$\begin{aligned} c_{k,0} & = -\frac{2k\left(8k^4+52k^3+112k^2+92k+24\right)}{2^{k+3}(k+3)!}(2k)! = -\frac{k(2k+1)(k+1)(k+2)(k+3)}{2^k(k+3)!}(2k)! \\ & = -\frac{k}{2^k k!}(2k+1)! = -k(2k+1)!!, \quad k \in \mathbb{N}\backslash\{1,2\}. \end{aligned} \tag{12.66}$$

For equation (12.65), 1 is substituted for μ.

$$
\begin{aligned}
c_{k,1} &= -\frac{2k\left(8k^4+28k^3+28k^2+8k\right)}{2^{k+2}\,(k+2)!}\,(2k-1)! = -\frac{2k^2\,(2k+1)\,(k+1)\,(k+2)}{2^k\,(k+2)!}\,(2k-1)! \\
&= -\frac{k}{2^k\,k!}\,(2k+1)! = -k\,(2k+1)!!, \quad k\in\mathbb{N}\backslash\{1,2\}.
\end{aligned} \tag{12.67}
$$

For equation (12.65), 2 is substituted for μ.

$$
\begin{aligned}
c_{k,2} &= -\frac{2k\left(8k^4-4k^3+16k^2+16k-12\right)}{2^{k+1}\,2!\,(k+1)!}\,(2k-2)! = -\frac{k\,(2k-1)\,(k+1)\left(k^2-k+3\right)}{2^{k-1}\,(k+1)!}\,(2k-2)! \\
&= -\frac{k^2-k+3}{2^{k-1}\,(k-1)!}\,(2k-1)! = -\left(k^2-k+3\right)(2k-1)!!, \quad k\in\mathbb{N}\backslash\{1,2\}.
\end{aligned} \tag{12.68}
$$

For equation (12.65), 3 is substituted for μ.

$$
\begin{aligned}
c_{k,3} &= -\frac{2k\left(8k^4-44k^3+124k^2-124k+36\right)}{2^k\,3!\,k!}\,(2k-3)! \\
&= -\frac{2k\,(2k-1)\,(k-1)\left(k^2-4k+9\right)}{3\cdot 2^{k-1}\cdot k!}\,(2k-3)! \\
&= -\frac{k^2-4k+9}{3\cdot 2^{k-1}\cdot (k-1)!}\,(2k-1)! = -\frac{k^2-4k+9}{3}\,(2k-1)!!, \quad k\in\mathbb{N}\backslash\{1,2\}.
\end{aligned} \tag{12.69}
$$

For equation (12.65), 1 is substituted for k, and 0,1, and 2 are substituted for μ.

$$
c_{1,\mu} = \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\left.\left(\begin{aligned}&\mu(\mu-1)^2\,(\mu-2)\,(\mu-3)\,(\mu+4) \\ &-2k\left(\begin{aligned}&8k^4-4k^3\left(\mu^2+5\mu-13\right) \\ &\quad+4k^2\left(2\mu^3+3\mu^2-26\mu+28\right) \\ &\quad-k\left(7\mu^4-2\mu^3-89\mu^2+168\mu-92\right) \\ &\quad+(\mu-1)\left(3\mu^4-2\mu^3-39\mu^2+68\mu-24\right)\end{aligned}\right)\end{aligned}\right)\right|_{k=1}, \quad (\mu=0,1,2)\,. \tag{12.70}
$$

The results of the above calculation are shown.

$$
\{c_{1,2},c_{1,1},c_{1,0}\} = \{-3,-3,-3\}. \tag{12.71}
$$

Because the coefficients $c_{1,4}$ and $c_{1,3}$ have already determined,

$$
\{c_{1,4},c_{1,3},c_{1,2},c_{1,1},c_{1,0}\} = \{1,-2,-3,-3,-3\}. \tag{12.72}
$$

This result is the same as the coefficient elements of the representation for $\zeta\,(3)$.
For equation (12.65), 2 is substituted for k, and 0,1,2,3, and 4 are substituted for μ.

$$
c_{2,\mu} = \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\left.\left(\begin{aligned}&\mu(\mu-1)^2\,(\mu-2)\,(\mu-3)\,(\mu+4) \\ &-2k\left(\begin{aligned}&8k^4-4k^3\left(\mu^2+5\mu-13\right) \\ &\quad+4k^2\left(2\mu^3+3\mu^2-26\mu+28\right) \\ &\quad-k\left(7\mu^4-2\mu^3-89\mu^2+168\mu-92\right) \\ &\quad+(\mu-1)\left(3\mu^4-2\mu^3-39\mu^2+68\mu-24\right)\end{aligned}\right)\end{aligned}\right)\right|_{k=2}, \quad (\mu=0,1,2,3,4)\,. \tag{12.73}
$$

The results of the above calculation are shown.

$$
\{c_{2,4},c_{2,3},c_{2,2},c_{2,1},c_{2,0}\} = \{0,-5,-15,-30,-30\}. \tag{12.74}
$$

Because the coefficient $c_{2,5}$ has already determined,

$$
\{c_{2,5},c_{2,4},c_{2,3},c_{2,2},c_{2,1},c_{2,0}\} = \{1,0,-5,-15,-30,-30\}. \tag{12.75}
$$

This result is the same as the coefficient elements of the representation for $\zeta\,(5)$.
The above results are combined with the results of equations (12.66), (12.67), (12.68), and (12.69). Therefore,

$$
c_{k,1} = c_{k,0} = -k\,(2k+1)!!, \quad k\in\mathbb{N}. \tag{12.76}
$$

(12.77)
$$c_{k,2} = -\left(k^2 - k + 3\right)(2k-1)!!, \quad k \in \mathbb{N}.$$

(12.78)
$$c_{k,3} = -\frac{k^2 - 4k + 9}{3}(2k-1)!!, \quad k \in \mathbb{N}.$$

Combining the above results, I arrive at one of the conclusions in this subsection.
The following is the second general representation for the zeta function for any odd number of 3 or more:

(12.79)
$$\zeta(2k+1) = \frac{2}{k(2k+1)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left(\begin{array}{r}(2\pi p)^{k+3} + \dfrac{(k-2)(k+3)}{2}(2\pi p)^{k+2} \\ + \displaystyle\sum_{\mu=0}^{k+1} c_{k,\mu}(2\pi p)^{\mu}\end{array}\right)e^{-2\pi p}, \quad k \in \mathbb{N}.$$

Where

(12.80)
$$c_{k,\mu} = \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\left(\begin{array}{l}\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) \\ -2k\left(\begin{array}{r}8k^4 - 4k^3\left(\mu^2+5\mu-13\right) \\ +4k^2\left(2\mu^3+3\mu^2-26\mu+28\right) \\ -k\left(7\mu^4-2\mu^3-89\mu^2+168\mu-92\right) \\ +(\mu-1)\left(3\mu^4-2\mu^3-39\mu^2+68\mu-24\right)\end{array}\right)\end{array}\right),$$
$$(k \in \mathbb{N}, \mu = 0, 1, 2, \cdots, k+1).$$

Hereafter, I consider the third general representation differently.
For equation (12.80), $k+3$ is substituted directly for μ under the condition of $k > 2$.
(12.81)
$$c_{k,k+3} = \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\left.\left(\begin{array}{l}\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) \\ -2k\left(\begin{array}{r}8k^4 - 4k^3\left(\mu^2+5\mu-13\right) \\ +4k^2\left(2\mu^3+3\mu^2-26\mu+28\right) \\ -k\left(7\mu^4-2\mu^3-89\mu^2+168\mu-92\right) \\ +(\mu-1)\left(3\mu^4-2\mu^3-39\mu^2+68\mu-24\right)\end{array}\right)\end{array}\right)\right|_{\mu=k+3},$$
$$k \in \mathbb{N}\backslash\{1, 2\}.$$

I show the final step of the calculation.

(12.82)
$$c_{k,k+3} = \frac{(k+3)(k+2)(k+1)k(k-1)(k-2)}{(k+3)!}(k-3)! = 1, \quad k \in \mathbb{N}\backslash\{1, 2\}.$$

For equation (12.80), $k+2$ is substituted directly for μ under the condition of $k > 1$.
(12.83)
$$c_{k,k+2} = \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\left.\left(\begin{array}{l}\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) \\ -2k\left(\begin{array}{r}8k^4 - 4k^3\left(\mu^2+5\mu-13\right) \\ +4k^2\left(2\mu^3+3\mu^2-26\mu+28\right) \\ -k\left(7\mu^4-2\mu^3-89\mu^2+168\mu-92\right) \\ +(\mu-1)\left(3\mu^4-2\mu^3-39\mu^2+68\mu-24\right)\end{array}\right)\end{array}\right)\right|_{\mu=k+2},$$
$$k \in \mathbb{N}\backslash\{1\}.$$

I show the final step of the calculation.

(12.84)
$$c_{k,k+2} = \frac{(k+3)(k+2)(k+1)k(k-1)(k-2)}{2(k+2)!}(k-2)! = \frac{(k-2)(k+3)}{2}, \quad k \in \mathbb{N}\backslash\{1\}.$$

The above results are the same as the coefficients of both $(2\pi p)^{k+3}$ and $(2\pi p)^{k+2}$ of equation (12.79) under the condition of $k > 2$. And I arrive at another conclusion in this subsection.
The following is the third general representation for the zeta function for any odd number of 7 or more:

(12.85)
$$\zeta(2k+1) = \frac{2}{k(2k+1)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left(\sum_{\mu=0}^{k+3} c_{k,\mu}(2\pi p)^{\mu}\right)e^{-2\pi p}, \quad k \in \mathbb{N}\backslash\{1, 2\}.$$

Where

(12.86)
$$c_{k,\mu}=\frac{(2k-\mu)!}{2^{k+3-\mu}\mu!\,(k+3-\mu)!}\left(\begin{array}{l}\mu(\mu-1)^2\,(\mu-2)\,(\mu-3)\,(\mu+4)\\ -\,2k\left(\begin{array}{l}8k^4-4k^3\left(\mu^2+5\mu-13\right)\\ \qquad +\,4k^2\left(2\mu^3+3\mu^2-26\mu+28\right)\\ \qquad -\,k\left(7\mu^4-2\mu^3-89\mu^2+168\mu-92\right)\\ \qquad +\,(\mu-1)\left(3\mu^4-2\mu^3-39\mu^2+68\mu-24\right)\end{array}\right)\end{array}\right),$$
$$(k\in\mathbb{N}\backslash\{1,2\},\mu=0,1,2,\cdots,k+3)\,.$$

And the coefficients deserving special mention are as follows:

(12.87)
$$c_{k,k+3}=1,\quad k\in\mathbb{N}.$$

(12.88)
$$c_{k,k+2}=\frac{(k-2)\,(k+3)}{2},\quad k\in\mathbb{N}.$$

(12.89)
$$c_{k,3}=-\frac{k^2-4k+9}{3}\,(2k-1)!!,\quad k\in\mathbb{N}.$$

(12.90)
$$c_{k,2}=-\left(k^2-k+3\right)(2k-1)!!,\quad k\in\mathbb{N}.$$

(12.91)
$$c_{k,1}=c_{k,0}=-k\,(2k+1)!!,\quad k\in\mathbb{N}.$$

Table of the coefficients of the infinite series

k	1	2	3	4	5	6	7	8
$\frac{2}{k(2k+1)!!}$	$\frac{2}{3}$	$\frac{1}{15}$	$\frac{2}{315}$	$\frac{1}{1890}$	$\frac{2}{51975}$	$\frac{1}{405405}$	$\frac{2}{14189175}$	$\frac{1}{137837700}$

Table. 12.1

Table of the coefficients $c_{k,\mu}$

k	$c_{k,11}$	$c_{k,10}$	$c_{k,9}$	$c_{k,8}$	$c_{k,7}$	$c_{k,6}$	$c_{k,5}$	$c_{k,4}$	$c_{k,3}$	$c_{k,2}$	$c_{k,1}$	$c_{k,0}$
1								1	-2	-3	-3	-3
2							1	0	-5	-15	-30	-30
3						1	3	0	-30	-135	-315	-315
4					1	7	21	0	-315	-1575	-3780	-3780
5				1	12	70	210	-105	-4410	-21735	-51975	-51975
6			1	18	162	882	2457	-3780	-72765	-343035	-810810	-810810
7		1	25	315	2520	12915	31185	-103950	-1351350	-6081075	-14189175	-14189175
8	1	33	550	5940	44055	211365	405405	-2702700	-27702675	-119594475	-275675400	-275675400

Table. 12.2

The approximate formula for the zeta function is written as $\zeta_\lambda\,(2k+1)$.
(12.92)
$$\zeta_\lambda\,(2k+1):=\frac{2}{k\,(2k+1)!!}\sum_{p=1}^{\lambda}\sigma_{-(2k+1)}\,(p)\left(\begin{array}{l}(2\pi p)^{k+3}+\dfrac{(k-2)\,(k+3)}{2}(2\pi p)^{k+2}\\ \qquad\qquad +\displaystyle\sum_{\mu=0}^{k+1}c_{k,\mu}(2\pi p)^{\mu}\end{array}\right)e^{-2\pi p},\quad(\lambda,k\in\mathbb{N})\,.$$

When calculating the values of the function $\zeta_\lambda\,(2k+1)$, the Mathematica inputs are shown as follows.

Alt + 9 (Creating a new input line on a notebook)

$$\mathrm{c}[\{\mathrm{k}\_,\mu\_\}] := \frac{(2\,\mathrm{k}-\mu)!}{2^{\mathrm{k}+3-\mu}\,\mu!\,(\mathrm{k}+3-\mu)!}\left(\begin{array}{l}\mu\,(\mu-1)^2\,(\mu-2)\,(\mu-3)\,(\mu+4)\\ -\,2\,\mathrm{k}\left(\begin{array}{r}8\,\mathrm{k}^4-4\,\mathrm{k}^3\,(\mu^2+5\,\mu-13)\\ +\,4\,\mathrm{k}^2\,(2\,\mu^3+3\,\mu^2-26\,\mu+28)\\ -\,\mathrm{k}\,(7\,\mu^4-2\,\mu^3-89\,\mu^2+168\,\mu-92)\\ +\,(\mu-1)\,(3\,\mu^4-2\,\mu^3-39\,\mu^2+68\,\mu-24)\end{array}\right)\end{array}\right);$$

Shift + Enter

Alt + 9 (Creating the second input line)

zeta [{k_, λ_}] :=

$$\frac{2}{\mathrm{k}\,(2\,\mathrm{k}{+}1)!!}\,\mathrm{Sum}\left[\begin{array}{l}\mathrm{DivisorSigma}\,[-\,(2\,\mathrm{k}{+}1)\,,\mathrm{p}]\\ \times\left(\begin{array}{l}(2\,\pi\,\mathrm{p})^{\mathrm{k}+3}+\dfrac{(\mathrm{k}{-}2)\,(\mathrm{k}{+}3)}{2}\,(2\,\pi\,\mathrm{p})^{\mathrm{k}+2}\\ \qquad +\,\mathrm{Sum}\,[\mathrm{c}\,[\{\mathrm{k},\mu\}]\,(2\,\pi\,\mathrm{p})^{\mu},\{\mu,0,\mathrm{k}{+}1\}]\end{array}\right)e^{-2\,\pi\,\mathrm{p}},\{\mathrm{p},1,\lambda\}\end{array}\right];$$

Shift + Enter

Alt + 9 (Creating the third input line)

N [{zeta [{1, 1}] , zeta [{1, 2}] , zeta [{1, 3}] , zeta [{1, 4}] , zeta [{1, 5}] , Zeta [3]}, 10] Shift + Enter

Note: Valid for Mathmatica Ver. 12.0 or later.

By increasing the degree of approximation, it asymptotes the true value of the zeta function for any odd number of 3 or more.

(12.93)
$$\{\zeta_1(3),\zeta_2(3),\zeta_3(3),\zeta_4(3),\zeta_5(3),\zeta(3)\}$$
$$\simeq\{1.148054268, 1.201550493, 1.202053511, 1.202056889, 1.202056903, 1.202056903\}.$$

(12.94)
$$\{\zeta_1(5),\zeta_2(5),\zeta_3(5),\zeta_4(5),\zeta_5(5),\zeta(5)\}$$
$$\simeq\{0.9638129674, 1.035899189, 1.036919392, 1.036927709, 1.036927755, 1.036927755\}.$$

(12.95)
$$\{\zeta_1(7),\zeta_2(7),\zeta_3(7),\zeta_4(7),\zeta_5(7),\zeta(7)\}$$
$$\simeq\{0.8992359468, 1.006186929, 1.008327215, 1.008349125, 1.008349277, 1.008349277\}.$$

(12.96)
$$\{\zeta_1(9),\zeta_2(9),\zeta_3(9),\zeta_4(9),\zeta_5(9),\zeta_6(9),\zeta(9)\}$$
$$\simeq\{0.8448946784, 0.9977962524, 1.001954409, 1.002007939, 1.002008390, 1.002008393, 1.002008393\}.$$

By choosing the approximation appropriately, for example, a value can be obtained with 30 decimal places of precision.

(12.97)
$$\{\zeta_{13}(3),\zeta(3)\}$$
$$\simeq\{1.202056903159594285399738161511, 1.202056903159594285399738161511\}.$$

(12.98)
$$\{\zeta_{14}(5),\zeta(5)\}$$
$$\simeq\{1.036927755143369926331365486457, 1.036927755143369926331365486457\}.$$

(12.99)
$$\{\zeta_{14}(7),\zeta(7)\}$$
$$\simeq\{1.008349277381922826839797549850, 1.008349277381922826839797549850\}.$$

(12.100)
$$\{\zeta_{14}(9),\zeta(9)\}$$
$$\simeq\{1.002008392826082214417852769232, 1.002008392826082214417852769232\}.$$

12.3 Two Special Series with Fixed Values

The left side of equation (12.30) can be written as follows:

(12.101)
$$\begin{aligned}\sum_{p=1}^{\infty}\sigma_{-5}(p)(2\pi p)^5 e^{-2\pi p} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^5}(2\pi mn)^5 e^{-2\pi mn}\\ &= (2\pi)^5\sum_{n=1}^{\infty}\frac{15}{8}\left(\frac{2}{15}+\frac{1}{\sinh^2(\pi n)}+\frac{1}{\sinh^4(\pi n)}\right)\frac{1}{\sinh^2(\pi n)}\\ &= 60\,\pi^5\sum_{n=1}^{\infty}\left(\frac{2}{15}+\frac{1}{\sinh^2(\pi n)}+\frac{1}{\sinh^4(\pi n)}\right)\frac{1}{\sinh^2(\pi n)}.\end{aligned}$$

Therefore

(12.102)
$$\sum_{n=1}^{\infty}\left(\frac{2}{15}+\frac{1}{\sinh^2(\pi n)}+\frac{1}{\sinh^4(\pi n)}\right)\frac{1}{\sinh^2(\pi n)}=\frac{1}{60\,\pi^5}\cdot\frac{4\,\pi^5}{63}=\frac{1}{945}.$$

The following formulae are known "[13]":

(12.103)
$$\sum_{n=1}^{\infty}\frac{1}{\sinh^2(\pi n)}=\frac{1}{6}-\frac{1}{2\pi}.$$

(12.104)
$$\sum_{n=1}^{\infty}\frac{1}{\sinh^4(\pi n)}=-\frac{11}{90}+\frac{1}{3\pi}+\frac{1}{1920\,\pi^6}\Gamma^8\left(\frac{1}{4}\right).$$

The above formulae are introduced into the equation (12.102) to obtain the following series with fixed value:

(12.105)
$$\sum_{n=1}^{\infty}\frac{1}{\sinh^6(\pi n)}=\frac{191}{1890}-\frac{4}{15\pi}-\frac{1}{1920\,\pi^6}\Gamma^8\left(\frac{1}{4}\right).$$

13. Function that Takes the Constant 0 Anywhere in the Whole Complex Plane, and Some Series that Give the Algebraic Number 0

The fifth-order $\mathrm{II_c}$ type functional equation is shown again.

(13.1)
$$\begin{aligned}&\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+4\theta+5\right)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)\\ &=\frac{\theta(\theta+1)(\theta+2)\left(\theta^2+6\theta+10\right)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)+H_{[5]}(\theta),\quad \theta\in\mathbb{C}.\end{aligned}$$

Where

(13.2)
$$H_{[5]}(\theta)=2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{aligned}&\left(\theta^5+4\theta^4+4\theta^3-4\theta^2-5\theta\right.\\ &\qquad\left.+\left(3\theta^3-11\theta^2-52\theta-150+(\theta-21)(2\pi p)^2\right)(2\pi p)^2\right)\\ &\qquad\times \mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ &+\left(2\theta^4+13\theta^3+34\theta^2+38\theta+15+2\left(3\theta^2+13\theta+39+(2\pi p)^2\right)(2\pi p)^2\right)\\ &\qquad\times(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{aligned}\right).$$

From (the third-order $\mathrm{II_c}$ type functional equation)$\times\left(\theta^2+6\theta+10\right)$,

(13.3)
$$\begin{aligned}&-\frac{\theta(\theta-1)(\theta+1)\left(\theta^2+6\theta+10\right)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)\\ &=-\frac{\theta(\theta+1)(\theta+2)\left(\theta^2+6\theta+10\right)}{2}\pi^{-\frac{1+\theta}{2}}\Gamma\left(\frac{1+\theta}{2}\right)\zeta(1+\theta)+\left(\theta^2+6\theta+10\right)H_{[3]}(\theta),\quad \theta\in\mathbb{C}.\end{aligned}$$

Where

$$H_{[3]}(\theta)=2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(\theta(\theta-1)(\theta+1)+(\theta-7)(2\pi p)^{2}\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+\left((2\theta+1)(\theta+1)+2(2\pi p)^{2}\right)(2\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{array}\right). \tag{13.4}$$

From (equation (13.1))−(equation (13.3)),

$$\frac{\theta(\theta-1)(\theta+1)(2\theta+5)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)=H_{[5]}(\theta)-\left(\theta^{2}+6\theta+10\right)H_{[3]}(\theta),\quad \theta\in\mathbb{C}. \tag{13.5}$$

In addition, from (equation (11.7))× $(\theta+1)(2\theta+5)$,

$$\begin{aligned}&-\frac{\theta(\theta-1)(\theta+1)(2\theta+5)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)\\&=2(\theta+1)(2\theta+5)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^{2}\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+\left((\theta-1)-2(2\pi p)^{2}\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}.\end{aligned} \tag{13.6}$$

And also, from (equation (13.5))+(equation (13.6)),

$$\begin{aligned}&H_{[5]}(\theta)-\left(\theta^{2}+6\theta+10\right)H_{[3]}(\theta)\\&+2(\theta+1)(2\theta+5)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^{2}\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+\left((\theta-1)-2(2\pi p)^{2}\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right)=0,\quad \theta\in\mathbb{C}.\end{aligned} \tag{13.7}$$

Substituting the results of $H_{[3]}(\theta)$ and $H_{[5]}(\theta)$ for the calculation, the following equation is obtained:

$$\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(-3(\theta-3)(\theta-5)+(\theta-21)(2\pi p)^{2}\right)(2\pi p)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+2\left(24+(2\pi p)^{2}\right)(2\pi p)^{3}\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right)=0,\quad \theta\in\mathbb{C}. \tag{13.8}$$

Let define the left side of equation (13.8) as the function $\omega(\theta)$.

$$\omega(\theta):=\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(-3(\theta-3)(\theta-5)+(\theta-21)(2\pi p)^{2}\right)(2\pi p)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+2\left(24+(2\pi p)^{2}\right)(2\pi p)^{3}\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}. \tag{13.9}$$

For above equation, $-\theta$ is substituted for θ, and I perform the calculation:

$$\begin{aligned}\omega(-\theta)&=\sum_{p=1}^{\infty}\sigma_{\theta}(p)\,p^{\frac{-\theta}{2}}\left(\begin{array}{r}\left(-3(-\theta-3)(-\theta-5)+(-\theta-21)(2\pi p)^{2}\right)(2\pi p)^{2}\,\mathrm{K}_{\frac{-\theta}{2}}(2\pi p)\\+4\left(24+(2\pi p)^{2}\right)(2\pi p)^{2}(\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\end{array}\right)\\&=\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(-3(\theta+3)(\theta+5)-(\theta+21)(2\pi p)^{2}\right)(2\pi p)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+4\left(24+(2\pi p)^{2}\right)(2\pi p)^{2}\left(\frac{\theta}{2}\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+(\pi p)\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\right)\end{array}\right)\\&=\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(-3(\theta-3)(\theta-5)+(\theta-21)(2\pi p)^{2}\right)(2\pi p)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+2\left(24+(2\pi p)^{2}\right)(2\pi p)^{3}\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}.\end{aligned} \tag{13.10}$$

As a result, I obtained the function $\omega(\theta)$ that takes the constant 0 anywhere in the whole complex plane.

$$\omega(\pm\theta)=\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}\left(-3(\theta-3)(\theta-5)+(\theta-21)(2\pi p)^{2}\right)(2\pi p)^{2}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\+2\left(24+(2\pi p)^{2}\right)(2\pi p)^{3}\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}. \tag{13.11}$$

In order to obtain a representation that has quadrable integrals, I modify the right side of equation (13.11) as follows:

$$\omega(\pm\theta) = \pi^{-\frac{\theta}{2}} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(\begin{array}{r} \left(-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2\right)(2\pi p)^2 (\pi p)^{\frac{\theta}{2}} \mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ + \left(24 + (2\pi p)^2\right)(2\pi p)^4 (\pi p)^{\frac{2-\theta}{2}} \mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \end{array} \right), \quad \theta \in \mathbb{C}. \tag{13.12}$$

From equations (12.3) and (12.4),

$$\omega(\theta) = \pi^{-\frac{\theta}{2}} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(\begin{array}{r} \left(-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2\right)(2\pi p)^2 \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \\ + \left(24 + (2\pi p)^2\right)(2\pi p)^4 \int_0^\infty x^{\theta-3} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right), \tag{13.13}$$

$\theta \in \mathbb{C}$.

Two integrals of the right side of equation (13.13) are quadrable when θ is any odd number of 1 or more. Some calculating examples will be shown. When $\theta = 1$,

$$\begin{aligned} \omega(1) &= \pi^{-\frac{1}{2}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(\begin{array}{r} \left(-24 - 20(2\pi p)^2\right)(2\pi p)^2 \int_0^\infty x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \\ + \left(24 + (2\pi p)^2\right)(2\pi p)^4 \int_0^\infty x^{-2} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right) \\ &= \frac{1}{\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(\left(-24 - 20(2\pi p)^2\right)(2\pi p)^2 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi p} + \left(24 + (2\pi p)^2\right)(2\pi p)^4 \cdot \frac{\sqrt{\pi}}{2\pi p} e^{-2\pi p} \right) \\ &= \sum_{p=1}^{\infty} \sigma_{-1}(p) \left((2\pi p)^3 - 10(2\pi p)^2 + 24(2\pi p) - 12\right)(2\pi p)^2 e^{-2\pi p}. \end{aligned} \tag{13.14}$$

Therefore

$$\sum_{p=1}^{\infty} \sigma_{-1}(p) \left((2\pi p)^3 - 10(2\pi p)^2 + 24(2\pi p) - 12\right)(2\pi p)^2 e^{-2\pi p} = \omega(\pm 1) = 0. \tag{13.15}$$

When $\theta = 3$,

$$\begin{aligned} \omega(3) &= \pi^{-\frac{3}{2}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(\begin{array}{r} \left(0 - 18(2\pi p)^2\right)(2\pi p)^2 \int_0^\infty x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \\ + \left(24 + (2\pi p)^2\right)(2\pi p)^4 \int_0^\infty x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right) \\ &= \frac{1}{\pi\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(-18(2\pi p)^4 \cdot \frac{(2\pi p) + 1}{4} \sqrt{\pi} e^{-2\pi p} + \left(24 + (2\pi p)^2\right)(2\pi p)^4 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi p} \right) \\ &= \frac{1}{2\pi} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^2 - 9(2\pi p) + 15\right)(2\pi p)^4 e^{-2\pi p}. \end{aligned} \tag{13.16}$$

Therfore

$$\sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^2 - 9(2\pi p) + 15\right)(2\pi p)^4 e^{-2\pi p} = 2\pi\omega(\pm 3) = 0. \tag{13.17}$$

When $\theta = 5$,

$$\omega(5) = \pi^{-\frac{5}{2}} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(\begin{array}{r} \left(0 - 16(2\pi p)^2\right)(2\pi p)^2 \int_0^\infty x^4 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \\ + \left(24 + (2\pi p)^2\right)(2\pi p)^4 \int_0^\infty x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \end{array} \right)$$

$$= \frac{1}{\pi^2\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-1}(p)\left(\begin{array}{r}-16(2\pi p)^4\cdot\dfrac{(2\pi p)^2+3(2\pi p)+3}{8}\sqrt{\pi}\,e^{-2\pi p}\\+\left(24+(2\pi p)^2\right)(2\pi p)^4\cdot\dfrac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p}\end{array}\right) \tag{13.18}$$

$$= \frac{1}{4\pi^2}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)-7\right)(2\pi p)^6 e^{-2\pi p}.$$

Therfore

$$\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)-7\right)(2\pi p)^6 e^{-2\pi p} = 4\pi^2\omega(\pm 5) = 0. \tag{13.19}$$

When $\theta = 7$,

$$\omega(7) = \pi^{-\frac{7}{2}}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left(\begin{array}{r}\left(-24-14(2\pi p)^2\right)(2\pi p)^2\displaystyle\int_0^{\infty}x^6 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\+\left(24+(2\pi p)^2\right)(2\pi p)^4\displaystyle\int_0^{\infty}x^4 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{array}\right)$$

$$= \frac{1}{\pi^3\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-1}(p)\left(\begin{array}{r}\left(-24-14(2\pi p)^2\right)(2\pi p)^2\cdot\dfrac{(2\pi p)^2+6(2\pi p)^2+15(2\pi p)+15}{16}\sqrt{\pi}\,e^{-2\pi p}\\+\left(24+(2\pi p)^2\right)(2\pi p)^4\cdot\dfrac{(2\pi p)^2+3(2\pi p)+3}{8}\sqrt{\pi}\,e^{-2\pi p}\end{array}\right) \tag{13.20}$$

$$= \frac{1}{8\pi^3}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left(\begin{array}{r}(2\pi p)^6-4(2\pi p)^5-15(2\pi p)^4-45(2\pi p)^3\\-105(2\pi p)^2-180(2\pi p)-180\end{array}\right)(2\pi p)^2 e^{-2\pi p}.$$

Therefore

$$\sum_{p=1}^{\infty}\sigma_{-7}(p)\left(\begin{array}{r}(2\pi p)^6-4(2\pi p)^5-15(2\pi p)^4-45(2\pi p)^3\\-105(2\pi p)^2-180(2\pi p)-180\end{array}\right)(2\pi p)^2 e^{-2\pi p} = 8\pi^3\omega(\pm 7) = 0. \tag{13.21}$$

When $\theta = 9$, 11, 13, 15, and 17, only the results will be shown.

$$\sum_{p=1}^{\infty}\sigma_{-9}(p)\left(\begin{array}{r}(2\pi p)^7-21(2\pi p)^5-147(2\pi p)^4-630(2\pi p)^3\\-1890(2\pi p)^2-3780(2\pi p)-3780\end{array}\right)(2\pi p)^2 e^{-2\pi p} = 16\pi^4\omega(\pm 9) = 0. \tag{13.22}$$

$$\sum_{p=1}^{\infty}\sigma_{-11}(p)\left(\begin{array}{r}(2\pi p)^8+5(2\pi p)^7-6(2\pi p)^6\\-252(2\pi p)^5-1995(2\pi p)^4-9765(2\pi p)^3\\-32445(2\pi p)^2-68040(2\pi p)-68040\end{array}\right)(2\pi p)^2 e^{-2\pi p} = 32\pi^5\omega(\pm 11) = 0. \tag{13.23}$$

$$\sum_{p=1}^{\infty}\sigma_{-13}(p)\left(\begin{array}{r}(2\pi p)^9+11(2\pi p)^8+45(2\pi p)^7-180(2\pi p)^6\\-4095(2\pi p)^5-33075(2\pi p)^4-170100(2\pi p)^3\\-585900(2\pi p)^2-1247400(2\pi p)-1247400\end{array}\right)(2\pi p)^2 e^{-2\pi p} = 64\pi^6\omega(\pm 13) = 0. \tag{13.24}$$

$$\sum_{p=1}^{\infty}\sigma_{-15}(p)\left(\begin{array}{r}(2\pi p)^{10}+18(2\pi p)^9+150(2\pi p)^8+450(2\pi p)^7\\-4725(2\pi p)^6-79380(2\pi p)^5-630315(2\pi p)^4-3274425(2\pi p)^3\\-11382525(2\pi p)^2-24324300(2\pi p)-24324300\end{array}\right)(2\pi p)^2 e^{-2\pi p} \tag{13.25}$$

$$= 128\pi^7\omega(\pm 15) = 0.$$

$$\sum_{p=1}^{\infty}\sigma_{-17}(p)\left(\begin{array}{r}(2\pi p)^{11}+26(2\pi p)^{10}+330(2\pi p)^9\\+2310(2\pi p)^8+3465(2\pi p)^7-124700(2\pi p)^6\\-1735965(2\pi p)^5-13357575(2\pi p)^4-68918850(2\pi p)^3\\-239188950(2\pi p)^2-510810300(2\pi p)-510810300\end{array}\right)(2\pi p)^2 e^{-2\pi p} \tag{13.26}$$

$$= 256\pi^8\omega(\pm 17) = 0.$$

It is recognized that some series that seem to give transcendental numbers, however, give the algebraic number 0 at times. And the function $\omega(\theta)$ is the null function in the whole complex plane.
Unfortunately, no one knows whether the zeta value for any odd number of 3 or more is a transcendental number at the moment.

14. Derivation of an Explicit Formula for a New Eta Function and the Proof of the Generalized Riemann Hypothesis for the Eta Function

14.1 Derivation of an Explicit Formula for a New Eta Function

The following equation defines the eta function $\eta(\theta)$:

$$\eta(\theta) := \frac{\pi^{\frac{\theta}{2}}}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)}\chi(\theta), \quad \theta \in \mathbb{C}\backslash\{1\}. \tag{14.1}$$

According to the definition, the non-trivial zeros of the eta function are the same as those of the Chi function. Thus, I also propose the generalized Riemann hypothesis for the eta function that states that all non-trivial zeros lie on the imaginary axis. it will be proved soon.
Equation (9.23) is introduced into the equation of definition.

$$\begin{aligned}\eta(\theta) &= \frac{\pi^{\frac{\theta}{2}}}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)}\left(\begin{array}{l}\frac{-\theta(1-\theta)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\zeta(\theta)\\ \quad -2(1-\theta)\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right)\end{array}\right)\\ &= \zeta(\theta)+\frac{2\pi^{\frac{\theta}{2}}}{\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\theta\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta\in\mathbb{C}\backslash\{1\}.\end{aligned} \tag{14.2}$$

The explicit formula for the zeta function is introduced into the equation (14.2).

$$\begin{aligned}\eta(\theta) =& \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad +\left((\theta-1)-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right)\\ &-\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\theta(\theta-1)\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad +(\theta-1)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right), \quad \theta\in\mathbb{C}\backslash\{1\}.\end{aligned} \tag{14.3}$$

By simplifying, I obtain the explicit formula for the eta function.

$$\eta(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((7-\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)-2(2\pi p)^3\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad \theta\in\mathbb{C}\backslash\{1\}. \tag{14.4}$$

For the explicit formula, $-\theta$ is substituted for θ.

$$\eta(-\theta) = \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{\theta}(p)\,p^{-\frac{\theta}{2}}\left(\begin{array}{l}(7+\theta)(2\pi p)^2\,\mathrm{K}_{-\frac{\theta}{2}}(2\pi p)\\ \quad -2(2\pi p)^3\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\end{array}\right), \quad \theta\in\mathbb{C}\backslash\{-1\}. \tag{14.5}$$

The relation between the equations (3.30) and (5.39) is applied to the equation (14.5).

$$\eta(-\theta) = \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}(7+\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad -4(2\pi p)^2(\pi p)\,\mathrm{K}_{\frac{2+\theta}{2}}(2\pi p)\end{array}\right), \quad \theta\in\mathbb{C}\backslash\{-1\}. \tag{14.6}$$

The recurrence formula for the modified Bessel function of the second kind is applied.

$$\begin{aligned}\eta(-\theta) &= \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}(7+\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad -4(2\pi p)^2\left(\left(\frac{\theta}{2}\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)+(\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right)\end{array}\right)\\ &= \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}(7-\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad -2(2\pi p)^3\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{array}\right), \quad \theta\in\mathbb{C}\backslash\{-1\}.\end{aligned} \tag{14.7}$$

The difference between $\eta(\theta)$ and $\eta(-\theta)$ is essentially reflected in each gamma factor.
And the result of equation (14.7) will be used in Section 16.

14.2 The Proof of the Generalized Riemann Hypothesis for the Eta Function

Assuming that there are an infinitely many non-trivial zeros on the imaginary axis in the eta function, and both x and y are assumed to be real numbers, and $(x + iy)$ is assumed to be a non-trivial zero of the eta function.
Based on the origin symmetry of the Chi function, $(-x - iy)$ is also a non-trivial zero of the eta function.
Because it contains trivial zeros of the eta function, the real axis must be excluded from the target region. I consider that non-trivial zeros are determined as solutions of simultaneous equations consisting of the following equations:

$$\begin{cases} \eta(x+iy)=0 & (x\in\mathbb{R}, y\in\mathbb{R}\backslash\{0\}), \\ \eta(-x-iy)=0 & (x\in\mathbb{R}, y\in\mathbb{R}\backslash\{0\}). \end{cases} \tag{14.8}$$

For the explicit formula for the eta function, $(x+iy)$ is substituted for θ.

$$\eta(x+iy)=\frac{2\pi^{\frac{x+iy}{2}}}{(1-x-iy)\,\Gamma\left(1+\frac{x+iy}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-x-iy}(p)\,p^{\frac{x+iy}{2}}\begin{pmatrix}(7-x-iy)(2\pi p)^2\,\mathrm{K}_{\frac{x+iy}{2}}(2\pi p)\\ -2(2\pi p)^3\,\mathrm{K}_{\frac{2-x-iy}{2}}(2\pi p)\end{pmatrix}=0, \quad (x\in\mathbb{R}, y\in\mathbb{R}\backslash\{0\}). \tag{14.9}$$

Similarly, in the explicit formula for the eta function, $(-x-iy)$ is substituted for θ.
(14.10)

$$\eta(-x-iy)=\frac{2\pi^{\frac{-x-iy}{2}}}{(1+x+iy)\,\Gamma\left(1+\frac{-x-iy}{2}\right)}\sum_{p=1}^{\infty}\sigma_{x+iy}(p)\,p^{\frac{-x-iy}{2}}\begin{pmatrix}(7+x+iy)(2\pi p)^2\,\mathrm{K}_{\frac{-x-iy}{2}}(2\pi p)\\ -2(2\pi p)^3\,\mathrm{K}_{\frac{2+x+iy}{2}}(2\pi p)\end{pmatrix}=0, \quad (x\in\mathbb{R}, y\in\mathbb{R}\backslash\{0\}).$$

The non-trivial zeros are determined as the solutions of the simultaneous equations comprising the equations (14.9) and (14.10).
Owing to the symmetric pair of these equations,

$$x=0 \tag{14.11}$$

is immediately determined.
For equation (14.9), 0 is substituted for x.

$$\eta(iy)=\frac{2\pi^{\frac{iy}{2}}}{(1-iy)\,\Gamma\left(1+\frac{iy}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\begin{pmatrix}(7-iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p)\\ -2(2\pi p)^3\,\mathrm{K}_{\frac{2-iy}{2}}(2\pi p)\end{pmatrix}=0, \quad y\in\mathbb{R}\backslash\{0\}. \tag{14.12}$$

For equation (14.10), 0 is substituted for x.

$$\eta(-iy)=\frac{2\pi^{-\frac{iy}{2}}}{(1+iy)\,\Gamma\left(1-\frac{iy}{2}\right)}\sum_{p=1}^{\infty}\sigma_{iy}(p)\,p^{-\frac{iy}{2}}\begin{pmatrix}(7+iy)(2\pi p)^2\,\mathrm{K}_{-\frac{iy}{2}}(2\pi p)\\ -2(2\pi p)^3\,\mathrm{K}_{\frac{2+iy}{2}}(2\pi p)\end{pmatrix}=0, \quad y\in\mathbb{R}\backslash\{0\}. \tag{14.13}$$

The equations (12.12) and (12.13) are complex conjugate of each other in relation. Hence, I can choose either of these equations as the determining equation of the real variable y.
Furthermore, the gamma factor of the eta function does not have a value of zero on the imaginary axis.

Additionally, the condition of $y=0$ can be excepted, because the eta function has no non-trivial zero on the real axis. Therefore, the following equation gives the determining equation of the real variable y.

$$\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\left((7-iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p)-2(2\pi p)^3\,\mathrm{K}_{\frac{2-iy}{2}}(2\pi p)\right)=0, \quad y\in\mathbb{R}. \tag{14.14}$$

The explicit formula for the Chi function is also found.

$$\chi(\theta)=(\theta-1)\,\pi^{\frac{\theta}{2}}\,\Gamma\left(1+\frac{\theta}{2}\right)$$

$$= -2\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((7-\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p)^3\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right),\quad \theta\in\mathbb{C}. \tag{14.15}$$

The function value of the Chi function on the imaginary axis is

$$\chi(iy) = -2\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\left((7-iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3\,\mathrm{K}_{\frac{2-iy}{2}}(2\pi p)\right),\quad y\in\mathbb{R}. \tag{14.16}$$

Applying the recurrence formula and the origin symmetry with respect to the index of the modified Bessel function of the second kind,

$$\begin{aligned}\chi(iy) &= -2\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\begin{pmatrix}(7-iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p)\\ \quad +2(2\pi p)^2\left(-(2\pi p)\,\mathrm{K}_{\frac{iy-2}{2}}(2\pi p)\right)\end{pmatrix}\\ &= -2\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\begin{pmatrix}(7-iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p)\\ \quad +2(2\pi p)^2\left(iy\,\mathrm{K}_{\frac{iy}{2}}(2\pi p) - (2\pi p)\,\mathrm{K}_{\frac{2+iy}{2}}(2\pi p)\right)\end{pmatrix}\\ &= -2\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\left((7+iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3\,\mathrm{K}_{\frac{2+iy}{2}}(2\pi p)\right),\quad y\in\mathbb{R}.\end{aligned} \tag{14.17}$$

Using the complex conjugate on both sides of the above equation,

$$\begin{aligned}\chi^*(iy) &= -2\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\left((7-iy)(2\pi p)^2\,\mathrm{K}_{-\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3\,\mathrm{K}_{\frac{2-iy}{2}}(2\pi p)\right)\\ &= -2\sum_{p=1}^{\infty}\sigma_{iy}(p)\,p^{-\frac{iy}{2}}\left((7-iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3\,\mathrm{K}_{\frac{2-iy}{2}}(2\pi p)\right),\quad y\in\mathbb{R}.\end{aligned} \tag{14.18}$$

Here, the equation (5.39) is applied.
Therefore, it is shown that the Chi function takes real value on the imaginary axis as follows:

$$\mathrm{Im}(\chi(iy)) = \frac{1}{2i}(\chi(iy) - \chi^*(iy)) = 0,\quad y\in\mathbb{R}. \tag{14.19}$$

The Hadamard product representation for the Chi function on the imaginary axis is

$$\chi(iy) = \left(\frac{1}{2} - 2\sum_{p=1}^{\infty}\sigma_0(p)(2\pi p)\,\mathrm{K}_1(2\pi p)\right)\prod_{m=1}^{\infty}\left(1-\frac{y^2}{(\tau_m)^2}\right),\quad y\in\mathbb{R}. \tag{14.20}$$

From equations (14.16) and (14.20),

$$\begin{aligned}&\sum_{p=1}^{\infty}\sigma_{-iy}(p)\,p^{\frac{iy}{2}}\left((7-iy)(2\pi p)^2\,\mathrm{K}_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3\,\mathrm{K}_{\frac{2-iy}{2}}(2\pi p)\right)\\ &= -\frac{1}{2}\left(\frac{1}{2} - 2\sum_{p=1}^{\infty}\sigma_0(p)(2\pi p)\,\mathrm{K}_1(2\pi p)\right)\prod_{m=1}^{\infty}\left(1-\frac{y^2}{(\tau_m)^2}\right)\in\mathbb{R},\quad y\in\mathbb{R}.\end{aligned} \tag{14.21}$$

On the imaginary axis, the left side of the determining equation takes a real value. In this case, the imaginary axis corresponds to the critical line of the eta function.

It is possible to assert that there is no non-trivial zero off the imaginary axis.
At this stage, the proof of the generalized Riemann hypothesis for the eta function is complete. It should be noted that the existence of an infinite number of non-trivial zeros of the eta function remains unproven.

Furthermore, it is possible to design an eta function variant that takes real values on the imaginary axis. The method is to obtain the absolute value for the gamma factor of the eta function's explicit formula. I decide to write the eta function variation as eta-tilde by adding the symbol ˜ (tilde) just above the Greek letter η.

$$\tilde{\eta}(\theta) := \left| \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \right| \left((7-\theta)(2\pi p)^2 \,\mathrm{K}_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p)^3 \,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \right), \quad \theta \in \mathbb{C}\backslash\{1\}. \tag{14.22}$$

14.3 Behavior of the Number-Theoretic Function $\sigma_{-\theta}(n)\, n^{\theta/2}$, Part2

From now, I consider the behavior of the number-theoretic function $\sigma_{-\theta}(n)\, n^{\theta/2}$ on the imaginary axis by presenting specific examples.
In the following discussions, a and b are assumed to be two different arbitrary prime numbers.
The magnitude correlation of a and b does not matter.
When $n = 1$,

$$n^{\frac{iy}{2}} \sigma_{-iy}(n)\Big|_{n=1} = 1. \quad y \in \mathbb{R}. \tag{14.23}$$

When n is a prime number a,

$$a^{\frac{iy}{2}} \sigma_{-iy}(a) = \left(1 + \frac{1}{a^{iy}}\right) a^{\frac{iy}{2}} = 2\cos\left(\frac{y}{2}\log(a)\right), \quad y \in \mathbb{R}. \tag{14.24}$$

When n is the $(2k-1)$ power of a prime number a,

$$\left(a^{2k-1}\right)^{\frac{iy}{2}} \sigma_{-iy}\left(a^{2k-1}\right) = 2\sum_{m=1}^{k} \cos\left(\frac{(2m-1)\,y}{2}\log(a)\right), \quad (k \in \mathbb{N}, y \in \mathbb{R}). \tag{14.25}$$

When n is the $2k$ power of a prime number a,

$$\left(a^{2k}\right)^{\frac{iy}{2}} \sigma_{-iy}\left(a^{2k}\right) = 1 + 2\sum_{m=1}^{k} \cos\left(my\log(a)\right), \quad (k \in \mathbb{N}, y \in \mathbb{R}). \tag{14.26}$$

When n is a composite number $a \cdot b$,

$$\begin{aligned} (a\cdot b)^{\frac{iy}{2}} \sigma_{-iy}(a\cdot b) &= \left(1 + \frac{1}{a^{iy}} + \frac{1}{b^{iy}} + \frac{1}{(a\cdot b)^{iy}}\right)(a\cdot b)^{\frac{iy}{2}} \\ &= \left(1+\frac{1}{a^{iy}}\right)\left(1+\frac{1}{b^{iy}}\right)(a\cdot b)^{\frac{iy}{2}} \\ &= 4\cos\left(\frac{y}{2}\log(a)\right)\cos\left(\frac{y}{2}\log(b)\right), \quad y \in \mathbb{R}. \end{aligned} \tag{14.27}$$

When n is a composite number $a \cdot b^2$,

$$\begin{aligned} \left(a\cdot b^2\right)^{\frac{iy}{2}} \sigma_{-iy}\left(a\cdot b^2\right) &= \left(1 + \frac{1}{a^{iy}} + \frac{1}{b^{iy}} + \frac{1}{(a\cdot b)^{iy}} + \frac{1}{(b^2)^{iy}} + \frac{1}{(a\cdot b^2)^{iy}}\right)\left(a\cdot b^2\right)^{\frac{iy}{2}} \\ &= \left(1+\frac{1}{a^{iy}}\right)\left(1+\frac{1}{b^{iy}}+\frac{1}{(b^2)^{iy}}\right)\left(a\cdot b^2\right)^{\frac{iy}{2}} \\ &= 2\cos\left(\frac{y}{2}\log(a)\right)\left(1+2\cos\left(y\log(b)\right)\right), \quad y \in \mathbb{R}. \end{aligned} \tag{14.28}$$

The number-theoretic function $\sigma_{-\theta}(n)\, n^{\theta/2}$ is described by real cosine functions except $n = 1$ in the explicit formula for the eta function on the imaginary axis.
This is the main reason why I chose θ instead of S as the complex variable. The variable θ is conventionally associated with trigonometric functions.

15. Visualization of the Zeta Functions and Eta Functions

Let start by preparing some formulae for visualization.
The rule in subsection 6.3 is applied to the explicit formula for the zeta function.
In this case, the approximate formula for the zeta function is written as $\zeta_{\lambda}(\theta)$.

$$\zeta_\lambda(\theta) := \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\lambda} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \begin{pmatrix} \left(\theta(\theta-1) + (7-\theta)(2\pi p)^2\right) \mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ + \left((\theta-1) - 2(2\pi p)^2\right)(2\pi p)\, \mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \end{pmatrix}, \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}). \tag{15.1}$$

The rule is also applied for defining equation (11.23).

$$\tilde{\zeta}_\lambda(\theta) := \left|\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\right| \sum_{p=1}^{\lambda} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \begin{pmatrix} \left(\theta(\theta-1) + (7-\theta)(2\pi p)^2\right) \mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ + \left((\theta-1) - 2(2\pi p)^2\right)(2\pi p)\, \mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \end{pmatrix}, \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}). \tag{15.2}$$

The relation between $\zeta_\lambda(\theta)$ and *(zeta-hat)$_\lambda$ (θ) is*

$$\lim_{\lambda\to+\infty} \zeta_\lambda(\theta) = \lim_{\lambda\to+\infty} \hat{\zeta}_\lambda(\theta), \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}). \tag{15.3}$$

The relation between $\zeta_\lambda(\theta)$ and *(zeta-tilde)$_\lambda$ (θ) is*

$$|\zeta_\lambda(\theta)| = \left|\tilde{\zeta}_\lambda(\theta)\right|, \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}). \tag{15.4}$$

The rule is also applied to the explicit formula for the eta function.

$$\eta_\lambda(\theta) := \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\lambda} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 \mathrm{K}_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p)^3 \mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}). \tag{15.5}$$

The rule is also applied for defining equation (14.22).

$$\tilde{\eta}_\lambda(\theta) := \left|\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\right| \sum_{p=1}^{\lambda} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 \mathrm{K}_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p)^3 \mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right), \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}). \tag{15.6}$$

The relation between $\eta_\lambda(\theta)$ and *(eta-tilde)$_\lambda$(θ) is*

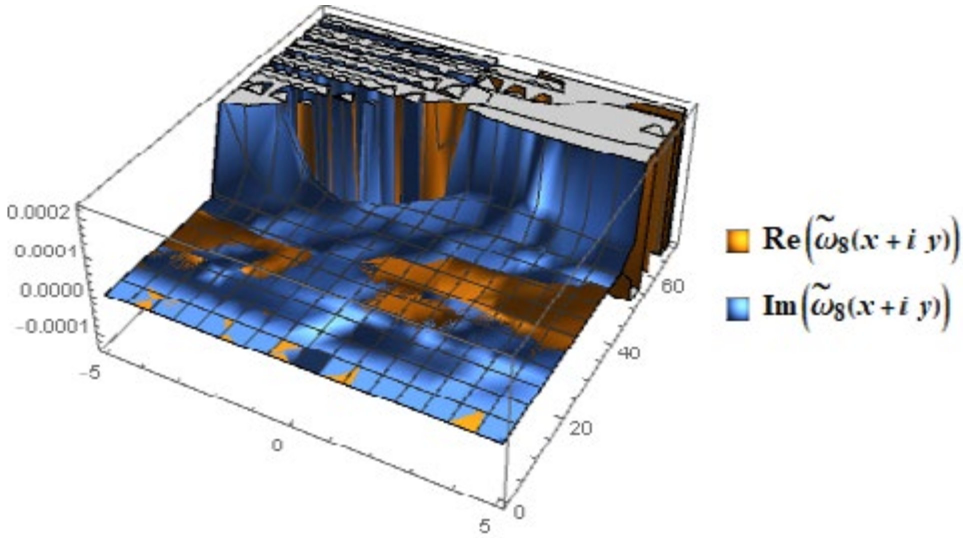
$$|\eta_\lambda(\theta)| = |\tilde{\eta}_\lambda(\theta)|, \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}). \tag{15.7}$$

The right side of the null function $\omega(\theta)$ is multiplied by the absolute value of the gamma factor of the zeta function to obtain the estimating function *(ω-tilde)$_\lambda$* (θ); moreover, the function*(ω-tilde)$_\infty$* (θ) is also the null function.

(15.8)

$$\tilde{\omega}_\lambda(\theta) := \left|\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\right| \sum_{p=1}^{\lambda} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \begin{pmatrix} \left(-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2\right)(2\pi p)^2 \mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ + 2\left(24 + 2(2\pi p)^2\right)(2\pi p)^3 \mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \end{pmatrix}, \quad (\theta \in \mathbb{C}\backslash\{1\}, \lambda \in \mathbb{N}).$$

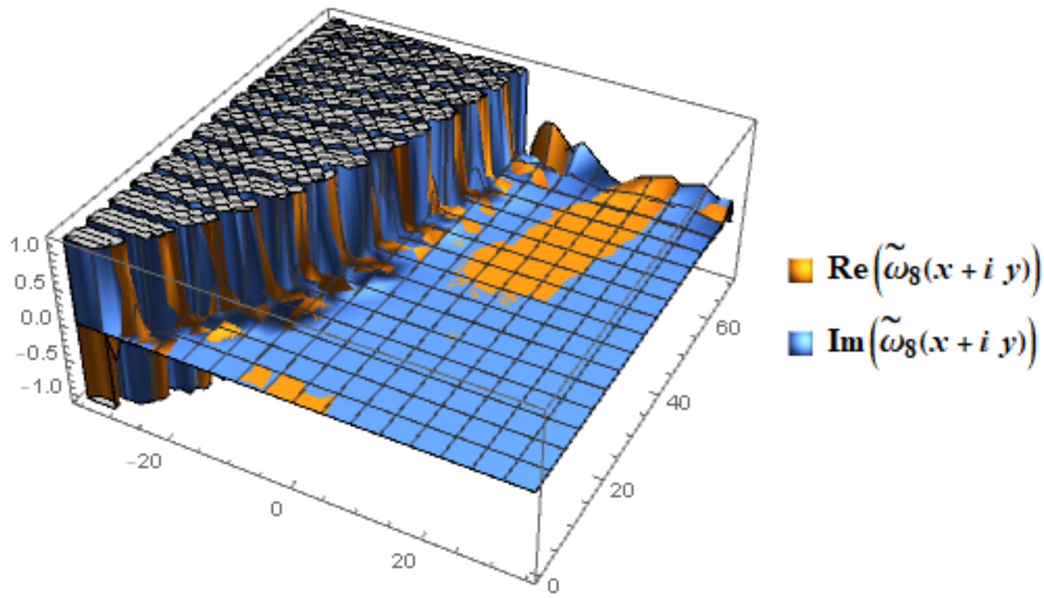
Three-dimensional graph of z=Re $(\tilde{\omega}_8\,(x+i\,y))$ and z=Im $(\tilde{\omega}_8\,(x+i\,y))$, $x \in [-5, 5]$, $y \in [0, 70]$



Graph. 15.1

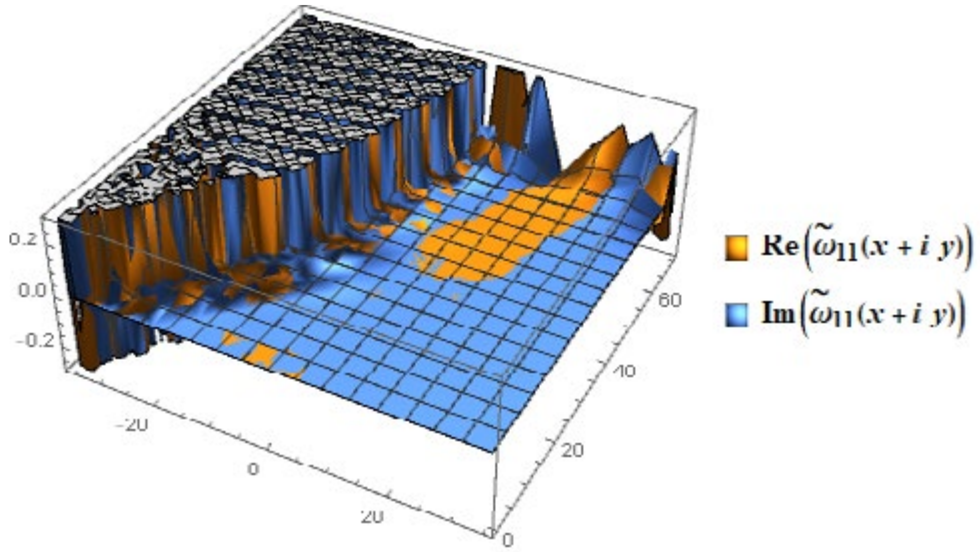
The above three-dimensional graph shows the computational error because the null function *(ω-tilde)*$_\infty$ (θ) takes the value zero anywhere in the whole complex plane.

Three-dimensional graph of z=Re $(\tilde{\omega}_8\,(x+i\,y))$ and z=Im $(\tilde{\omega}_8\,(x+i\,y))$, $x \in [-35, 35]$, $y \in [0, 70]$



Graph. 15.2

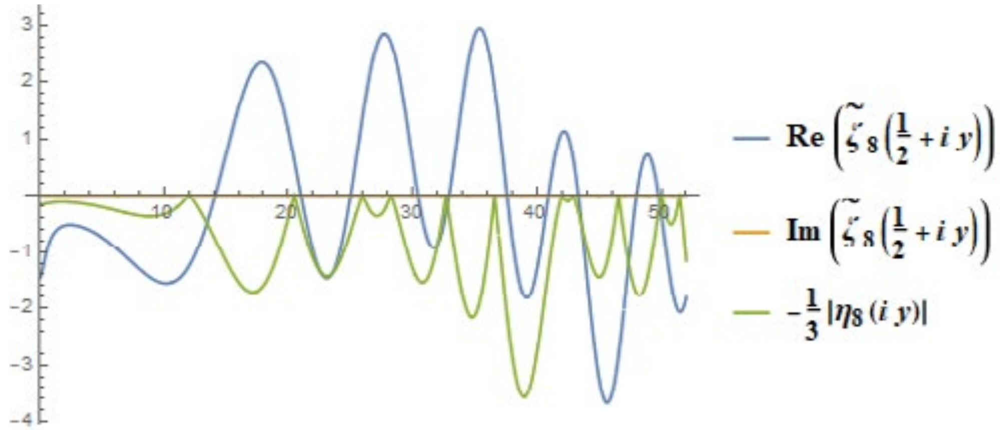
Three-dimensional graph of z=Re $(\tilde{\omega}_{11}(x+iy))$ and z=Im $(\tilde{\omega}_{11}(x+iy))$, $x \in [-35, 35]$, $y \in [0, 70]$



Graph. 15.3

When the above three graphs are compared, it is observed that the computational error decreases with a larger degree of approximation at each point in the observational domain. In comparison to the same domain, the above results emphasize that the degree of approximation is needed to draw another kind of graph.

Graph of z=Re $\left(\tilde{\zeta}_8\left(\frac{1}{2}+iy\right)\right)$, z=Im $\left(\tilde{\zeta}_8\left(\frac{1}{2}+iy\right)\right)$, and $z=-\frac{1}{3}\left|(\tilde{\eta}_8(iy))\right|$, $y \in [0, 52]$



Graph. 15.4

Both the functions,

$$\tilde{\zeta}_8\left(\frac{1}{2}+iy\right) \text{ and } -\frac{1}{3}\left|(\tilde{\eta}_8(iy))\right|$$

are projected onto the same surface.

I can see a relation between the imaginary parts of non-trivial zeros of the zeta function and those of the eta function that is neither too close nor too far away. The imaginary part of the function *(zeta-tilde)* (θ) is zero anywhere along the critical line because the absolute value is taken for the gamma factor of the zeta function. On the critical line, the function *(zeta-tilde)* (θ) takes a real value, and 10 non-trivial zeros of the zeta function lie on the interval $[1/2+0i, 1/2+52i]$.

Here are the details. (The positive imaginary parts of the 11 non-trivial zeros of the eta function will be shown later.)

(15.9)
$$\{\mathrm{Re}\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-5.82730\times10^{-10},1.00359\times10^{-9}\}.$$

(15.10)
$$\{\mathrm{Re}\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{8.77134\times10^{-10},-1.39654\times10^{-9}\}.$$

(15.11)
$$\{\mathrm{Re}\left(\tilde{\zeta}_9\left(\frac{1}{2}+i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_9\left(\frac{1}{2}+i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-1.57157\times10^{-9},1.17188\times10^{-9}\}.$$

(15.12)
$$\{\mathrm{Re}\left(\tilde{\zeta}_9\left(\frac{1}{2}+i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_9\left(\frac{1}{2}+i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{1.12075\times10^{-9},-1.48713\times10^{-9}\}.$$

(15.13)
$$\{\mathrm{Re}\left(\tilde{\zeta}_9\left(\frac{1}{2}+i\left(\frac{32\,935\,061\,588}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_9\left(\frac{1}{2}+i\left(\frac{32\,935\,061\,588}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-1.02165\times10^{-9},1.74259\times10^{-9}\}.$$

(15.14)
$$\{\mathrm{Re}\left(\tilde{\zeta}_{10}\left(\frac{1}{2}+i\left(\frac{37\,586\,178\,159}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_{10}\left(\frac{1}{2}+i\left(\frac{37\,586\,178\,159}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{1.59892\times10^{-9},-2.27410\times10^{-9}\}.$$

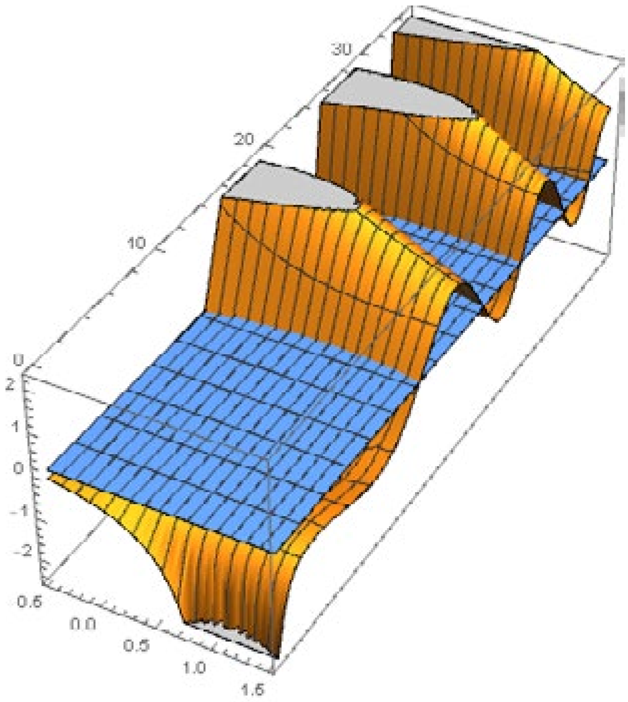
(15.15)
$$\{\mathrm{Re}\left(\tilde{\zeta}_{10}\left(\frac{1}{2}+i\left(\frac{40\,918\,719\,012}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_{10}\left(\frac{1}{2}+i\left(\frac{40\,918\,719\,012}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-1.71047\times10^{-9},1.27075\times10^{-9}\}.$$

(15.16)
$$\{\mathrm{Re}\left(\tilde{\zeta}_{10}\left(\frac{1}{2}+i\left(\frac{43\,327\,073\,281}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_{10}\left(\frac{1}{2}+i\left(\frac{43\,327\,073\,281}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{1.67766\times10^{-9},-1.98936\times10^{-9}\}.$$

(15.17)
$$\{\mathrm{Re}\left(\tilde{\zeta}_{11}\left(\frac{1}{2}+i\left(\frac{48\,005\,150\,881}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_{11}\left(\frac{1}{2}+i\left(\frac{48\,005\,150\,881}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-1.83014\times10^{-9},1.30592\times10^{-9}\}.$$

(15.18)
$$\{\mathrm{Re}\left(\tilde{\zeta}_{12}\left(\frac{1}{2}+i\left(\frac{49\,773\,832\,478}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\zeta}_{12}\left(\frac{1}{2}+i\left(\frac{49\,773\,832\,478}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{9.53951\times10^{-10},-1.88391\times10^{-9}\}.$$

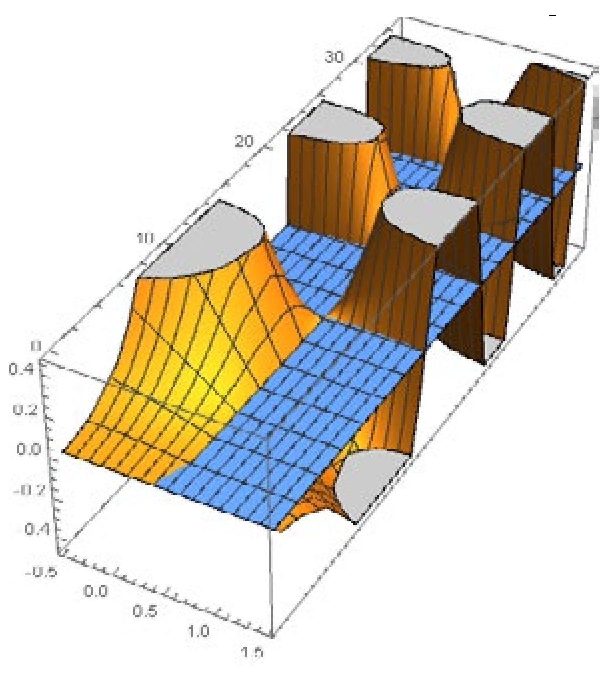
Three-dimensional graph of $z = \mathrm{Re}\left(\tilde{\zeta}_5\,(x + i\,y)\right)$ and $z = 0$, $x \in \left[-\frac{1}{2}, \frac{3}{2}\right]$, $y \in [0, 35]$



Graph. 15.5

Three-dimensional graphs will be hereafter drawn using the plane $z = 0$ as the auxiliary plane.
Using topography as an analogy, the plane $z = 0$ (colored blue) is suited for the surface of the sea, and the candidate sites for the non-trivial zeros of the zeta function are along the coast lines.

Three-dimensional graph of $z = \mathrm{Im}\left(\tilde{\zeta}_5\,(x + i\,y)\right)$ and $z = 0$, $x \in \left[-\frac{1}{2}, \frac{3}{2}\right]$, $y \in [0, 35]$



Graph. 15.6

On the sea's complicated surface, the critical line is clearly visible.
The intersections of the coast line and the critical line in the two graphs above are the non-trivial zeros.

As reference, the same graphs above (Graph. 15.5 and Graph. 15.6) can also be drawn using only the well-known built-in Mathematica functions. The Mathematica inputs are shown as follows:

Alt + 9 (Creating a new input line on a notebook)

$$\text{zetaxx}[\theta\_] := \text{Abs}\left[\frac{2\,\pi^{\frac{\theta}{2}}}{(1-\theta)\ \text{Gamma}\left[1+\frac{\theta}{2}\right]}\right]\text{Zeta}[\theta]/\frac{2\,\pi^{\frac{\theta}{2}}}{(1-\theta)\ \text{Gamma}\left[1+\frac{\theta}{2}\right]};$$ Shift + Enter

Alt + 9 (Creating the second input line)

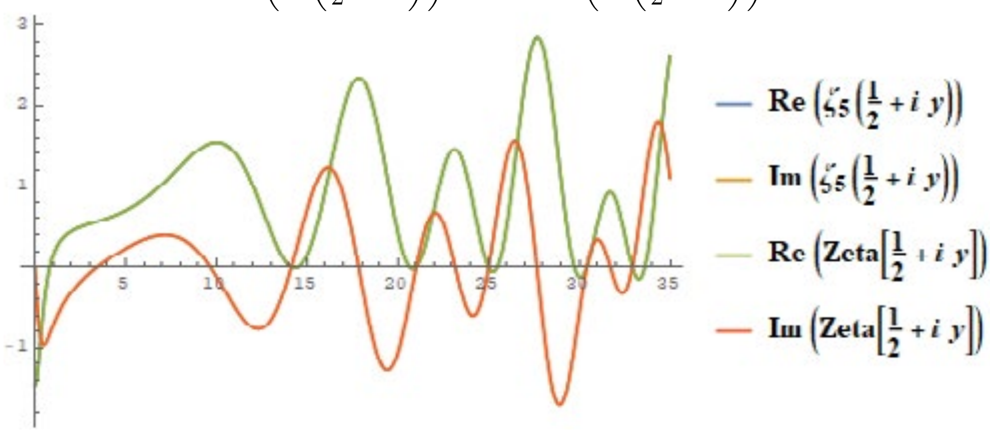
$$\text{Plot3D}\left[\{\text{Re}\,[\text{zetaxx}\,[\text{x}+i\,\text{y}]]\,,0\},\{\text{x},-\frac{1}{2},\frac{3}{2}\},\{\text{y},0,35\},\text{BoxRatios}\rightarrow\{1,3,1\}\right]$$ Shift + Enter

Alt + 9 (Creating the third input line)

$$\text{Plot3D}\left[\{\text{Im}\,[\text{zetaxx}\,[\text{x}+i\,\text{y}]]\,,0\},\{\text{x},-\frac{1}{2},\frac{3}{2}\},\{\text{y},0,35\},\text{BoxRatios}\rightarrow\{1,3,1\}\right]$$ Shift + Enter

Note: Valid for Mathmatica Ver. 12.0 or later.

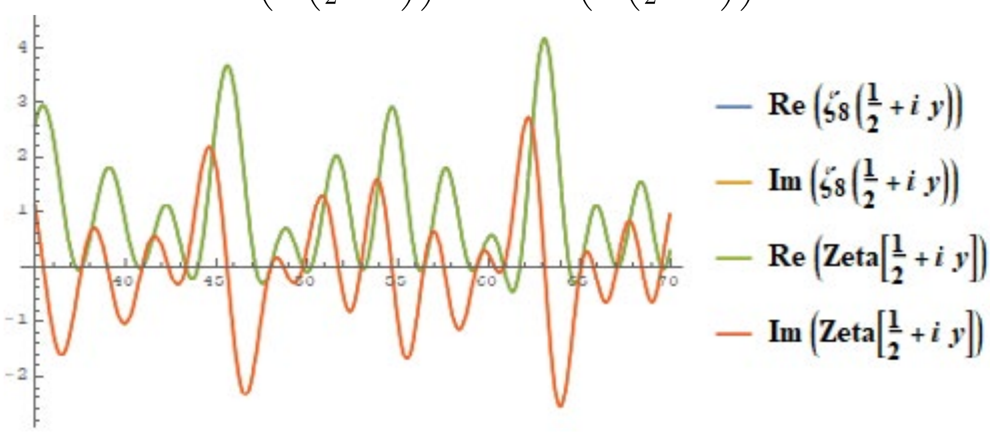
Graph of z=Re $\left(\zeta_5\left(\frac{1}{2}+i\,y\right)\right)$ and z=Im $\left(\zeta_5\left(\frac{1}{2}+i\,y\right)\right)$, $y\in[0,35]$



Graph. 15.7

To evaluate the error, real and imaginary values of Zeta $[1/2+i\,y]$, i.e., Mathematica's built-in zeta function, are superposed on the graph. A five degree of approximation is appropriate for the interval $[1/2+0\,i,1/2+35\,i]$. The built-in zeta function should be superposed if necessary for all graphs.

Graph of z=Re $\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\,y\right)\right)$ and z=Im $\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\,y\right)\right)$, $y\in[35,70]$

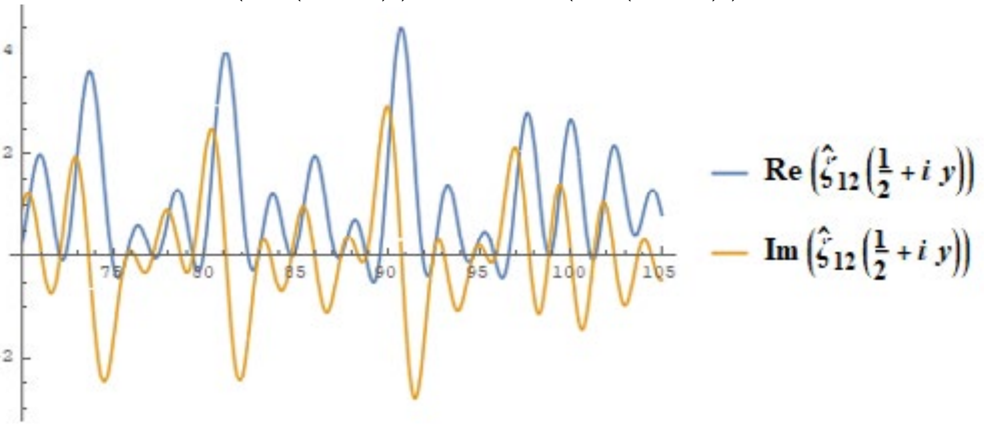


Graph. 15.8

An eight degree of approximation is appropriate for the interval $[1/2+35\,i,1/2+70\,i]$.

I change to the function $(zeta\text{-}hat)_{12}(\theta)$ for drawing in the interval $[1/2+70\,i, 1/2+105\,i]$ for accuracy.

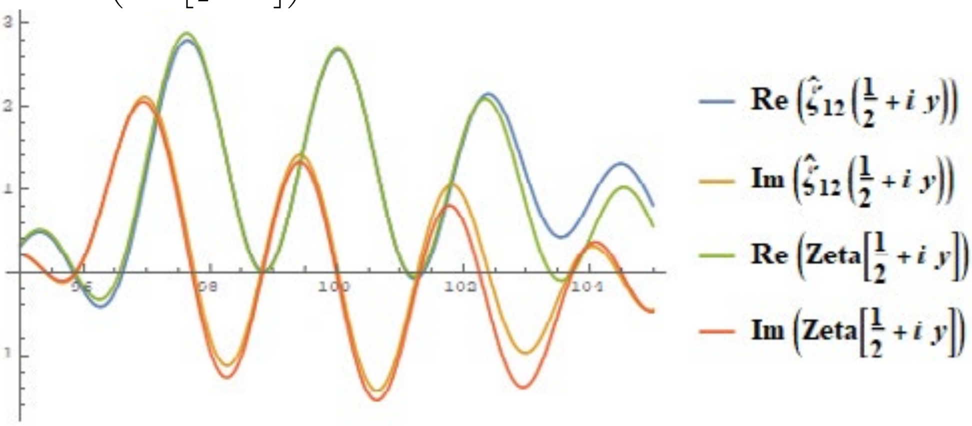
Graph of z=Re $\left(\hat{\zeta}_{12}\left(\frac{1}{2}+i\,y\right)\right)$ and z=Im $\left(\hat{\zeta}_{12}\left(\frac{1}{2}+i\,y\right)\right)$, $y\in[70,105]$



Graph. 15.9

It would be short on accuracy in the interval $[1/2+95\,i, 1/2+105\,i]$. The next graph is prepared to confirm this error.

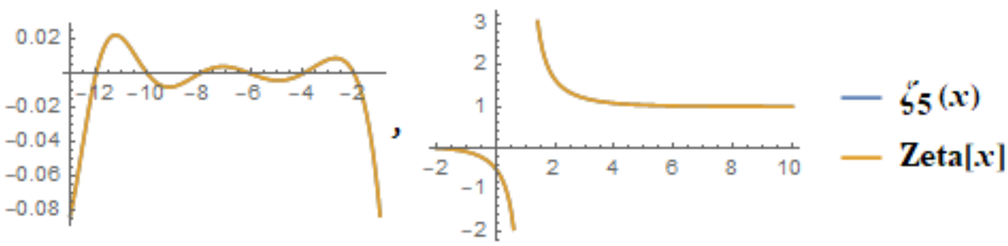
Graph of z=Re $\left(\hat{\zeta}_{12}\left(\frac{1}{2}+i\,y\right)\right)$, z=Im $\left(\hat{\zeta}_{12}\left(\frac{1}{2}+i\,y\right)\right)$, z=Re $\left(\text{Zeta}\left[\frac{1}{2}+i\,y\right]\right)$,
and z=Im $\left(\text{Zeta}\left[\frac{1}{2}+i\,y\right]\right)$, $y\in[95,105]$



Graph. 15.10

From the above four graphs, I can recognize that in total, 29 non-trivial zeros of the zeta function lie on the interval $[1/2+0\,i, 1/2+100\,i]$.

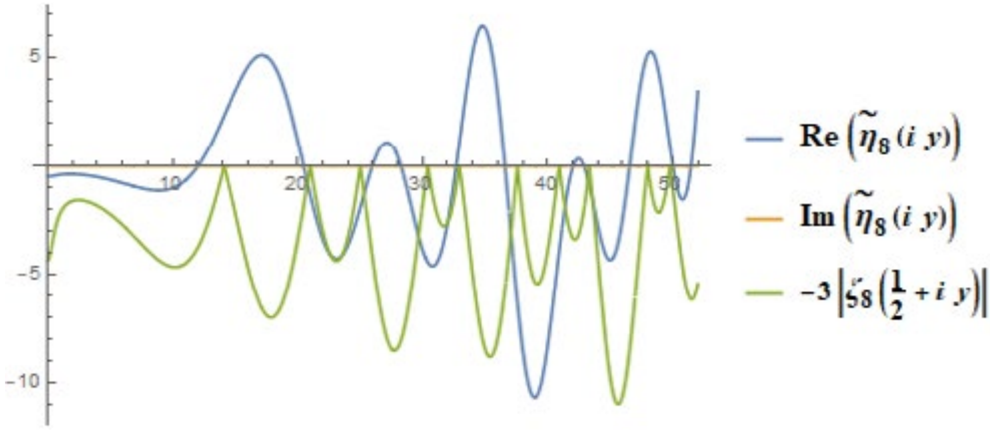
Graph of z=$\zeta_5(x)$ and z=Zeta$[x]$, $x\in[-13,10]$



Graph. 15.11

I can see that the pole and the trivial zeros of the zeta function on the real axis are accurate.

$$\text{Graph of z=Re}\left(\tilde{\eta}_8\left(i\,y\right)\right),\ \text{z=Im}\left(\tilde{\eta}_8\left(i\,y\right)\right),\ \text{and z} = -3\left|\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\,y\right)\right)\right|,\ y\in[0,52]$$



Graph. 15.12

Both the functions,

$$\tilde{\eta}_8\left(i\,y\right)\ \text{and}\ -3\left|\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\,y\right)\right)\right|$$

are projected onto the same surface.
I can also see the relation neither too close nor too far away between the imaginary parts of the zeta function's non-trivial zeros and those of the eta function. Since the gamma factor of the eta function is taken as an absolute value, the imaginary part of the function $(eta\text{-}tilde)\,(\theta)$ is zero anywhere on the imaginary axis. Function $(eta\text{-}tilde)\,(\theta)$ is observed to take real values on the imaginary axis; moreover, 11 non-trivial zeros of the eta function lie on the interval $[0\,i, 52\,i]$.
Here are the details.

(15.19)
$$\{\text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-1.151464\times10^{-9},4.99152\times10^{-10}\}.$$

(15.20)
$$\{\text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{3.39148\times10^{-9},-1.69660\times10^{-9}\}.$$

(15.21)
$$\{\text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-1.074615\times10^{-9},2.36501\times10^{-9}\}.$$

(15.22)
$$\{\text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{28\,269\,450\,283}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{28\,269\,450\,283}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{2.75628\times10^{-9},-9.95323\times10^{-9}\}.$$

(15.23)
$$\{\text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{32\,685\,214\,209}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{32\,685\,214\,209}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{-5.02927\times10^{-9},3.53524\times10^{-9}\}.$$

(15.24)
$$\{\text{Re}\left(\tilde{\eta}_{10}\left(i\left(\frac{36\,583\,986\,392}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\text{Re}\left(\tilde{\eta}_{10}\left(i\left(\frac{36\,583\,986\,392}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}$$
$$\simeq\{7.85814\times10^{-9},-4.74443\times10^{-9}\}.$$

$$\left\{\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,044\,079\,278}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,044\,079\,278}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\right\} \simeq \{-8.94336\times 10^{-9}, 2.39905\times 10^{-9}\}. \tag{15.25}$$

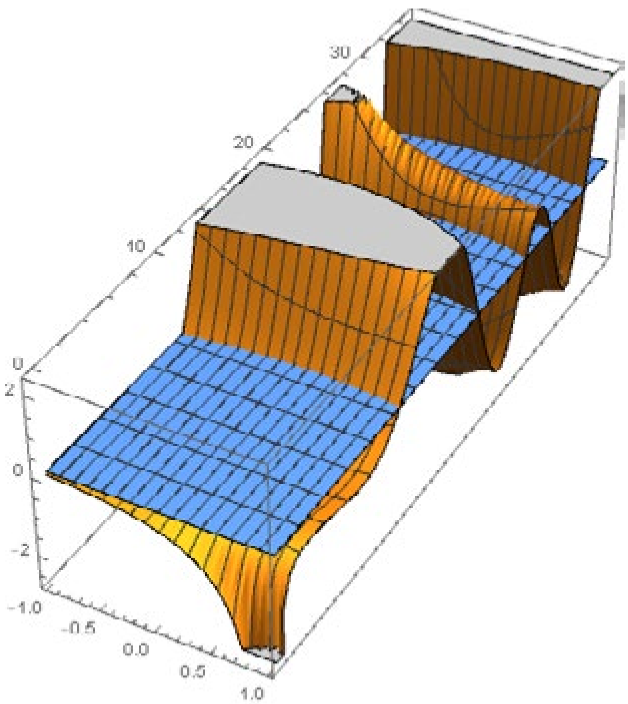
$$\left\{\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,901\,222\,688}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,901\,222\,688}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\right\} \simeq \{1.46300\times 10^{-9}, -1.56310\times 10^{-9}\}. \tag{15.26}$$

$$\left\{\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{46\,556\,753\,041}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{46\,556\,753\,041}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\right\} \simeq \{-4.30492\times 10^{-9}, 5.06314\times 10^{-9}\}. \tag{15.27}$$

$$\left\{\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{50\,021\,715\,160}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{50\,021\,715\,160}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\right\} \simeq \{4.64200\times 10^{-9}, -2.54722\times 10^{-9}\}. \tag{15.28}$$

$$\left\{\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{51\,457\,512\,850}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{51\,457\,512\,850}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\right\} \simeq \{-5.47820\times 10^{-9}, 3.82695\times 10^{-9}\}. \tag{15.29}$$

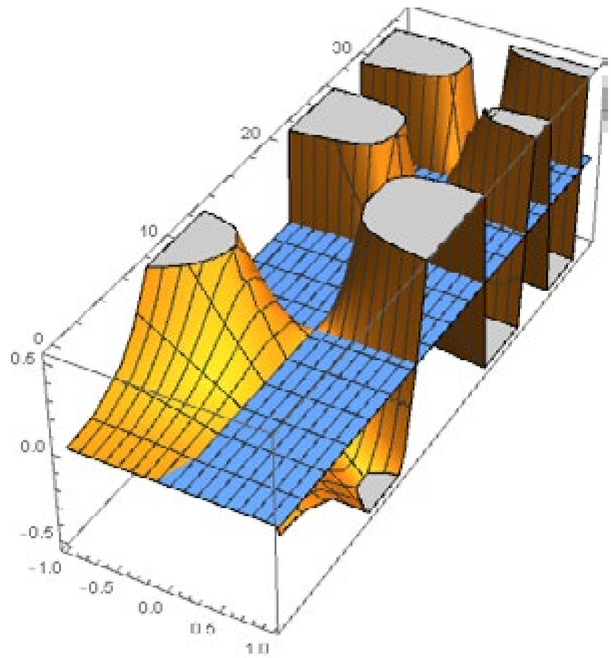
Three-dimensional graph of $z = \mathrm{Re}\left(\tilde{\eta}_5\left(x + i\,y\right)\right)$ and $z = 0$, $x \in [-1, 1]$, $y \in [0, 35]$



Graph. 15.13

The coast lines are candidate sites for the non-trivial zeros of the eta function.

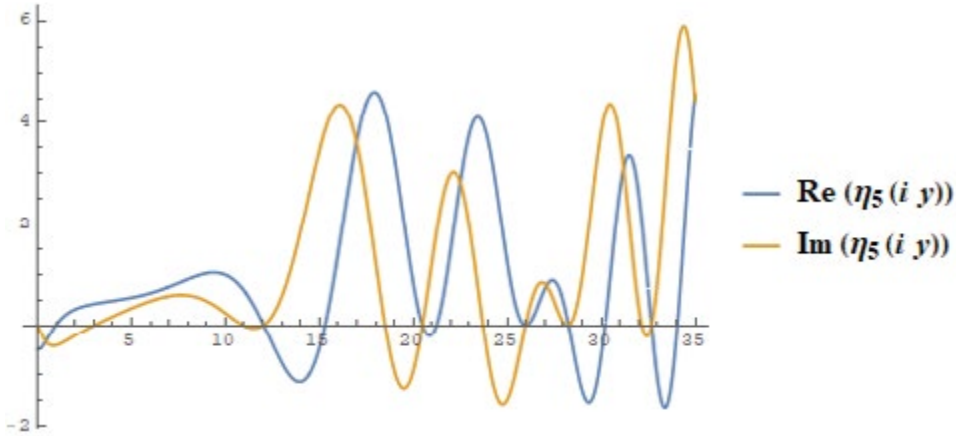
Three-dimensional graph of $z = \mathrm{Im}\left(\tilde{\eta}_5\left(x + i\,y\right)\right)$ and $z = 0,\ x \in [-1, 1]\,,\ y \in [0, 35]$



Graph. 15.14

On the intricate surface of the sea, the imaginary axis is clearly visible. The intersections of the coast lines and the imaginary axis in the following two graphs are the non-trivial zeros. Here, the imaginary axis corresponds to the critical line of the eta function.

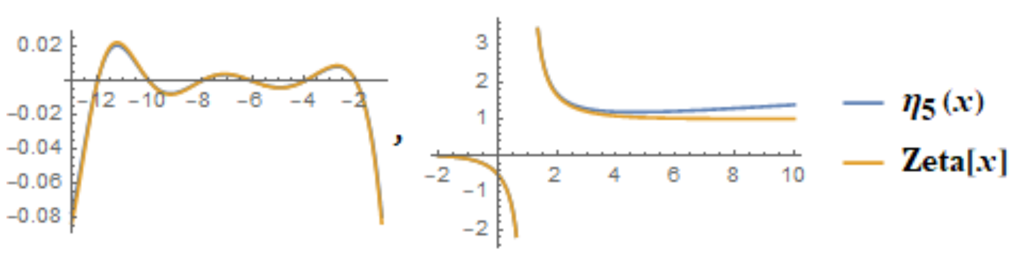
Graph of z=$\mathrm{Re}\left(\eta_5\left(i\,y\right)\right)$ and z=$\mathrm{Im}\left(\eta_5\left(i\,y\right)\right)\,,\ y \in [0, 35]$



Graph. 15.15

The eta function on the imaginary axis has a similar shape as that of the zeta function on the critical line.

Graph of z=$\eta_5\left(x\right)$ and z=$\mathrm{Zeta}\left[x\right]\,,\ x \in [-13, 10]$



Graph. 15.16

On the real axis, I can also see the pole and the trivial zeros of the eta function.
On the real axis, there is a little difference between the eta and the zeta functions.

16 Similarities and Differences between the Zeta and the Eta Functions, and Other Knowledge

16.1 Similarities and Differences between the Zeta and the Eta Functions

The explicit formulae for the zeta and the eta functions are shown again.

$$(16.1)\quad \zeta(\theta)=\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{l}\left(\theta(\theta-1)+(7-\theta)(2\pi p)^2\right)\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad+\left((\theta-1)-2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}\backslash\{1\}.$$

$$(16.2)\quad \eta(\theta)=\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left((7-\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)-2(2\pi p)^3\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\right),\quad \theta\in\mathbb{C}\backslash\{1\}.$$

An easily recognized common point here is that they have the same gamma factor. Thus, having only a simple pole at $\theta=1$ is one of the common points.
The residue of the eta function at $\theta=1$ is as follows:

$$(16.3)\quad \mathrm{Res}\,(1;\eta(\theta))=\left.(\theta-1)\,\eta(\theta)\right|_{\theta=1}=\left.(\theta-1)\,\zeta(\theta)\right|_{\theta=1}=1.$$

The calculation is performed for the second term of equation (16.3).

$$(16.4)\quad \begin{aligned}\left.(\theta-1)\,\eta(\theta)\right|_{\theta=1}&=-\frac{2\pi^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-1}(p)\,p^{\frac{1}{2}}\left(6(2\pi p)^2\,\mathrm{K}_{\frac{1}{2}}(2\pi p)-2(2\pi p)^3\,\mathrm{K}_{\frac{1}{2}}(2\pi p)\right)\\&=\frac{4\pi^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-1}(p)\,p^{\frac{1}{2}}\left((2\pi p)-3\right)(2\pi p)^2\,\mathrm{K}_{\frac{1}{2}}(2\pi p)\\&=\frac{8\pi^{\frac{1}{2}}}{\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-1}(p)\,p^{\frac{1}{2}}\left((2\pi p)-3\right)(2\pi p)^2\cdot\sqrt{\frac{\pi}{2\,(2\pi p)}}\,e^{-2\pi p}\\&=4\sum_{p=1}^{\infty}\sigma_{-1}(p)\left((2\pi p)-3\right)(2\pi p)^2\,e^{-2\pi p}.\end{aligned}$$

Thus, the following infinite series that gives the algebraic number $1/4$ is obtained:

$$(16.5)\quad \sum_{p=1}^{\infty}\sigma_{-1}(p)\left((2\pi p)-3\right)(2\pi p)^2\,e^{-2\pi p}=\frac{1}{4}.$$

In addition, another common point is that they have trivial zeros for any even number of -2 or less.
That is

$$(16.6)\quad \zeta(-2k)=\eta(-2k)=0.\ \ k\in\mathbb{N}.$$

When k is assumed to be any positive integer again, the value of $\zeta(1-2k)$ corresponds to the value of $\zeta(2k)$ due to the point symmetry $(\theta=1/2)$ of the Riemann Xi function.
In contrast, in case of the eta function, the value of $\eta(-2k-1)$ corresponds to the value of $\eta(2k+1)$ due to the origin symmetry $(\theta=0)$ of the Chi function.
Furthermore, the mathematical phrase can define any relation between of $\eta(-2k-1)$ and $\eta(2k+1)$.
The modified equation is performed to calculate the function value on the real axis of the eta function (16.2).

$$(16.7)\quad \eta(\theta)=\frac{2}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(\begin{array}{l}(7-\theta)(2\pi p)^2(\pi p)^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ \quad-(2\pi p)^4(\pi p)^{\frac{2-\theta}{2}}\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}\backslash\{1\}.$$

When θ is any odd number of 3 or more, the following integral representation is used:

$$\eta(\theta)=\frac{2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(\begin{array}{r}(7-\theta)(2\pi p)^2\int_0^\infty x^{\theta-1}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ -(2\pi p)^4\int_0^\infty x^{\theta-3}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{array}\right),\quad \theta\in\mathbb{C}\backslash\{1\}. \tag{16.8}$$

For equation (16.2), $-\theta$ is substituted for θ, which is obtained earlier as equation (14.7).

$$\eta(-\theta)=\frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\,p^{\frac{\theta}{2}}\left(\begin{array}{r}(7-\theta)(2\pi p)^2\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ -2(2\pi p)^3\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}\backslash\{-1\}. \tag{16.9}$$

The modification is performed for equation (16.9).

$$\eta(-\theta)=\frac{2\pi^{-\theta}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(\begin{array}{r}(7-\theta)(2\pi p)^2(\pi p)^{\frac{\theta}{2}}\,\mathrm{K}_{\frac{\theta}{2}}(2\pi p)\\ -(2\pi p)^4(\pi p)^{\frac{\theta-2}{2}}\,\mathrm{K}_{\frac{\theta-2}{2}}(2\pi p)\end{array}\right),\quad \theta\in\mathbb{C}\backslash\{-1\}. \tag{16.10}$$

When θ is any odd number of 1 or more, the following integral representation is used:

$$\eta(-\theta)=\frac{2\pi^{-\theta}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}(p)\left(\begin{array}{r}(7-\theta)(2\pi p)^2\int_0^\infty x^{\theta-1}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ -(2\pi p)^4\int_0^\infty x^{\theta-3}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{array}\right),\quad \theta\in\mathbb{C}\backslash\{-1\}. \tag{16.11}$$

For equation (16.11), 1 is substituted for θ.

$$\begin{aligned}\eta(-1)&=\frac{2\pi^{-1}}{2\Gamma\left(1-\frac{1}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-1}(p)\left(\begin{array}{r}6(2\pi p)^2\int_0^\infty x^{0}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ -(2\pi p)^4\int_0^\infty x^{-2}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{array}\right)\\ &=\frac{1}{\pi\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-1}(p)\sum_{p=1}^{\infty}\sigma_{-1}(p)\left(6(2\pi p)^2\cdot\frac{\sqrt{\pi}}{2}e^{-2\pi p}-(2\pi p)^4\cdot\frac{\sqrt{\pi}}{2\pi p}e^{-2\pi p}\right)\\ &=-\frac{1}{\pi}\sum_{p=1}^{\infty}\sigma_{-1}(p)\left((2\pi p)-3\right)(2\pi p)^2e^{-2\pi p}.\end{aligned} \tag{16.12}$$

The representation for $\eta(-1)$ and its exact value are shown.

$$\begin{aligned}\eta(-1)&=-\frac{1}{\pi}\sum_{p=1}^{\infty}\sigma_{-1}(p)\left((2\pi p)-3\right)(2\pi p)^2e^{-2\pi p}\\ &=-\frac{1}{4\pi}.\end{aligned} \tag{16.13}$$

For equation (16.11), 3 is substituted for θ.

$$\begin{aligned}\eta(-3)&=\frac{2\pi^{-3}}{4\Gamma\left(1-\frac{3}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-3}(p)\left(\begin{array}{r}4(2\pi p)^2\int_0^\infty x^{2}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ -(2\pi p)^4\int_0^\infty x^{0}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{array}\right)\\ &=-\frac{1}{4\pi^3\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-3}(p)\left(4(2\pi p)^2\cdot\frac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p}-(2\pi p)^4\cdot\frac{\sqrt{\pi}}{2}e^{-2\pi p}\right)\\ &=\frac{1}{8\pi^3}\sum_{p=1}^{\infty}\sigma_{-3}(p)\left((2\pi p)^2-2(2\pi p)-2\right)(2\pi p)^2e^{-2\pi p}.\end{aligned} \tag{16.14}$$

The representation for $\eta(-3)$ and its approximate value are shown.

$$\eta(-3) = \frac{1}{8\,\pi^3}\sum_{p=1}^{\infty}\sigma_{-3}(p)\left((2\pi p)^2 - 2(2\pi p) - 2\right)(2\pi p)^2 e^{-2\pi p}$$
$$\simeq 0.0077338931815. \tag{16.15}$$

In addition, in the equation (16.8), 3 is substituted for θ.

$$\begin{aligned}\eta(3) &= \frac{2}{-2\,\Gamma\left(1+\frac{3}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-3}(p)\begin{pmatrix}4(2\pi p)^2\displaystyle\int_0^{\infty}x^2 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx \\ -(2\pi p)^4\displaystyle\int_0^{\infty}x^0 e^{-\left(x^0+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix} \\ &= -\frac{4}{3\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-3}(p)\begin{pmatrix}4(2\pi p)^2\cdot\dfrac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p} \\ -(2\pi p)^4\cdot\dfrac{\sqrt{\pi}}{2}\,e^{-2\pi p}\end{pmatrix} \\ &= \frac{2}{3}\sum_{p=1}^{\infty}\sigma_{-3}(p)\left((2\pi p)^2 - 2(2\pi p) - 2\right)(2\pi p)^2 e^{-2\pi p} \\ &\simeq 1.2789292363.\end{aligned} \tag{16.16}$$

Equations (16.15) and (16.16) are integrated to obtain a relation between $\eta(3)$ and $\eta(-3)$.

$$\eta(3) = \frac{16\,\pi^3}{3}\eta(-3)\,. \tag{16.17}$$

For equation (16.11), 5 is substituted for θ.

$$\begin{aligned}\eta(-5) &= \frac{2\pi^{-5}}{6\,\Gamma\left(1-\frac{5}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-5}(p)\begin{pmatrix}2(2\pi p)^2\displaystyle\int_0^{\infty}x^4 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx \\ -(2\pi p)^4\displaystyle\int_0^{\infty}x^2 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix} \\ &= -\frac{1}{4\,\pi^5\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-3}(p)\begin{pmatrix}2(2\pi p)^2\cdot\dfrac{(2\pi p)^2+3(2\pi p)+3}{8}\sqrt{\pi}\,e^{-2\pi p} \\ -(2\pi p)^4\cdot\dfrac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p}\end{pmatrix} \\ &= -\frac{1}{16\,\pi^5}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^3 - 3(2\pi p) - 3\right)(2\pi p)^2 e^{-2\pi p} \\ &\simeq -0.0036345168378.\end{aligned} \tag{16.18}$$

For equation (16.8), 5 is substituted for θ.

$$\begin{aligned}\eta(5) &= \frac{2}{-4\,\Gamma\left(1+\frac{5}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-5}(p)\begin{pmatrix}2(2\pi p)^2\displaystyle\int_0^{\infty}x^4 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx \\ -(2\pi p)^4\displaystyle\int_0^{\infty}x^2 e^{-\left(x^0+\frac{(\pi p)^2}{x^2}\right)}dx\end{pmatrix} \\ &= -\frac{4}{15\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-5}(p)\begin{pmatrix}2(2\pi p)^2\cdot\dfrac{(2\pi p)^2+3(2\pi p)+3}{8}\sqrt{\pi}\,e^{-2\pi p} \\ -(2\pi p)^4\cdot\dfrac{(2\pi p)+1}{4}\sqrt{\pi}\,e^{-2\pi p}\end{pmatrix} \\ &= \frac{1}{15}\sum_{p=1}^{\infty}\sigma_{-5}(p)\left((2\pi p)^3 - 3(2\pi p) - 3\right)(2\pi p)^2 e^{-2\pi p} \simeq 1.1863826102.\end{aligned} \tag{16.19}$$

Equations (16.18) and (16.19) are integrated to obtain a relation between $\eta(5)$ and $\eta(-5)$.

$$\eta(5) = -\frac{16\,\pi^5}{15}\eta(-5)\,. \tag{16.20}$$

Hereafter, the calculations can be performed similarly.
In cases where $\theta = 7, 9, 11, 13, 15$, and 17, each representation by the elementary functions and the approximate value are shown all together.

(16.21)
$$\eta(7) = \frac{512\,\pi^7}{4725}\eta(-7) = \frac{2}{315}\sum_{p=1}^{\infty}\sigma_{-7}(p)\left((2\pi p)^3 + 3(2\pi p) + 3\right)(2\pi p)^4 e^{-2\pi p} \simeq 1.2460084172.$$

(16.22)
$$\begin{aligned}\eta(9) &= -\frac{128\,\pi^9}{19845}\eta(-9)\\ &= \frac{1}{1890}\sum_{p=1}^{\infty}\sigma_{-9}(p)\begin{pmatrix}(2\pi p)^5 + 7(2\pi p)^4 + 25(2\pi p)^3\\ + 60(2\pi p)^2 + 105(2\pi p) + 105\end{pmatrix}(2\pi p)^2 e^{-2\pi p} \simeq 1.3360607156.\end{aligned}$$

(16.23)
$$\begin{aligned}\eta(11) &= \frac{4096\,\pi^{11}}{16372125}\eta(-11)\\ &= \frac{2}{51975}\sum_{p=1}^{\infty}\sigma_{-11}(p)\begin{pmatrix}(2\pi p)^6 + 12(2\pi p)^5 + 75(2\pi p)^4 + 315(2\pi p)^3\\ + 945(2\pi p)^2 + 1890(2\pi p) + 1890\end{pmatrix}(2\pi p)^2 e^{-2\pi p}\\ &\simeq 1.4336203440.\end{aligned}$$

(16.24)
$$\begin{aligned}\eta(13) &= -\frac{4096\,\pi^{13}}{602026425}\eta(-13)\\ &= \frac{1}{405405}\sum_{p=1}^{\infty}\sigma_{-13}(p)\begin{pmatrix}(2\pi p)^7 + 18(2\pi p)^6 + 168(2\pi p)^5 + 1050(2\pi p)^4\\ + 4725(2\pi p)^3 + 15120(2\pi p)^2 + 31185(2\pi p) + 31185\end{pmatrix}(2\pi p)^2 e^{-2\pi p}\\ &\simeq 1.5315820258.\end{aligned}$$

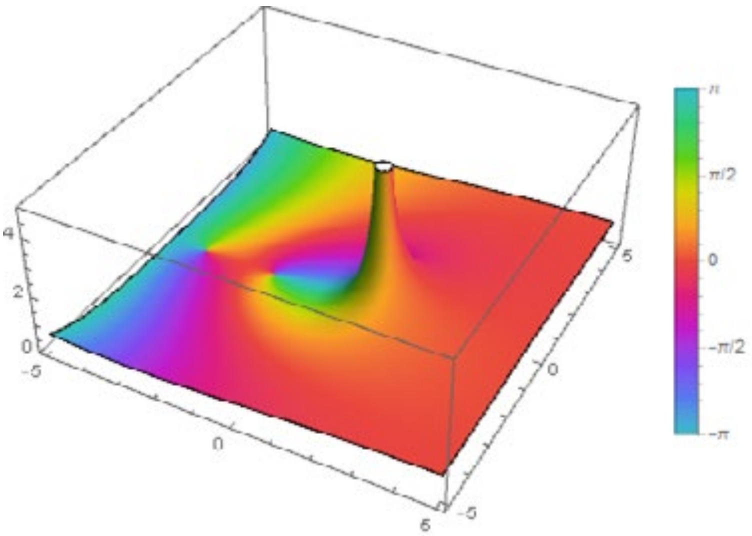
(16.25)
$$\begin{aligned}\eta(15) &= \frac{262144\,\pi^{15}}{1917454163625}\eta(-15)\\ &= \frac{2}{14189175}\sum_{p=1}^{\infty}\sigma_{-15}(p)\begin{pmatrix}(2\pi p)^8 + 25(2\pi p)^7 + 322(2\pi p)^6\\ + 2772(2\pi p)^5 + 17325(2\pi p)^4 + 79695(2\pi p)^3\\ + 259875(2\pi p)^2 + 540540(2\pi p) + 540540\end{pmatrix}(2\pi p)^2 e^{-2\pi p}\\ &\simeq 1.6272471919.\end{aligned}$$

(16.26)
$$\begin{aligned}\eta(17) &= -\frac{16384\,\pi^{17}}{7761123995625}\eta(-17)\\ &= \frac{1}{137837700}\sum_{p=1}^{\infty}\sigma_{-17}(p)\begin{pmatrix}(2\pi p)^9 + 33(2\pi p)^8 + 558(2\pi p)^7 + 6300(2\pi p)^6\\ + 51975(2\pi p)^5 + 322245(2\pi p)^4 + 1486485(2\pi p)^3\\ + 4864860(2\pi p)^2 + 10135125(2\pi p) + 10135125\end{pmatrix}(2\pi p)^2 e^{-2\pi p}\\ &\simeq 1.7195887751.\end{aligned}$$

In the domain of $\theta > 1$ on the real axis, the zeta function is strictly monotonically decreasing.
Conversely, the eta function has a minimal value at $\theta > 1$ in the same domain. Although the eta function is strictly monotonically decreasing in $1 < \theta < \kappa$, it turns to be strictly monotonically increasing in $\theta > \kappa$.
The range of the minimal value and κ value (that gives the minimal value) are shown.

(16.27)
$$1.183937412932 < \eta(\kappa) < 1.183937412933, \quad (4.62270 < \kappa < 4.62271).$$

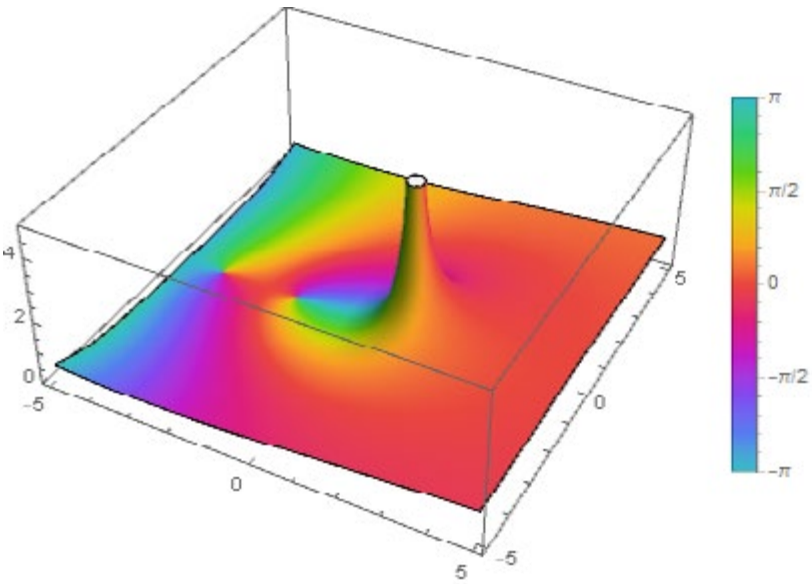
Three-dimensional complex plot of the zeta function in the square domain $[-5-5\,i,\,5+5\,i]$



Graph. 16.1

It was drawn by assuming the degree of approximation as 5 in the explicit formula for the zeta function.
In the small, it can be seen the first trivial zero at $\theta = -2$, the second trivial zero at $\theta = -4$, and the pole at $\theta = 1$ in this domain. Around each zero, the argument rotates CCW (counterclockwise) from 0 to 2π [rad]. Around the pole, the argument rotates CW (clockwise) from 0 to -2π [rad].

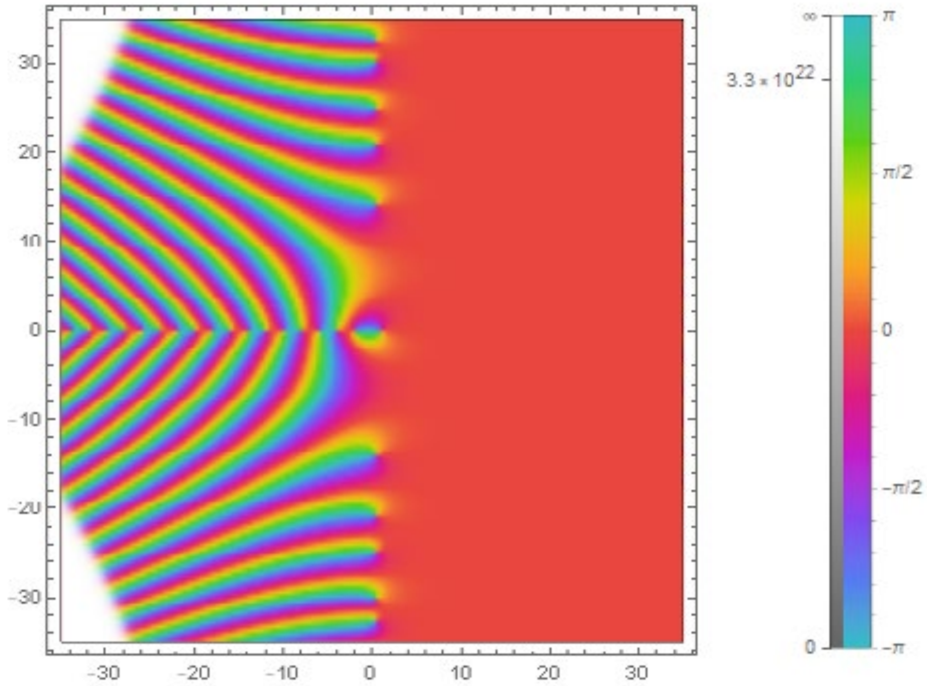
Three-dimensional complex plot of the eta function in the square domain $[-5-5\,i,\,5+5\,i]$



Graph. 16.2

It was also drawn by assuming the degree of approximation as 5 in the explicit formula for the eta function.
In the small, it is hard to distinguish the eta function from the zeta function.

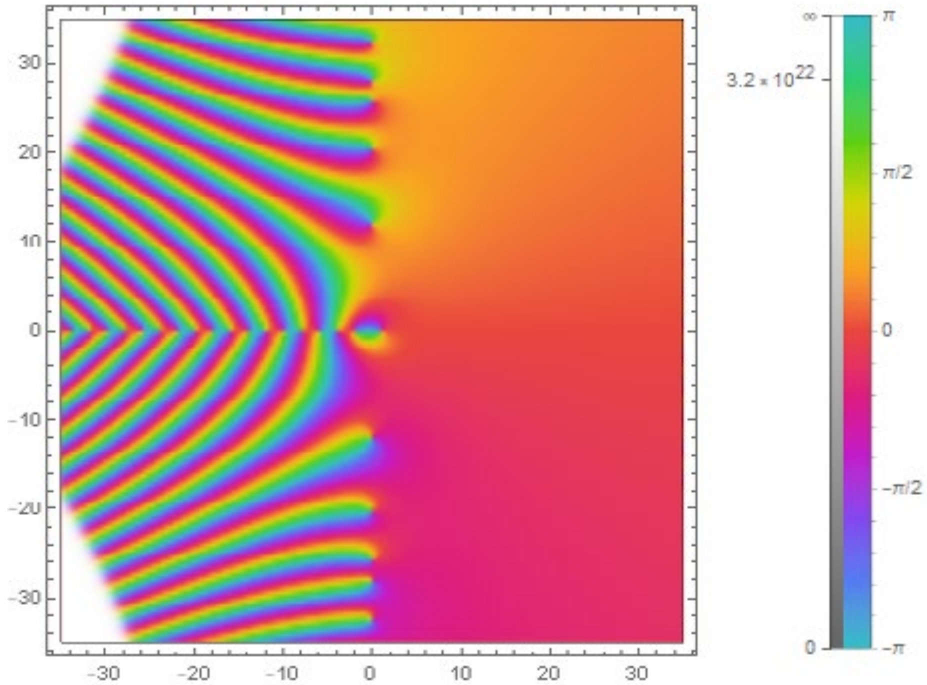
Complex plot of the zeta function in the square domain $[-35 - 35\,i,\ 35 + 35\,i]$



Graph. 16.3

Complex plot is expressed by coloring the argument, and the absolute value is expressed by shading the color.
The color bar displayed on the right of the complex plot is shown as the legends of the above.
It was drawn by assuming the degree of approximation as 8 in the explicit formula for the zeta function.

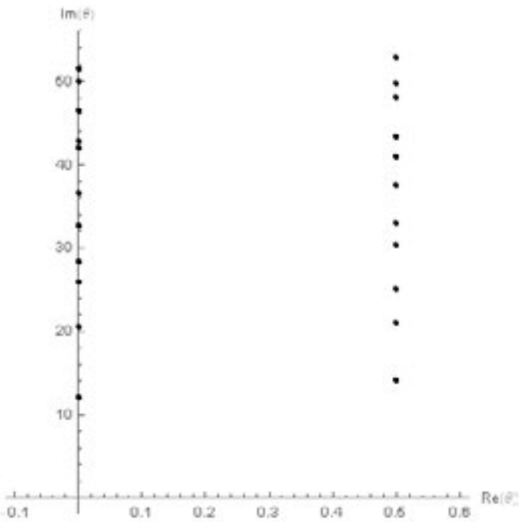
Complex plot of the eta function in the square domain $[-35 - 35\,i,\ 35 + 35\,i]$



Graph. 16.4

It was also drawn by assuming the degree of approximation as 8 in the explicit formula for the eta function.
When taking a broad view, it is easy to distinguish the eta function from the zeta function.

Graph of the non-trivial zeros of the eta function and those of the zeta function



Graph. 16.5

The dots on the imaginary axis ($\mathrm{Re}(\theta)=0$) indicate the non-trivial zeros of the eta function, while those on the critical line ($\mathrm{Re}(\theta)=1/2$) indicate the non-trivial zeros of the zeta function.
In the region up to height 56, both the zeta and eta functions each have 11 non-trivial zeros.

16.2 The General Representation for the Eta Function for any Odd Number of 3 or More

For equation (16.8), any odd number of 3 or more is substituted for θ,

(16.28)
$$\eta(2k+1)=\frac{2}{-2k\,\Gamma\left(1+\frac{2k+1}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left(\begin{array}{r}(6-2k)(2\pi p)^2\int_0^\infty x^{2k}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ -(2\pi p)^4\int_0^\infty x^{2(k-1)}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{array}\right)$$
$$=\frac{2^{k+1}}{k(2k+1)!!\sqrt{\pi}}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left(\begin{array}{r}2(k-3)(2\pi p)^2\int_0^\infty x^{2k}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\\ +(2\pi p)^4\int_0^\infty x^{2(k-1)}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx\end{array}\right),\quad k\in\mathbb{N}.$$

The general representation for the two integrals of equation (16.28) is given by the description in the subsection 19-4 i.e.,

(16.29)
$$\int_0^\infty x^{2k}e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)}dx=\frac{1}{2^{k+1}}\sum_{\mu=0}^{k}a_{k,\mu}(2\pi p)^{\mu}\sqrt{\pi}\,e^{-2\pi p},\quad (k,p\in\mathbb{N}).$$

Where

(16.30)
$$a_{k,\mu}=\frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!},\quad (k\in\mathbb{N},\mu=\{0,1,2,\cdots,k\}).$$

And the coefficients deserving special mention are as follows:

(16.31)
$$a_{k,k}=1,\quad k\in\mathbb{N}.$$

(16.32)
$$a_{k,1}=a_{k,0}=(2k-1)!!,\quad k\in\mathbb{N}.$$

The above general representation for the two integrals is introduced into equation (16.28).

$$\eta(2k+1) = \frac{2^{k+1}}{k(2k+1)!!\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} 2(k-3)(2\pi p)^2 \cdot \frac{1}{2^{k+1}} \sum_{\mu=0}^{k} a_{k,\mu} (2\pi p)^{\mu} \sqrt{\pi}\, e^{-2\pi p} \\ + (2\pi p)^4 \cdot \frac{1}{2^k} \sum_{\mu=0}^{k-1} a_{k-1,\mu} (2\pi p)^{\mu} \sqrt{\pi}\, e^{-2\pi p} \end{pmatrix}$$

(16.33)

$$= \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} (k-3) \sum_{\mu=0}^{k} a_{k,\mu} (2\pi p)^{\mu} \\ + \sum_{\mu=2}^{k+1} a_{k-1,\mu-2} (2\pi p)^{\mu} \end{pmatrix} (2\pi p)^2 e^{-2\pi p}, \quad k \in \mathbb{N}.$$

Therfore,

$$\eta(2k+1)$$

(16.34)
$$= \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} a_{k-1,k-1} (2\pi p)^{k+1} \\ + \sum_{\mu=2}^{k} (a_{k-1,\mu-2} + (k-3) a_{k,\mu}) (2\pi p)^{\mu} \\ + (k-3) a_{k,1} (2\pi p) + (k-3) a_{k,0} \end{pmatrix} (2\pi p)^2 e^{-2\pi p}, \quad k \in \mathbb{N}.$$

From equation (12.56),

(16.35)
$$a_{k-1,k-1} = 1, \quad k \in \mathbb{N}.$$

From the definition of $a_{k,\mu}$,

(16.36)
$$(k-3) a_{k,1} = (k-3) a_{k,0} = (k-3)(2k-1)!!, \quad k \in \mathbb{N}.$$

The above results are introduced into equation (16.34).

$$\eta(2k+1)$$

(16.37)
$$= \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} (2\pi p)^{k+1} \\ + \sum_{\mu=2}^{k} (a_{k-1,\mu-2} + (k-3) a_{k,\mu}) (2\pi p)^{\mu} \\ + (k-3)(2k-1)!!\,(2\pi p) \\ + (k-3)(2k-1)!! \end{pmatrix} (2\pi p)^2 e^{-2\pi p}, \quad k \in \mathbb{N}.$$

Here, new coefficients $d_{k,\mu}$ are used for the general representation for the eta function for any odd number of 3 or more.
The coefficients $d_{k,\mu}$ are determined by the following equation of definition:

(16.38)
$$\eta(2k+1) =: \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \begin{pmatrix} (2\pi p)^{k+1} + \sum_{\mu=2}^{k} d_{k,\mu} (2\pi p)^{\mu} \\ + (k-3)(2k-1)!!\,(2\pi p) \\ + (k-3)(2k-1)!! \end{pmatrix} (2\pi p)^2 e^{-2\pi p}, \quad k \in \mathbb{N}.$$

The coefficients $d_{k,k+1}$, $d_{k,1}$, and $d_{k,0}$ are immediately determined as follows:

(16.39)
$$d_{k,k+1} = 1, \quad k \in \mathbb{N}.$$

(16.40)
$$d_{k,1} = d_{k,0} = (k-3)(2k-1)!!, \quad k \in \mathbb{N}.$$

The coefficients $d_{k,\mu}$ are calculated as follows:

(16.41)
$$d_{k,\mu} = a_{k-1,\mu-2} + (k-3) a_{k,\mu} = \left. \frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!} \right|_{k=k-1,\mu=\mu-2} + (k-3) \cdot \frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}$$
$$= \frac{(\mu-1)(\mu+6) + 2k(k-(\mu+2))}{2^{k-\mu+1}\mu!\,(k-\mu+1)!} (2k-\mu)!, \quad (k \in \mathbb{N}\backslash\{1\}, \mu = 2, 3, 4, \cdots, k).$$

For equation (16.41), 1 is substituted for k, and 2 is substituted for μ.

$$d_{1,2} = 1. \tag{16.42}$$

This result is equivalent to the result of substituting 1 for k in equation (16.39).
For equation (16.41), 1 is substituted for k, and 0 and 1 are substituted for μ.

$$\{d_{1,1}, d_{1,0}\} = \{-2, -2\}. \tag{16.43}$$

This results are equivalent to the results of substituting 1 for k in equations (16.40).
Thus, the right side of equation (16.41) holds even for the case when $k = 1$.

$$d_{k,\mu} = \frac{(\mu-1)(\mu+6) + 2k(k-(\mu+2))}{2^{k-\mu+1}\mu!\,(k-\mu+1)!}(2k-\mu)!, \quad (k \in \mathbb{N}, \mu = 2, 3, 4, \cdots, k). \tag{16.44}$$

For equation (16.41), k is substituted for μ.

$$\begin{aligned} d_{k,k} &= \left.\frac{(\mu-1)(\mu+6) + 2k(k-(\mu+2))}{2^{k-\mu+1}\mu!\,(k-\mu+1)!}(2k-\mu)!\right|_{\mu=k} = \frac{k^2+k-6}{2^1 \cdot k! \cdot 1!}k! \\ &= \frac{(k-2)(k+3)}{2}, \quad k \in \mathbb{N}. \end{aligned} \tag{16.45}$$

Combining the above results, I arrive at the conclusions in this subsection.
As a result, the general representation for the eta function for any odd number of 3 or more is obtained.

$$\eta(2k+1) = \frac{2}{k(2k+1)!!}\sum_{p=1}^{\infty}\sigma_{-(2k+1)}(p)\left(\sum_{\mu=0}^{k+1} d_{k,\mu}(2\pi p)^{\mu}\right)(2\pi p)^2 e^{-2\pi p}, \quad k \in \mathbb{N}. \tag{16.46}$$

Where

$$d_{k,\mu} = \frac{(\mu-1)(\mu+6) + 2k(k-(\mu+2))}{2^{k-\mu+1}\mu!\,(k-\mu+1)!}(2k-\mu)!, \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \cdots, k+1). \tag{16.47}$$

And the coefficients deserving special mention are shown as follows:

$$d_{k,k+1} = 1, \quad k \in \mathbb{N}. \tag{16.48}$$

$$d_{k,k} = \frac{(k-2)(k+3)}{2}, \quad k \in \mathbb{N}. \tag{16.49}$$

$$d_{k,1} = d_{k,0} = (k-3)(2k-1)!!, \quad k \in \mathbb{N}. \tag{16.50}$$

The coefficients of the infinite sum of the function $\eta(2k+1)$ are the same as those of the function $\zeta(2k+1)$ (Not $\hat{\zeta}(2k+1)$).

Table of the coefficients $d_{k,\mu}$

k	$d_{k,9}$	$d_{k,8}$	$d_{k,7}$	$d_{k,6}$	$d_{k,5}$	$d_{k,4}$	$d_{k,3}$	$d_{k,2}$	$d_{k,1}$	$d_{k,0}$
1								1	−2	−2
2							1	0	−3	−3
3						1	3	3	0	0
4					1	7	25	60	105	105
5				1	12	75	315	945	1890	1890
6			1	18	168	1050	4725	15120	31185	31185
7		1	25	322	2772	17325	79695	259875	540540	540540
8	1	33	558	6300	51975	322245	1486485	4864860	10135125	10135125

Table. 16.1

16.3 Observational Results About Relationships between Prime Numbers and the Zeta Function

This subsection describes the known relationships between the prime numbers and the zeta function that are necessary "[15]." The purpose is to compare the relationships between the prime numbers and the eta function.
Let consider the second Chebyshev function "[16]."

$$\Psi(x) := \sum_{\substack{p^k \le x \\ p:\,\text{prime},\, k\in\mathbb{N}}} \log(p), \quad x > 0. \tag{16.51}$$

The i-th prime number is assumed to be p_{i}. This function is a step function where the function value steps up by $\log(p_{\mathrm{i}})$ as the positive real variable x continuously increases and reaches the k-th power of the prime number p_{i}. Until it reaches the first prime number 2, the function value on the interval of $0 < x < 2$ is zero.
This function can be written using the Mangoldt function $\Lambda(n)$.

$$\Psi(x) = \sum_{n \le x} \Lambda(n), \quad x > 0. \tag{16.52}$$

Where

$$\Lambda(n) := \begin{cases} \log(n) & n = p^k, \\ 0 & \text{otherwise.} \end{cases} \tag{16.53}$$

Let $\Psi^*(x)$ be the modified function so that its value at the step-up point corresponds to the midpoint.

$$\Psi^*(x) := \frac{1}{2}\lim_{h\to 0}\left(\Psi(x-h) + \Psi(x+h)\right), \quad x > 0. \tag{16.54}$$

The modified second Chebyshev function is described by the second Chebyshev function and the Mangoldt function as follows:

$$\Psi^*(x) = \begin{cases} \Psi(x) - \frac{1}{2}\Lambda(n) & x = 2, 3, 2^2, 5, 7, 2^3, 3^2, 11, 13, 2^4, 17, \cdots, \\ \Psi(x) & \text{otherwise.} \end{cases} \tag{16.55}$$

The Riemann-von Mangold explicit formula states that the modified second Chebyshev function is described by the information on the zeta function as follows:

$$\Psi^*(x) = \mathrm{s}(x) - \sum_{\substack{\zeta(\rho)=0 \\ \rho:\,\text{non-trivial zero}}} \frac{x^\rho}{\rho}, \quad x > 0. \tag{16.56}$$

Here, $\mathrm{s}(x)$ is called the approximate term and is given by the following equations:

$$\mathrm{s}(x) = x - \frac{\log\left(1 - x^{-2}\right)}{2} - \frac{\zeta'(0)}{\zeta(0)}, \quad x > 0. \tag{16.57}$$

$$\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi) \simeq 1.8378770664. \tag{16.58}$$

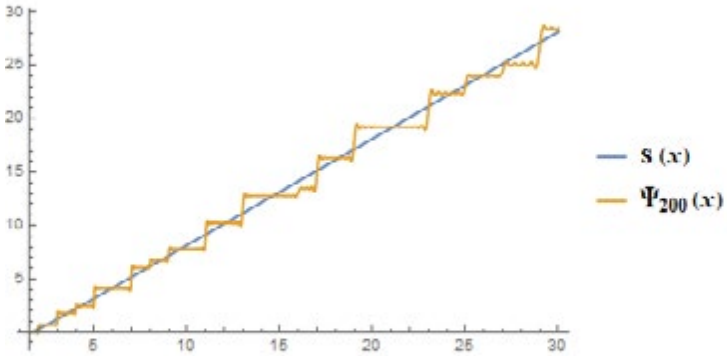
The periodic terms are the second term on the right side of equation (16.56), and the sum of the periodic terms crosses all non-trivial zeros of the zeta function. The function $\Psi_{\mathrm{N}}(x)$ is defined such that the infinite sum of periodic terms is restricted to the sum of N matching pairs. Here, the matching pair consists of a non-trivial zero and its complex conjugate.

$$\Psi_{\mathrm{N}}(x) := \mathrm{s}(x) - \sum_{m=1}^{\mathrm{N}}\left(\frac{x^{1/2+i\rho_m}}{1/2+i\rho_m} + \frac{x^{1/2-i\rho_m}}{1/2-i\rho_m}\right), \quad (x > 0, \mathrm{N} \in \mathbb{N}). \tag{16.59}$$

Here, ρ_m is the positive imaginary part of the m-th non-trivial zero of the zeta function to which the number is allocated in an order that is closer to the real axis. The values of 201 imaginary parts of non-trivial zeros of the zeta function were calculated with 25-decimal accuracy for the following drawing "[17]."

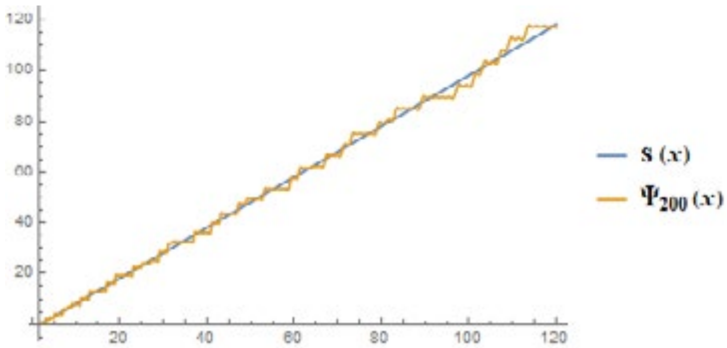
The numerical data will be presented at the end of subsection 16.5.

Graph of the approximate term $\mathrm{s}(x)$ and $\Psi_{200}(x)$ on the interval [3/2,30]



Graph. 16.6

Graph of the approximate term $\mathrm{s}(x)$ and $\Psi_{200}(x)$ on the interval [3/2,120]



Graph. 16.7

It is shown that the step function is generated from the information on the non-trivial zeros of the zeta function.

As a main spectrum, the imaginary parts of non-trivial zeros of the zeta function generate prime numbers. In principle, the points that the step function steps up (i.e., the positive integer power of the prime numbers) can be extracted by differentiating the periodic term of $\Psi_{\mathrm{N}}(x)$ by x.
According to the known result, the positive integer powers of the prime numbers are given as the maximal values of the following function $\Phi_{\mathrm{N}}(x)$, which is generated using the information on finite numbers of the imaginary parts of non-trivial zeros of the zeta function:

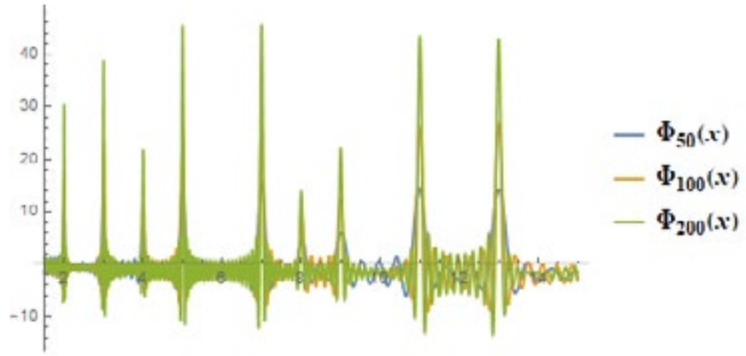
(16.60)
$$\begin{aligned}\Phi_{\mathrm{N}}(x) &:= -\frac{d}{dx}\sum_{m=1}^{\mathrm{N}}\left(\frac{x^{1/2+i\rho_m}}{1/2+i\rho_m}+\frac{x^{1/2-i\rho_m}}{1/2-i\rho_m}\right)\\ &= -\frac{2}{\sqrt{x}}\sum_{m=1}^{\mathrm{N}}\cos\left(\rho_m\log(x)\right), \quad (x>0, \mathrm{N}\in\mathbb{N}).\end{aligned}$$

The damping factor $(2/\sqrt{x})$ written before the summation symbol can be replaced with 1 because the essence of the function is not lost even if it is absent. Therefore, the function can be redefined as follows:

(16.61)
$$\Phi_{\mathrm{N}}(x) := -\sum_{m=1}^{\mathrm{N}}\cos\left(\rho_m\log(x)\right), \quad (x>0, \mathrm{N}\in\mathbb{N}).$$

Since the maximal values at the square or more of a sufficiently large prime number tend to be sufficiently small in comparison with the maximal value at the first power of the prime number, I use the expression ”main spectrum.”

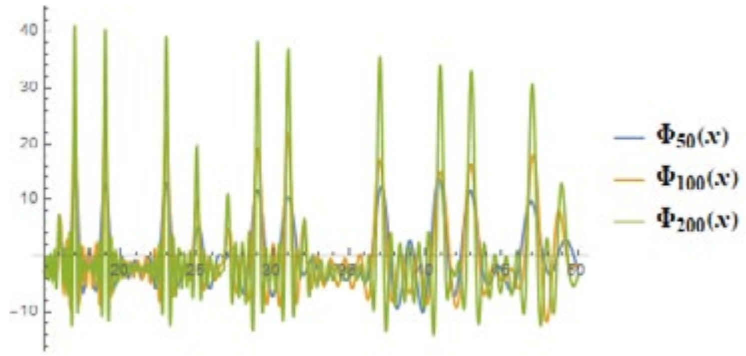
Graph of $\Phi_{50}(x)$, $\Phi_{100}(x)$, and $\Phi_{200}(x)$ on the interval [3/2,15]



Graph. 16.8

The peaks are observed at the points where x is one of the prime numbers 2, 3, 5, 7, 11, and 13; 4 and 9 are the squares of prime numbers 2 and 3, respectively; and 8 is the cube of prime number 2.

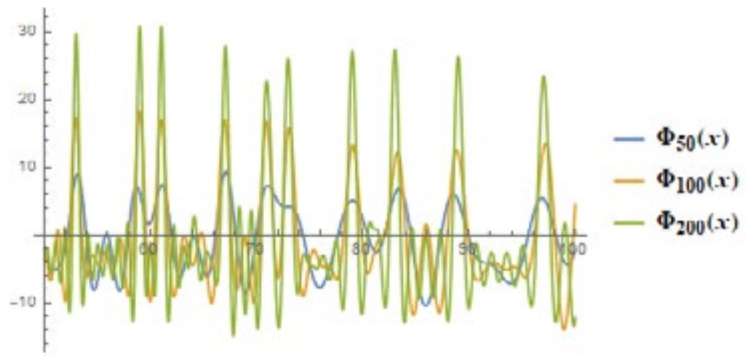
Graph of $\Phi_{50}(x)$, $\Phi_{100}(x)$, and $\Phi_{200}(x)$ on the interval [15,50]



Graph. 16.9

The peaks are found at the points where x is one of the following prime numbers: 17, 19, 23, 29, 31, 37, 41, 43, and 47; 25 and 49 are the squares of prime numbers 5 and 7, respectively; 27 is the cube of prime number 3; and 16 and 32 are the fourth and fifth powers of prime number 2, respectively.

Graph of $\Phi_{50}(x)$, $\Phi_{100}(x)$, and $\Phi_{200}(x)$ on the interval [50,100]



Graph. 16.10

Peaks are recognized where x is one of the following prime numbers: 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97. However, at the points where x is 64 (the sixth power of prime number 2) and 81 (the fourth power of prime number 3), the peak values are so small that their detection is difficult.

The imaginary parts of non-trivial zeros of the zeta function are shown to generate prime numbers as a main spectrum.

When I consider another relation between the prime numbers and the zeta function's non-trivial zeros, the following equation, whose peaks correspond to the imaginary parts of non-trivial zeros, is also known as "[16]":

$$\Omega_{\mathrm{C}}(y) := -\sum_{\substack{p^n \le \mathrm{C} \\ p\text{: prime},\, n\in\mathbb{N}}} \frac{\log(p)}{p^n}\cos\left(y\log\left(p^n\right)\right), \quad (y\in\mathbb{R}, \mathrm{C}\in\mathbb{N}). \tag{16.62}$$

Here, the sum crosses all the integers for which the positive integer powers of the conditional prime number p are less than or equal to the sufficiently large arbitrary positive integer C.
Assuming the k-th prime number as p_k, the conditional partial sum is given as follows:

$$\omega_{\mathrm{C}}(y,m,n) := -\sum_{k=1}^{m}\frac{\log(p_k)}{{p_k}^n}\cos\left(y\log\left({p_k}^n\right)\right), \quad (\mathrm{C},m,n\in\mathbb{N}, y\in\mathbb{R}, {p_m}^n \le \mathrm{C}). \tag{16.63}$$

For the positive integer C, 3 numbers (1260, 110 880, and 10 810 800) are selected to display graphs.
Some prime numbers p_k for the calculations are shown as follows:

$$\{p_1,p_2,p_3,p_4,p_5,p_{205},p_{10\,522},p_{714\,823}\} = \{2,3,5,7,11,1259,110\,879,10\,810\,763\}. \tag{16.64}$$

When C=1260,

$$\begin{aligned}\Omega_{1260}(y) &= \omega_{1260}(y,205,1)+\omega_{1260}(y,11,2)+\omega_{1260}(y,4,3) \\ &+\omega_{1260}(y,3,4)+\sum_{n=5}^{6}\omega_{1260}(y,2,n)+\sum_{n=7}^{10}\omega_{1260}(y,1,n), \quad y\in\mathbb{R}.\end{aligned} \tag{16.65}$$

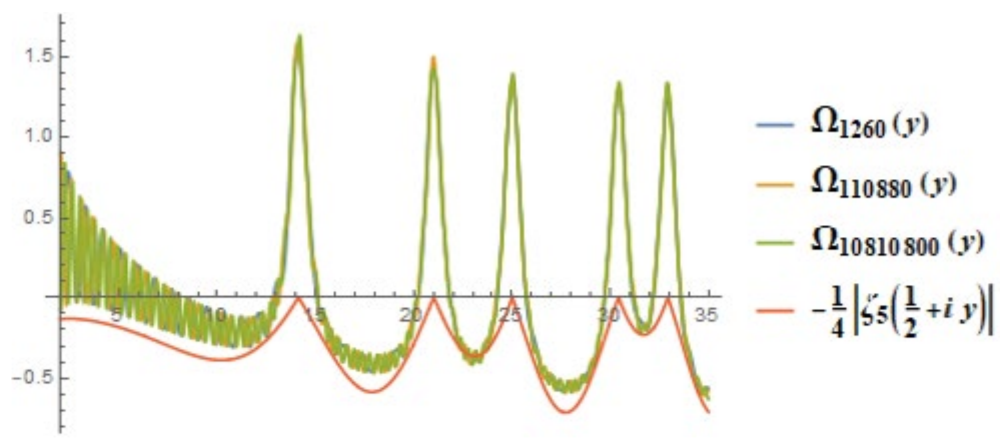
When C=110 880,

$$\begin{aligned}\Omega_{110\,880}(y) &= \omega_{110\,880}(y,10\,522,1)+\omega_{110\,880}(y,67,2)+\omega_{110\,880}(y,15,3)+\omega_{110880}(y,7,4) \\ &+\omega_{110\,880}(y,4,5)+\sum_{n=6}^{7}\omega_{110\,880}(y,3,n)+\sum_{n=8}^{10}\omega_{110\,880}(y,2,n)+\sum_{n=11}^{16}\omega_{110\,880}(y,1,n), \quad y\in\mathbb{R}.\end{aligned} \tag{16.66}$$

When C=10 810 800,

$$\begin{aligned}\Omega_{10\,810\,800}(y) &= \omega_{10\,810\,800}(y,714\,823,1)+\omega_{10\,810\,800}(y,462,2)+\omega_{10\,810\,800}(y,47,3) \\ &+\omega_{10\,810\,800}(y,16,4)+\omega_{10\,810\,800}(y,9,5)+\omega_{10\,810\,800}(y,6,6)+\sum_{n=7}^{8}\omega_{10\,810\,800}(y,4,n) \\ &+\sum_{n=9}^{10}\omega_{10\,810\,800}(y,3,n)+\sum_{n=11}^{14}\omega_{10\,810\,800}(y,2,n)+\sum_{n=15}^{23}\omega_{10\,810\,800}(y,1,n), \quad y\in\mathbb{R}.\end{aligned} \tag{16.67}$$

Graph of $\Omega_{\mathrm{C}}(y)$ $(\mathrm{C} = 1\,260, 110\,880, 10\,810\,800)$ and $-\frac{1}{4}\left|\zeta_5\left(\frac{1}{2}+i\,y\right)\right|$, $y\in[2,35]$



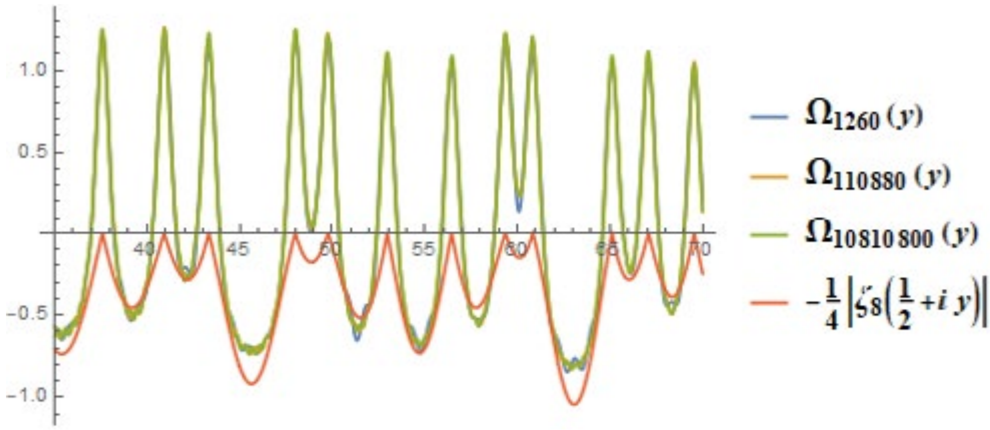
Graph. 16.11

To clearly show the imaginary parts of non-trivial zeros of the zeta function, the function

$$-\frac{1}{4}\left|\zeta_5\left(\frac{1}{2}+i\,y\right)\right|$$

is superposed on the above graph. Hereafter, functions showing the imaginary parts of non-trivial zeros of the zeta functions or those of the eta functions are superposed on all graphs as needed.
The imaginary parts of non-trivial zeros of the zeta function are shown to exist under each peak, i.e., the maximal values of the functions $\Omega_{\mathrm{C}}(y)$.

Graph of $\Omega_{\mathrm{C}}(y)$ $(\mathrm{C}=1\,260, 110\,880, 10\,810\,800)$ and $-\frac{1}{4}\left|\zeta_8\left(\frac{1}{2}+i\,y\right)\right|$, $y\in[35,70]$



Graph. 16.12

I demonstrated that the imaginary parts of the zeta function's non-trivial zeros are generated as a spectrum using only prime number information.

16.4 Observational Results about Relationships between Prime Numbers and the Eta Function Without a Proof

I define the following equation as an analogy to the equation (16.56):

$$\tilde{\Psi}^*(x) := \tilde{s}(x) - \sum_{\substack{\eta(\tau)=0 \\ \tau:\,\text{non-trivial zero}}} \frac{\left(\sqrt{x}\right)^{\tau}}{\tau}, \quad x>0. \tag{16.68}$$

In comparison to equations (16.57) and (16.58), the positive real variable x is replaced with $x^{1/2}$ and the non-trivial zeros of the zeta function are replaced with those of the eta function.

$$\tilde{s}(x) := \sqrt{x} - \frac{\log\left(1-x^{-1}\right)}{2} - \frac{\eta'(0)}{\eta(0)}, \quad x>0. \tag{16.69}$$

$$\frac{\eta'(0)}{\eta(0)} \simeq 1.8609727754. \tag{16.70}$$

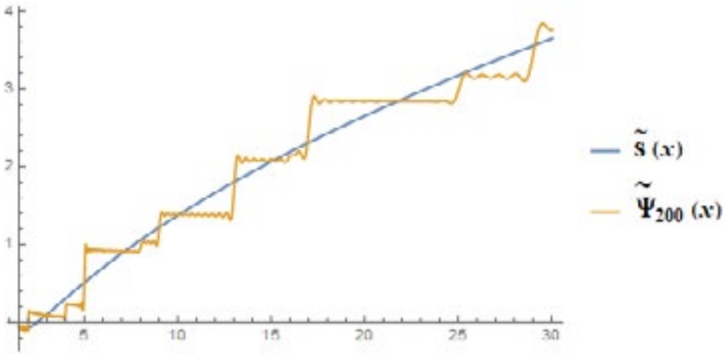
The sum of the periodic terms crosses all non-trivial zeros of the eta function.
As an analogy to the equation (16.59), I define the following equation:

$$\tilde{\Psi}_{\mathrm{N}}(x) := \tilde{s}(x) - \sum_{m=1}^{\mathrm{N}} \left(\frac{\left(\sqrt{x}\right)^{i\tau_m}}{i\tau_m} + \frac{\left(\sqrt{x}\right)^{-i\tau_m}}{-i\tau_m} \right), \quad \left(x>0, \mathrm{N}\in\mathbb{N}\right). \tag{16.71}$$

Here, τ_m is the positive imaginary part of the eta function's m-th non-trivial zero, to which the number is allocated in order that is closer to the real axis.

The values of 201 imaginary parts of non-trivial zeros of the eta function were obtained with 25-decimal accuracy following the drawing. Specifically, as part of the search for the non-trivial zeros of the eta function, the sign changing was investigated at a step width of 0.1 (finely adjusted if necessary) using $\tilde{\eta}_\lambda(\theta)$, which takes a real value on the imaginary axis.
At the end of subsection 16.5, the numerical data will be presented.

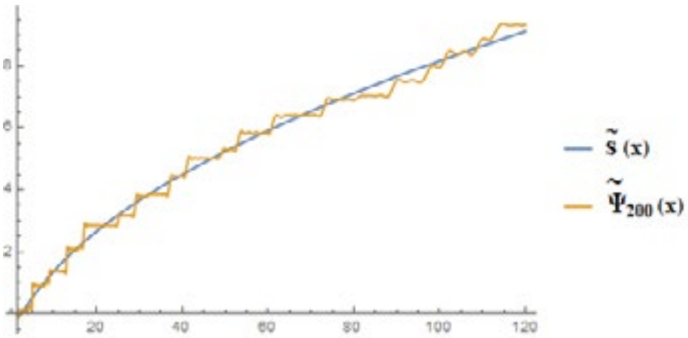
Graph of the approximate term $\tilde{\mathrm{s}}(x)$ and $\tilde{\Psi}_{200}$ on the interval [3/2,30]



Graph. 16.13

The step-up is not recognized at the non-Pythagorean prime points of 3, 7, 11, 19, and 23 at all.

Graph of the approximate term $\tilde{\mathrm{s}}(x)$ and $\tilde{\Psi}_{200}$ on the interval [3/2,120]



Graph. 16.14

The information on the non-trivial zeros of the eta function is used to produce the different step function.

In principle, the points that the step function steps up can be extracted by differentiating the periodic terms of $\tilde{\Psi}_{\mathrm{N}}(x)$ by x.
As a counterpart to the equation (16.60), I define the following equation:

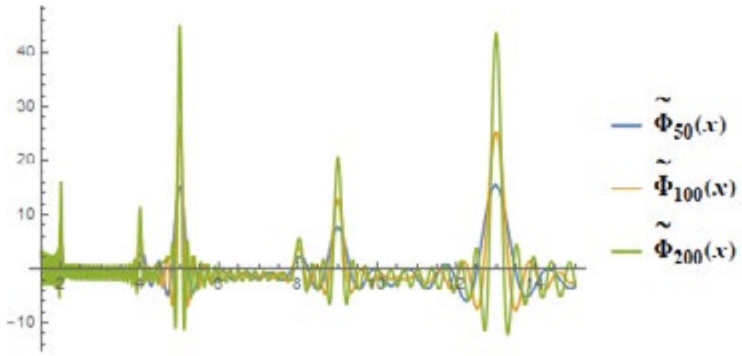
(16.72)
$$\begin{aligned}\tilde{\Phi}_{\mathrm{N}}(x) &:= -\frac{d}{dx}\sum_{m=1}^{\mathrm{N}}\left(\frac{(\sqrt{x})^{i\tau_m}}{i\tau_m}+\frac{(\sqrt{x})^{-i\tau_m}}{-i\tau_m}\right)\\ &= -\frac{1}{x}\sum_{m=1}^{\mathrm{N}}\cos\left(\frac{\tau_m}{2}\log(x)\right), \quad (x>0, \mathrm{N}\in\mathbb{N}).\end{aligned}$$

The damping factor $1/x$ written before the summation symbol can also be replaced with 1 because the essence of the function is not lost even if it is absent. Therefore, the function can be redefined as follows:

(16.73)
$$\tilde{\Phi}_{\mathrm{N}}(x) := -\sum_{m=1}^{\mathrm{N}}\cos\left(\frac{\tau_m}{2}\log(x)\right), \quad (x>0, \mathrm{N}\in\mathbb{N}).$$

In contrast to the equation (16.61), the essential difference is that ρ_m is replaced by $\tau_m/2$.

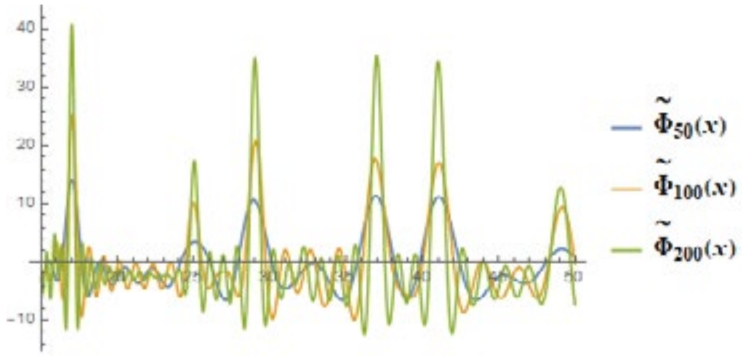
Graph of $\tilde{\Phi}_{50}(x)$, $\tilde{\Phi}_{100}(x)$, and $\tilde{\Phi}_{200}(x)$ on the interval [3/2,15]



Graph. 16.15

The peaks are recognized at the points where x is the Pythagorean primes 5 and 13; 2, 4, and 8 are the first, second, and third powers of prime number 2; and 9 is the square of prime number 3.

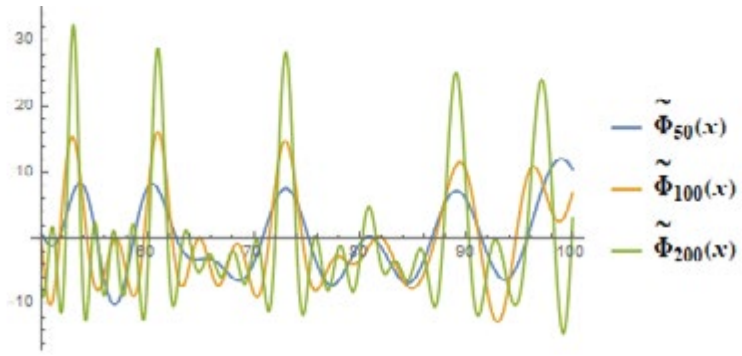
Graph of $\tilde{\Phi}_{50}(x)$, $\tilde{\Phi}_{100}(x)$, and $\tilde{\Phi}_{200}(x)$ on the interval [15,50]



Graph. 16.16

The peaks are recognized at the points where x is one of the following Pythagorean primes: 17, 29, 37, and 41; 16 is the fourth power of prime number 2; and 25 and 49 are the squares of prime numbers 5 and 7, respectively. However, the peaks are not recognized at the points where x is 27(the cube of prime number 3) and 32(fifth power of prime number 2).

Graph of $\tilde{\Phi}_{50}(x)$, $\tilde{\Phi}_{100}(x)$, and $\tilde{\Phi}_{200}(x)$ on the interval [50,100]



Graph. 16.17

The peaks are recognized at the points where x is one of the Pythagorean primes 53, 61, 73, 89, and 97; 81 is the fourth power of prime number 3. However, the recognition of the peak is difficult when x is 64(sixth power of prime number 2). It is obvious that the imaginary parts of the eta function's non-trivial zeros generate Pythagorean primes, squares of non-Pythagorean primes, and even prime 2 as a main spectrum.

Based on the research results so far, let define the following function:

(16.74)
$$\tilde{\Omega}_{\mathrm{C}}(y) := -\frac{1}{2}\sum_{2^n\le \mathrm{C},\, n\in\mathbb{N}}\frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log(2^n)\right) - \sum_{\substack{q^n\le \mathrm{C} \\ q:\,\text{Pythagorean prime},\, n\in\mathbb{N}}}\frac{\log(q)}{q^n}\cos\left(\frac{y}{2}\log(q^n)\right) - \sum_{\substack{r^{2n}\le \mathrm{C} \\ r:\,\text{non-Pythagorean prime},\, n\in\mathbb{N}}}\frac{\log(r)}{r^{2n}}\cos\left(\frac{y}{2}\log\left(r^{2n}\right)\right), \quad (y\in\mathbb{R}, \mathrm{C}\in\mathbb{N}).$$

Here, each sum crosses all integers whose positive integer powers corresponding to even prime 2 or the conditional Pythagorean prime q or the conditional non-Pythagorean prime r are less than or equal to the sufficiently large arbitrary positive integer C.
Assuming the k-th Pythagorean prime as q_k, the conditional partial sum of the second term on the right side of equation (16.74) is defined as follows:

(16.75)
$$\tilde{\omega}_{\mathrm{C}}(y,m,n) := -\sum_{k=1}^{m}\frac{\log(q_k)}{{q_k}^n}\cos\left(\frac{y}{2}\log\left({q_k}^n\right)\right), \quad (\mathrm{C},m,n\in\mathbb{N}, y\in\mathbb{R}, {q_m}^n\le \mathrm{C}).$$

For the positive integer C, 3 numbers (5040, 554 400, and 61 261 200) are selected to display graphs.
Some Pythagorean primes q_k for the calculations are shown as follows:

(16.76)
$$\{q_1,q_2,q_3,q_4,q_5,q_{331},q_{22\,799},q_{1\,815\,875}\} = \{5,13,17,29,37,5021,554\,377,61\,261\,169\}.$$

Assuming the k-th non-Pythagorean prime as r_k, the conditional partial sum of the third term on the right side of equation (16.74) is defined as follows:

(16.77)
$$\hat{\omega}_{\mathrm{C}}(y,m,n) := -\sum_{k=1}^{m}\frac{\log(r_k)}{{r_k}^{2n}}\cos\left(\frac{y}{2}\log\left({r_k}^{2n}\right)\right), \quad \left(\mathrm{C},m,n\in\mathbb{N}, y\in\mathbb{R}, {r_m}^{2n}\le \mathrm{C}\right).$$

Some Pythagorean primes r_k for the calculations are shown as follows:

(16.78)
$$\{r_1,r_2,r_3,r_4,r_5,r_{10},r_{69},r_{499}\} = \{3,7,11,19,23,67,743,7823\}.$$

When $\mathrm{C} = 5040$,

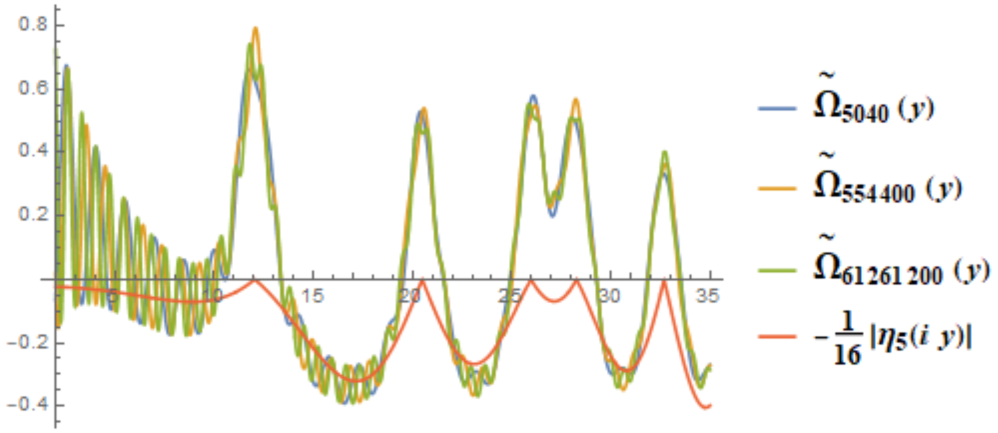
(16.79)
$$\tilde{\Omega}_{5040}(y) = -\frac{1}{2}\sum_{n=1}^{12}\frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log(2^n)\right) + \tilde{\omega}_{5040}(y,331,1) + \tilde{\omega}_{5040}(y,8,2) + \tilde{\omega}_{5040}(y,3,3) + \tilde{\omega}_{5040}(y,2,4) + \sum_{n=5}^{7}\tilde{\omega}_{5040}(y,1,n) + \hat{\omega}_{5040}(y,10,1) + \hat{\omega}_{5040}(y,2,2) + \hat{\omega}_{5040}(y,1,3), \quad y\in\mathbb{R}.$$

When $\mathrm{C} = 554\,400$,

(16.80)
$$\tilde{\Omega}_{554\,400}(y) = -\frac{1}{2}\sum_{n=1}^{19}\frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log(2^n)\right) + \tilde{\omega}_{554\,400}(y,22\,799,1) + \tilde{\omega}_{554\,400}(y,62,2) + \tilde{\omega}_{554\,400}(y,9,3) + \tilde{\omega}_{554\,400}(y,3,4) + \tilde{\omega}_{554\,400}(y,2,5) + \sum_{n=6}^{8}\tilde{\omega}_{554\,400}(y,1,n) + \hat{\omega}_{554\,400}(y,69,1) + \hat{\omega}_{554\,400}(y,5,2) + \hat{\omega}_{554\,400}(y,2,3) + \sum_{n=4}^{6}\hat{\omega}_{554\,400}(y,1,n), \quad y\in\mathbb{R}.$$

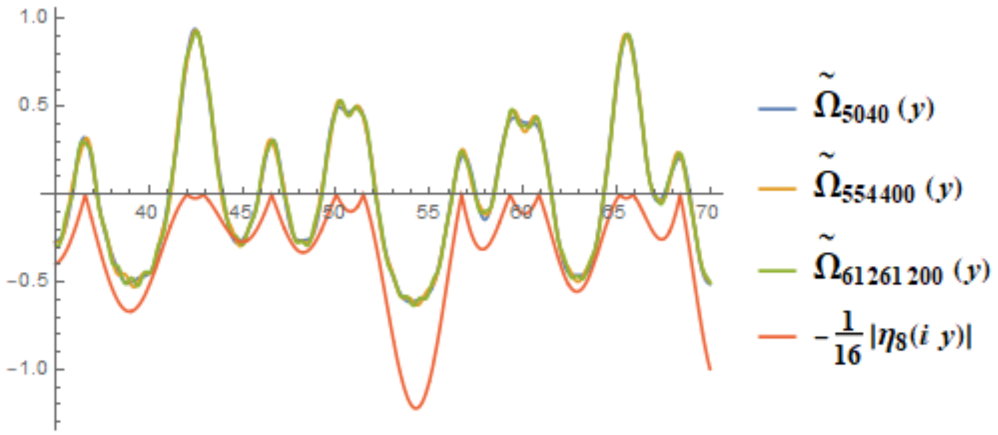
When $\mathrm{C} = 61\,261\,200$,
(16.81)
$$\tilde{\Omega}_{61\,261\,200}(y) = -\frac{1}{2}\sum_{n=1}^{25}\frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log(2^n)\right) + \tilde{\omega}_{61\,261\,200}(y,1\,815\,875,1) + \tilde{\omega}_{61\,261\,200}(y,489,2) + \tilde{\omega}_{61\,261\,200}(y,36,3) + \tilde{\omega}_{61\,261\,200}(y,9,4) + \tilde{\omega}_{61\,261\,200}(y,4,5) + \tilde{\omega}_{61\,261\,200}(y,3,6) + \sum_{n=7}^{11}\tilde{\omega}_{61\,261\,200}(y,1,n) + \hat{\omega}_{61\,261\,200}(y,499,1) + \hat{\omega}_{61\,261\,200}(y,13,2) + \hat{\omega}_{61\,261\,200}(y,4,3) + \hat{\omega}_{61\,261\,200}(y,2,4) + \sum_{n=5}^{8}\hat{\omega}_{61\,261\,200}(y,1,n), \quad y\in\mathbb{R}.$$

Graph of $\tilde{\Omega}_{\mathrm{C}}(y)$ $(\mathrm{C} = 5040, 554\,400, 61\,261\,200)$ and $-\dfrac{1}{16}\left|\eta_5(i\,y)\right|, \; y \in [2, 35]$



Graph. 16.18

Graph of $\tilde{\Omega}_{\mathrm{C}}(y)$ $(\mathrm{C} = 5040, 554\,400, 61\,261\,200)$ and $-\dfrac{1}{16}\left|\eta_8(i\,y)\right|, \; y \in [35, 70]$



Graph. 16.19

Below each peak, there are one or two imaginary parts of the eta function's non-trivial zeros. It is shown that the imaginary parts of non-trivial zeros of the eta function are generated as a quasi-spectrum using the information on Pythagorean primes, squares of non-Pythagorean primes, and even prime 2.

For further exploration, I newly define the following function:
(16.82)

$$\bar{\Omega}_{\mathrm{C}}(y) := -\frac{1}{2}\sum_{2^n \le \mathrm{C},\, n\in\mathbb{N}} \frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log\left(2^n\right)\right) - \sum_{\substack{r^{2n-1}\le \mathrm{C} \\ r:\,\text{non-Pythagorean prime},\, n\in\mathbb{N}}} \frac{\log(r)}{r^{2n-1}}\cos\left(\frac{y}{2}\log\left(r^{2n-1}\right)\right), \quad (y\in\mathbb{R}, \mathrm{C}\in\mathbb{N}).$$

By using this, I obtain the following obvious relation:

(16.83) $$\Omega_{\mathrm{C}}(y) = \tilde{\Omega}_{\mathrm{C}}(2y) + \bar{\Omega}_{\mathrm{C}}(2y), \quad (y\in\mathbb{R}, \mathrm{C}\in\mathbb{N}).$$

The conditional partial sum of the second term on the right side of equation (16.83) is defined as follows:

(16.84) $$\bar{\omega}_{\mathrm{C}}(y, m, n) := -\sum_{k=1}^{m} \frac{\log(r_k)}{{r_k}^{2n-1}}\cos\left(\frac{y}{2}\log\left({r_k}^{2n-1}\right)\right), \quad \left(\mathrm{C}, m, n\in\mathbb{N}, y\in\mathbb{R}, {r_m}^{2n-1}\le \mathrm{C}\right).$$

When $\mathrm{C} = 5040$,

(16.85) $$\bar{\Omega}_{5040}(y) = -\frac{1}{2}\sum_{n=1}^{12}\frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log\left(2^n\right)\right) + \bar{\omega}_{5040}(y, 343, 1) + \bar{\omega}_{5040}(y, 3, 2) + \sum_{n=3}^{4}\bar{\omega}_{5040}(y, 1, n), \quad y\in\mathbb{R}.$$

When C = 554 400,

$$\bar{\Omega}_{554\,400}(y) = -\frac{1}{2}\sum_{n=1}^{19}\frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log(2^n)\right) + \bar{\omega}_{554\,400}(y, 22855, 1) + \bar{\omega}_{554\,400}(y, 12, 2)$$

(16.86)

$$+\,\bar{\omega}_{554\,400}(y, 3, 3) + \sum_{n=4}^{6}\bar{\omega}_{554\,400}(y, 1, n), \quad y \in \mathbb{R}.$$

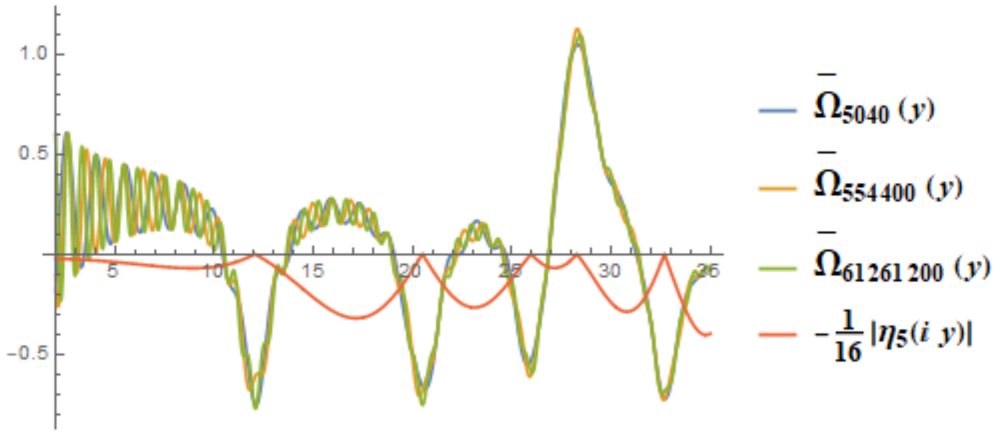
When C = 61 261 200,

$$\bar{\Omega}_{61\,261\,200}(y) = -\frac{1}{2}\sum_{n=1}^{25}\frac{\log(2)}{2^n}\cos\left(\frac{y}{2}\log(2^n)\right) + \bar{\omega}_{61\,261\,200}(y, 1\,816\,644, 1) + \bar{\omega}_{61\,261\,200}(y, 40, 2)$$

(16.87)

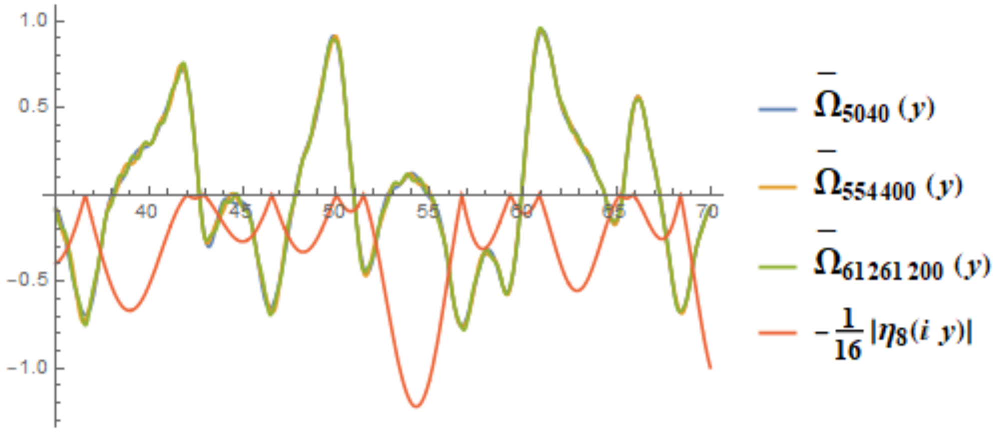
$$+\,\bar{\omega}_{61\,261\,200}(y, 6, 3) + \bar{\omega}_{61\,261\,200}(y, 3, 4) + \bar{\omega}_{61\,261\,200}(y, 2, 5) + \sum_{n=6}^{8}\bar{\omega}_{61\,261\,200}(y, 1, n), \quad y \in \mathbb{R}.$$

Graph of $\bar{\Omega}_{\mathrm{C}}(y)$ $(\mathrm{C} \in \{5040, 554\,400, 61\,261\,200\})$ and $-\frac{1}{16}|\eta_5(i\,y)|$, $y \in [2, 35]$



Graph. 16.20

Graph of $\bar{\Omega}_{\mathrm{C}}(y)$ $(\mathrm{C} = 5040, 554\,400, 61\,261\,200)$ and $-\frac{1}{16}|\eta_8(i\,y)|$, $y \in [35, 70]$

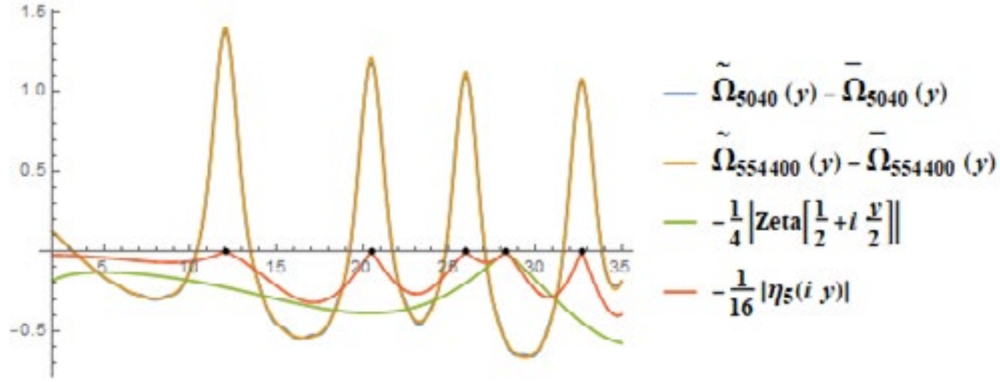


Graph. 16.21

It is shown shortly after that the $\tilde{\Omega}_{\mathrm{C}}(y)$ and $\bar{\Omega}_{\mathrm{C}}(y)$ functions are important when considering their difference and sum.

Graph of $\tilde{\Omega}_{\mathrm{C}}(y) - \bar{\Omega}_{\mathrm{C}}(y)$ $(\mathrm{C} \in \{5040, 554\,400\})$, $-\frac{1}{4}\left|\mathrm{Zeta}\left[\frac{1}{2} + i\,\frac{y}{2}\right]\right|$ and $-\frac{1}{16}\left|\eta_5(i\,y)\right|$, $y \in [2, 35]$

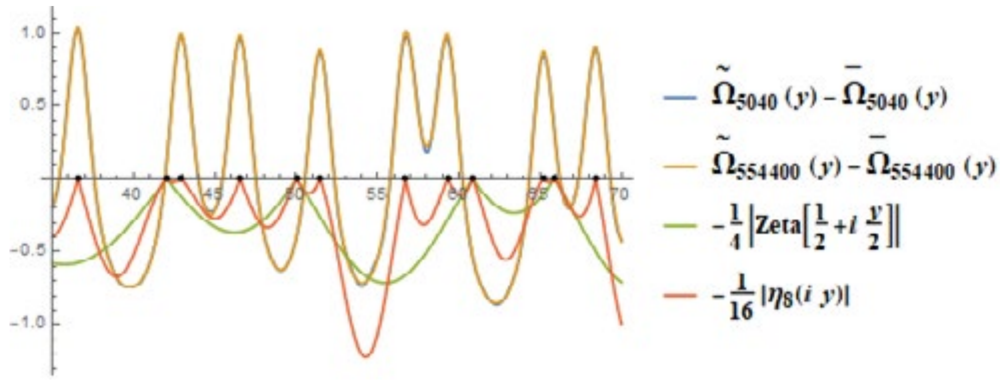
$\bullet$: τ_m, $m \in \{1, 2, 3, 4, 5\}$



Graph. 16.22

Graph of $\tilde{\Omega}_{\mathrm{C}}(y) - \bar{\Omega}_{\mathrm{C}}(y)$ $(\mathrm{C} \in \{5040, 554\,400\})$, $-\frac{1}{4}\left|\mathrm{Zeta}\left[\frac{1}{2} + i\,\frac{y}{2}\right]\right|$ and $-\frac{1}{16}\left|\eta_8(i\,y)\right|$, $y \in [35, 70]$

$\bullet$: τ_m, $m \in \{6, 7, 8, \cdots, 17\}$

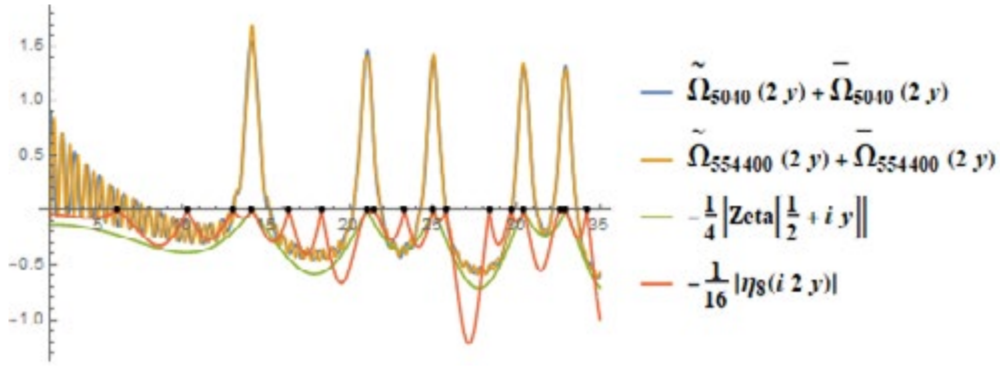


Graph. 16.23

The dots on the horizontal axis indicate the imaginary parts of the eta function's non-trivial zeros.
Below each peak, there is always one imaginary part of the eta function's non-trivial zero. In the observation range, the imaginary parts of non-trivial zeros of the modified zeta function $\zeta(\theta/2 + 1/4)$ belong to the subset of those of the eta function. In other words, twice the imaginary part of each zeta function's non-trivial zero belongs to the subset of imaginary parts of the eta function's non-trivial zeros. Twice the imaginary parts of the zeta function's non-trivial zeros are far from the y-coordinates of the peaks, respectively.

Graph of $\tilde{\Omega}_{\mathrm{C}}(2\,y) + \bar{\Omega}_{\mathrm{C}}(2\,y)$ $(\mathrm{C} \in \{5040, 554\,400\})$, $-\frac{1}{4}\left|\mathrm{Zeta}\left[\frac{1}{2} + i\,y\right]\right|$ and $-\frac{1}{16}\left|\eta_8(2\,i\,y)\right|$, $y \in [2, 35]$

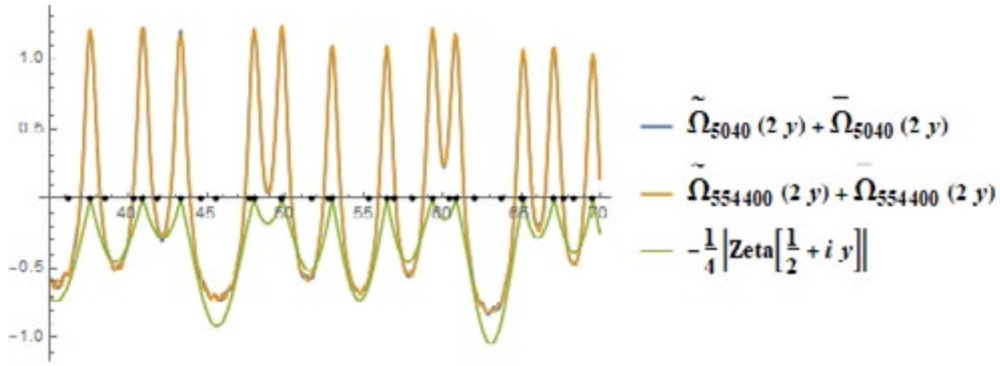
$\bullet$: $\frac{\tau_m}{2}$, $m \in \{1, 2, 3, \cdots, 17\}$



Graph. 16.24

Graph of $\tilde{\Omega}_{\mathrm{C}}(2\,y) + \bar{\Omega}_{\mathrm{C}}(2\,y)$ $(\mathrm{C} \in \{5040, 554\,400\})$ and $-\frac{1}{4}\left|\mathrm{Zeta}\left[\frac{1}{2} + i\,y\right]\right|$, $y \in [35, 70]$

$\bullet$: $\frac{\tau_m}{2}$, $m \in \{18, 19, 20, \cdots, 48\}$



Graph. 16.25

The dots on the horizontal axis indicate the imaginary parts of non-trivial zeros of the modified eta function $\eta(2\theta)$. Below each peak, there is always one imaginary part of non-trivial zero of the modified eta function. Furthermore, it is also an imaginary part of non-trivial zero of the zeta function.

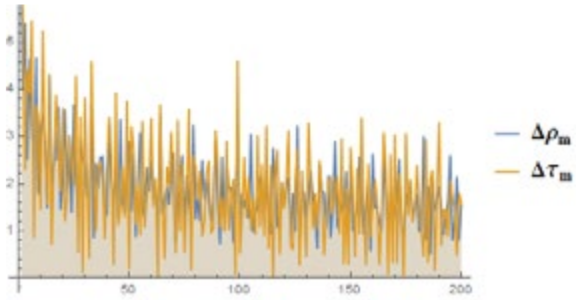
In this margin, I will write down one topic that I could not write about in the previous subsections. Numbers such as 1260, 5040, 110 880, 554 400, 10 810 800, and 61 261 200 are all called highly composite numbers.
A highly composite number is a positive integer with more divisors than any smaller positive integer has.
Highly composite numbers were chosen as an option for the number C because it is easy to understand all the divisors of the highly composite number C.

16.5 Two Kinds of the Intervals of Adjacent Zeros Part1

The intervals of adjacent zeros of the zeta function on the critical line and the intervals of adjacent zeros of the eta function on the imaginary axis are shown on the same graph.

(16.88) $$\Delta\rho_m := \rho_{m+1} - \rho_m, \quad \Delta\tau_m := \tau_{m+1} - \tau_m, \quad (m \in \mathbb{N}, \rho_m, \tau_m > 0, \zeta(1/2 + i\rho_m) = \eta(i\tau_m) = 0).$$

Graph of $\Delta\rho_m$ and $\Delta\tau_m$, $m \in \{1, 2, 3, \cdots, 200\}$

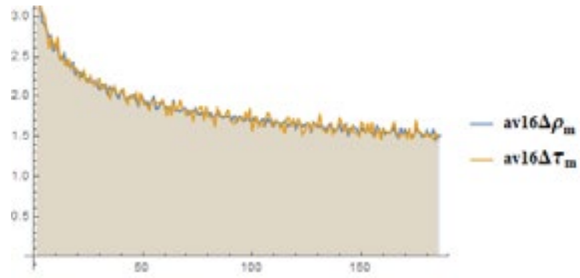


Graph. 16.26

Via taking the moving averages of 16 elements for the intervals of adjacent zeros,

$$\text{av16}\Delta\rho_m := \frac{1}{16}\sum_{k=m}^{m+15}\Delta\rho_k, \quad \text{av16}\Delta\tau_m := \frac{1}{16}\sum_{k=m}^{m+15}\Delta\tau_k, \quad m \in \mathbb{N}. \tag{16.89}$$

Graph of av16$\Delta\rho_m$ and av16$\Delta\tau_m$, $m \in \{1, 2, 3, \cdots, 186\}$

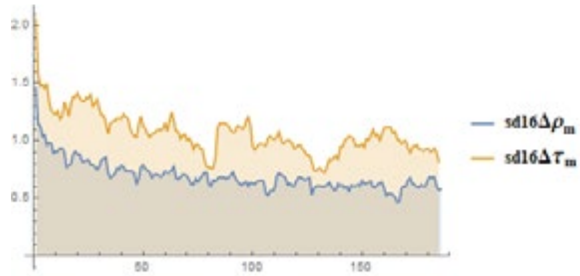


Graph. 16.27

Via taking the moving standard deviations of 16 elements for the intervals of adjacent zeros,

$$\text{sd16}\Delta\rho_m := \sqrt{\frac{1}{15}\sum_{k=m}^{m+15}(\Delta\rho_k - \text{av16}\Delta\rho_k)^2}, \quad \text{sd16}\Delta\tau_m := \sqrt{\frac{1}{15}\sum_{k=m}^{m+15}(\Delta\tau_k - \text{av16}\Delta\tau_k)^2}, \quad m \in \mathbb{N}. \tag{16.90}$$

Graph of sd16$\Delta\rho_m$ and sd16$\Delta\tau_m$, $m \in \{1, 2, 3, \cdots, 186\}$



Graph. 16.28

It is recognized that the moving standard deviations for the intervals of adjacent zeros of the zeta function are smaller than those of the eta function at each point in the observation range.

List of 201 non-trivial zeros of the zeta function (Positive imaginary part, 25-decimal accuracy), Part 1

k	ρ_{5k-4}	ρ_{5k-3}	ρ_{5k-2}	ρ_{5k-1}	ρ_{5k}	$5k-4 \sim 5k$
1	14.1347251417346937904572 5	21.0220396387715549926284 8	25.0108575801456887632137 9	30.4248761258595132103119 0	32.9350615877391896906623 7	$1 \sim 5$
2	37.5861781588256712572177 6	40.9187190121474951873981 3	43.3270732809149995194961 2	48.0051508811671597279424 7	49.7738324776723021819167 8	$6 \sim 10$
3	52.9703214777144606441473 0	56.4462476970633948043677 6	59.3470440026023530796536 5	60.8317785246098098442599 0	65.1125440480816066608750 5	$11 \sim 15$
4	67.0798105294941737144788 3	69.5464017111739792529268 6	72.0671576744819075825221 1	75.7046906990839331683269 2	77.1448400688748053726826 6	$16 \sim 20$
5	79.3373750202493679227635 9	82.9103808540860301831648 4	84.7354929805170501057353 1	87.4252746131252294065316 7	88.8091112076344654236823 5	$21 \sim 25$
6	92.4918992705584842962597 3	94.6513440405198869665979 3	95.8706342282453097587410 3	98.8311942181936922333244 2	101.3178510057313912287855	$26 \sim 30$
7	103.7255380404783394163984	105.4466230523260944936708	107.1686111842764075151234	111.0295355431696745246565	111.8746591769926370856121	$31 \sim 35$
8	114.3202209154527127658909	116.2266803208575543821608	118.7907828659762173229791	121.3701250024206459189455	122.9468292935525882008175	$36 \sim 40$
9	124.2568185543457671847320	127.5166838795964951242793	129.5787041999560509857680	131.0876885309326567235664	133.4977372029975864501305	$41 \sim 45$
10	134.7565097533738713313261	138.1160420545334432001916	139.7362089521213889504501	141.1237074040211237619404	143.1118458076206327394051	$46 \sim 50$
11	146.0009824867655185474025	147.4227653425596020495212	150.0535204207848803514325	150.9252576122414667618525	153.0246938111988961982565	$51 \sim 55$
12	156.1129092942378675697502	157.5975918175940598875305	158.8499881714204987241750	161.1889641375960275194373	163.0307096871819872433110	$56 \sim 60$
13	165.5370691879004188300389	167.1844399781745134409578	169.0945154155688214895059	169.9119764794116989666998	173.4115365195915529598461	$61 \sim 65$
14	174.7541915233657258133788	176.4414342977104188888926	178.3774077760999772858309	179.9164840202569961393400	182.2070784843664619154070	$66 \sim 70$
15	184.8744678483875088009607	185.5987836777074714665277	187.2289225835018519916415	189.4161586560169370848523	192.0266563607137865472836	$71 \sim 75$
16	193.0797266038457040474022	195.2653966795292353214632	196.8764818409583169486223	198.0153096762519124249199	201.2647519437037887330161	$76 \sim 80$
17	202.4935945141405342776867	204.1896718031045543307164	205.3946972021632860252124	207.9062588878062098615020	209.5765097168562598528356	$81 \sim 85$
18	211.6908625953653075639075	213.3479193597126661906391	214.5470447834914232229442	216.1695385082637002658696	219.0675963490213789856773	$86 \sim 90$
19	220.7149188393140033691156	221.4307055546933387320975	224.0070002546043352117289	224.9833246695822875037825	227.4214442796792913104614	$91 \sim 95$
20	229.3374133055253481077601	231.2501887004991647738062	231.9872352531802486037717	233.6934041789083006407045	236.5242296658162058024755	$95 \sim 100$

Table. 16.2

It can be confirmed that 79 non-trivial zeros of the zeta function clored in blue satisfy the following equations:

(16.91)
$$\zeta\left(\frac{1}{2}+i\rho_m\right)=\eta\left(2\,i\,\rho_m\right)=0,\quad m\in\{1,2,3,\cdots,79\}.$$

List of 201 non-trivial zeros of the zeta function (Positive imaginary part, 25-decimal accuracy), Part 2

k	ρ_{5k-4}	ρ_{5k-3}	ρ_{5k-2}	ρ_{5k-1}	ρ_{5k}	$5k-4 \sim 5k$
21	237.7698204809252040032366	239.5554775733276287402689	241.0491577962165864128379	242.8232719342226000168265	244.0708984970781582368165	101 ∼ 105
22	247.1369900748974994675510	248.1019900601484592567621	249.5736896447072091923298	251.0149477950160011429542	253.0699867479994771945990	106 ∼ 110
23	255.3062564549140227530865	256.3807136944344777893584	258.6104394915313682089831	259.8744069896780003506728	260.8050845045968701859312	111 ∼ 115
24	263.5738939048701322330816	265.5578518388763202924773	266.6149737815010724957201	267.9219150828240594403790	269.9704490239976025946935	116 ∼ 120
25	271.4940556416449990181794	273.4596091884032870457143	275.5874926493438412487407	276.4520495031329386798873	278.2507435298419544927483	121 ∼ 125
26	279.2292509277451892284099	282.4651147650520962330272	283.2111857332338674204938	284.8359639809047241331576	286.6674453630028842928476	126 ∼ 130
27	287.9119205014221871552541	289.5798549292188341527380	291.8462913290673958355131	293.5584341393562853567767	294.9653696192655421750665	131 ∼ 135
28	295.5732548789582923884608	297.9792770619434152099297	299.8403260537213129600271	301.6493254621941836234701	302.6967495896069170517515	136 ∼ 140
29	304.8643713408572977001488	305.7289126020368092892228	307.2194961281700547894100	310.1094631467018988047862	311.1651415303560032709427	141 ∼ 145
30	312.4278011806008919804860	313.9852857311589229790490	315.4756160894757338685961	317.7348059423701803956455	318.8531042563165979066892	146 ∼ 150
31	321.1601343091135782919215	322.1445586724829322988375	323.4669695575120505062120	324.8628660517396132649801	327.4439012619054573434693	151 ∼ 155
32	329.0330716804809340336147	329.9532397282338663438921	331.4744675826634243756618	333.6453785248698505849617	334.2113548332443832324034	156 ∼ 160
33	336.8418504283906847946548	338.3399928508066118862573	339.8582167253635401923266	341.0422611110465604825978	342.0548775103635854514038	161 ∼ 165
34	344.6617029402523370441812	346.3478705660099473959365	347.2726775844204844757971	349.3162608706961441231556	350.4084193491920991876720	166 ∼ 170
35	351.8786490253592804367134	353.4889004887188067836038	356.0175749772649473179604	357.1513022520396248096029	357.9526851016322737551289	171 ∼ 175
36	359.7437549531144487992920	361.2893616958046503902913	363.3313305789738347473345	364.7360241140889937162621	366.2127102883313168610772	176 ∼ 180
37	367.9935754817403033261833	368.9684380957343898915769	370.0509192121060003396512	373.0619283721128384491194	373.8648739109085697447564	181 ∼ 185
38	375.8259127667393341079077	376.3240922306680521171908	378.4366802499654797240910	379.8729753465323466510241	381.4844686171865249196625	186 ∼ 190
39	383.4435294495364877043458	384.9561168148636871037516	385.8613008459742291805620	387.2228902223879809759485	388.8461283542322546008094	191 ∼ 195
40	391.4560835636380457705782	392.2450833395190967490152	393.4277438444340259366990	395.5828700109937209708777	396.3818542225921869319995	195 ∼ 200
41	397.9187362096142433869812	Not calculated	Not calculated	Not calculated	Not calculated	201 ∼ 205

Table. 16.3

List of 201 non-trivial zeros of the eta function (Positive imaginary part, 25-decimal accuracy), Part 1

k	τ_{5k-4}	τ_{5k-3}	τ_{5k-2}	τ_{5k-1}	τ_{5k}	$5k-4 \sim 5k$
1	12.0418978093951933098040 0	20.4875406083331091042755 1	25.9761960246248450149062 2	28.2694502834693875809145 0	32.6852142091744443899537 2	1 ∼ 5
2	36.5839863922470696770520 1	42.0440792775431099852569 6	42.9012226879669209944019 0	46.5567530409190630636391 2	50.0217151602913775264275 8	6 ∼ 10
3	51.4575128501774551345301 8	56.7192686860506555713032 2	59.3127680291863054436198 1	60.8497522517190264206238 0	65.1843730542343102616303 9	11 ∼ 15
4	65.8701231754783793813247 4	68.3999150184262938260895 9	72.2857609166062756611316 3	75.1723563176513425144355 3	77.0238462834373825875530 1	16 ∼ 20
5	80.6453481333810883606887 9	81.8374380242949903747962 5	83.6141692400091246743150 4	86.6541465618299990389922 4	89.2357821173246067869641 0	21 ∼ 25
6	91.1991687935831334918754 0	95.4831245618782825015626 9	96.0103017623343194558849 5	99.4462586475651721331391 4	99.5476649553446043638335 7	26 ∼ 30
7	103.3721869057410568790676	105.5376415356094585300702	105.9406429554289212882946	110.5350871693984496934365	112.8924953941267896087355	31 ∼ 35
8	113.8687481104045937736034	116.2334142213478359545247	118.6940880052047061593073	120.8434278980156693460372	121.6635570492196196885198	36 ∼ 40
9	124.0172645715355389038612	127.4292822375708662470366	129.9523411461919986972118	130.2250880961632133217501	134.1596210589883474289577	41 ∼ 45
10	135.2738417270921367961100	136.7317690076688459224675	139.0928034223479585058537	140.3717598176042241227426	144.1343153489638151650442	46 ∼ 50
11	144.3109699487637624293831	147.5352710429717866723093	150.2862432948662228115960	151.4093813981678663366538	153.3926064068603981291333	51 ∼ 55
12	154.2896801377496107453653	157.6199966286418260073836	158.6747500404987358455272	160.4202624767332778302961	162.4279032537663023154697	56 ∼ 60
13	165.8207617081720603663297	167.3333120289411433025666	169.4634807275632572164352	169.4709859610341002114706	173.1553203367805288204211	61 ∼ 65
14	174.8505492262504588130633	175.2594362391757993780789	177.6182224152689308473647	179.6022632333916226519388	182.6994076293951469478620	66 ∼ 70
15	184.4749998209085160920084	184.9837985411169685925195	188.3332391719200426141061	189.3026880810397739331959	191.7412684564906195174821	71 ∼ 75
16	192.2720223235611163705490	193.9234831588349671552155	197.5106008315090553372079	197.6623884363873844666488	200.2697734061353705803846	76 ∼ 80
17	202.6357020114627824575709	204.2827616537792276535200	206.5761507633580054031867	207.4510760809566788327968	208.6665396885349090099017	81 ∼ 85
18	210.8932461046521889873417	213.3889178177916834592008	214.3372223685528150302467	215.3804139502934004162589	218.5198859478563849825795	86 ∼ 90
19	220.9992163528581895609698	222.0590710863393490493129	223.7493183539852741712242	224.7356334880243599143332	227.6295910979853470791262	91 ∼ 95
20	228.6404418309054255317819	230.2847392952986558070189	232.3864093169224379792603	232.4533606417151087643216	237.0751087576937233156002	95 ∼ 100

Table. 16.4

In this page, it can be confirmed that 37 non-trivial zeros of the eta function clored in magenta satisfy the following equations:

$$\eta\left(i\,\tau_m\right) = \zeta\left(\frac{1}{2} + i\,\frac{\tau_m}{2}\right) = 0, \quad m \in \{4,\ 7,\ 10,\ 14,\ 16,\ 19,\ 22,\ 24,\ 28,\ 30,\ 33,\ 35,\ 38,\ 40,\ 44,\ 45,\ 48,\ 50,\ 54,\ 56,\ 58,\ 61,\ 64,\ 66,\ 68,\ 72,\ 74,\ 75,\ 79,\ 81,\ 84,\ 86,\ 88,\ 92,\ 93,\ 96,\ 99\}. \tag{16.92}$$

List of 201 non-trivial zeros of the eta function (Positive imaginary part, 25-decimal accuracy), Part 2

k	τ_{5k-4}	τ_{5k-3}	τ_{5k-2}	τ_{5k-1}	τ_{5k}	$5k-4 \sim 5k$
21	237.5815657319524346459583	238.9059797524181807164837	241.4625872373207538673698	242.7402500048412918378911	244.8949227581374238393590	101 ∼ 105
22	245.8936585871051764016349	247.5890975206390121975876	248.5136371086915343694640	251.5370391198311875249928	252.5975520498828181958003	106 ∼ 110
23	255.0333677591929902485586	255.9188153661265987661118	259.1574083999121019715361	259.7712467172909386524701	262.1753770618653134471327	111 ∼ 115
24	262.1871575081679921504687	264.2871532019756557610026	266.9954744059951729002610	267.4883629279496263828475	269.5130195067477426626522	116 ∼ 120
25	270.9816745051140044951081	273.0946245345651167786625	276.2320841090668864003831	276.9145890193925153301808	277.5003554092228802590954	121 ∼ 125
26	279.4724179042427779009001	282.2474148080422475238807	282.5072650195035682561334	284.7888350443955645966940	286.2236916152412654788102	126 ∼ 130
27	286.6581254832354640231872	289.9563325554227243635735	292.0019649735310370948050	293.0440105698166769799168	294.8455306851192040990424	131 ∼ 135
28	295.8690616243618340696205	298.3769154346050727635361	300.1070408415697607028649	300.5927180030862429160328	301.8505152244829335237050	136 ∼ 140
29	303.9239753409673814558369	306.0493876223977923965131	307.3992252470305210510120	309.1509828574398346725859	311.3004973264868717362307	141 ∼ 145
30	312.2258185884757351395004	315.1951836351881197750610	315.4966106158058566381491	317.4100422510587022498614	317.6999763428409974483500	146 ∼ 150
31	320.4729681935420846675554	322.3779282751920550388747	322.8142939523134146905627	325.1320933791980792525484	326.0614193743639744866221	151 ∼ 155
32	329.4623292331537938398151	330.8028385717263081892296	331.0741383758008376600778	333.5077583382168542293586	334.3688799563490268819155	156 ∼ 160
33	336.0888415634194892414857	338.1890308311376429790117	339.8239529588233979333997	340.1022623750533804296962	341.4695339891576109129115	161 ∼ 165
34	344.5609635432426762367639	346.8230730391831059196922	346.8859576722700314339416	349.5083830467314516267575	349.8301761708011468088310	166 ∼ 170
35	352.8828685954208377777853	353.1946042838361348859912	355.4024251491475516160485	356.7247498179713112188241	356.7548155521999545716619	171 ∼ 175
36	359.8329680405139922786801	361.1386207705710627955890	363.2298274634087799618162	364.4141569687329238308141	365.8335366579379344544134	176 ∼ 180
37	368.2300647507367775313421	369.7489356967750176019213	370.7479932154009498344296	371.1975673554149429330554	374.1375211813888190829118	181 ∼ 185
38	374.4578451670037039832831	376.5427486645415013718075	378.8323173120338741697046	378.9834629842570850271637	380.7423752215115454866728	186 ∼ 190
39	384.0533127214275730945673	384.7222757521126368148321	386.1594532076914080948044	387.5941458522933619891087	388.4637664587217984439057	191 ∼ 195
40	390.5307933590584706429264	392.2640113093685696467059	393.7529636819166338972445	394.2268924431528170755268	396.0306193525038248498398	195 ∼ 200
41	397.6129514722938134226983	Not calculated	Not calculated	Not calculated	Not calculated	201 ∼ 205

Table. 16.5

In this page, it can be confirmed that 42 non-trivial zeros of the eta function clored in magenta satisfy the following equations:

$$\eta\left(i\,\tau_m\right) = \zeta\left(\frac{1}{2} + i\,\frac{\tau_m}{2}\right) = 0, \quad m \in \{101, 104, 106, 108, 111, 113, 115, 118, 120, 123, 126, 127, 130, 133, 135, 138, 140, 142, 146, 147, 150, 152, 155, 158, 160, 162, 163, 167, 169, 171, 175, 176, 179, 182, 184, 186, 188, 191, 193, 196, 198, 200\}. \tag{16.93}$$

List of 122 conditional non-trivial zeros of the eta function after renumbering (Positive imaginary part, 25-decimal accuracy)

k	$\hat{\tau}_{5k-4}$	$\hat{\tau}_{5k-3}$	$\hat{\tau}_{5k-2}$	$\hat{\tau}_{5k-1}$	$\hat{\tau}_{5k}$	$5k-4 \sim 5k$
1	12.04189780939519330980400	20.48754060833310910427551	25.97619602462484501490622	32.68521420917444438995372	36.58398639224706967705201	$1 \sim 5$
2	42.90122268796692099440190	46.55675304091906306363912	51.45751285017745513453018	56.71926868605065557130322	59.31276802918630544361981	$6 \sim 10$
3	65.18437305423431026163039	68.39991501842629382608959	72.28576091660627566113163	77.02384628343738258755301	80.64534813338108836068879	$11 \sim 15$
4	83.61416924000912467431504	89.23578211732460678696410	91.19916879358313349187540	95.48312456187828250156269	99.44625864756517213313914	$16 \sim 20$
5	103.3721869057410568790676	105.5376415356094585300702	110.5350871693984496934365	113.8687481104045937736034	116.2334142213478359545247	$21 \sim 25$
6	120.8434278980156693460372	124.0172645715355389038612	127.4292822375708662470366	129.9523411461919986972118	135.2738417270921367961100	$26 \sim 30$
7	136.7317690076688459224675	140.3717598176042241227426	144.3109699487637624293831	147.5352710429717866723093	150.2862432948662228115960	$31 \sim 35$
8	153.3926064068603981291333	157.6199966286418260073836	160.4202624767332778302961	162.4279032537663023154697	167.3333120289411433025666	$36 \sim 40$
9	169.4634807275632572164352	173.1553203367805288204211	175.2594362391757993780789	179.6022632333916226519388	182.6994076293951469478620	$41 \sim 45$
10	184.4749998209085160920084	188.3332391719200426141061	192.2720223235611163705490	193.9234831588349671552155	197.5106008315090553372079	$46 \sim 50$
11	200.2697734061353705803846	204.2827616537792276535200	206.5761507633580054031867	208.6665396885349090099017	213.3889178177916834592008	$51 \sim 55$
12	215.3804139502934004162589	218.5198859478563849825795	220.9992163528581895609698	224.7356334880243599143332	227.6295910979853470791262	$56 \sim 60$
13	230.2847392952986558070189	232.3864093169224379792603	237.0751087576937233156002	238.9059797524181807164837	241.4625872373207538673698	$61 \sim 65$
14	244.8949227581374238393590	247.5890975206390121975876	251.5370391198311875249928	252.5975520498828181958003	255.9188153661265987661118	$66 \sim 70$
15	259.7712467172909386524701	262.1871575081679921504687	264.2871532019756557610026	267.4883629279496263828475	270.9816745051140044951081	$71 \sim 75$
16	273.0946245345651167786625	276.9145890193925153301808	277.5003554092228802590954	282.5072650195035682561334	284.7888350443955645966940	$76 \sim 80$
17	286.6581254832354640231872	289.9563325554227243635735	293.0440105698166769799168	295.8690616243618340696205	298.3769154346050727635361	$81 \sim 85$
18	300.5927180030862429160328	303.9239753409673814558369	307.3992252470305210510120	309.1509828574398346725859	311.3004973264868717362307	$86 \sim 90$
19	315.4966106158058566381491	317.4100422510587022498614	320.4729681935420846675554	322.8142939523134146905627	325.1320933791980792525484	$91 \sim 95$
20	329.4623292331537938398151	330.8028385717263081892296	333.5077583382168542293586	336.0888415634194892414857	340.1022623750533804296962	$95 \sim 100$
21	341.4695339891576109129115	344.5609635432426762367639	346.8859576722700314339416	349.8301761708011468088310	353.1946042838361348859912	$101 \sim 105$
22	355.4024251491475516160485	356.7247498179713112188241	361.1386207705710627955890	363.2298274634087799618162	365.8335366579379344544134	$106 \sim 110$
23	368.2300647507367775313421	370.7479932154009498344296	374.1375211813888190829118	376.5427486645415013718075	378.9834629842570850271637	$111 \sim 115$
24	380.7423752215115454866728	384.7222757521126368148321	387.5941458522933619891087	388.4637664587217984439057	392.2640113093685696467059	$116 \sim 120$
25	394.2268924431528170755268	397.6129514722938134226983	Not calculated	Not calculated	Not calculated	$121 \sim 125$

Table. 16.6

Each conditional non-trivial zero of the eta function after renumbering satisfies the following simultaneous expressions an equation and a non-equation:

$$\begin{cases} \eta\left(i\,\hat{\tau}_m\right) = 0 \\ \zeta\left(1/2 + i\,\hat{\tau}_m/2\right) \neq 0 \end{cases}, \quad m \in \{1, 2, 3, \cdots, 122\}. \tag{16.94}$$

Redefine the function $\hat{\omega}_{\mathrm{C}}(y,m,n)$ as follows:

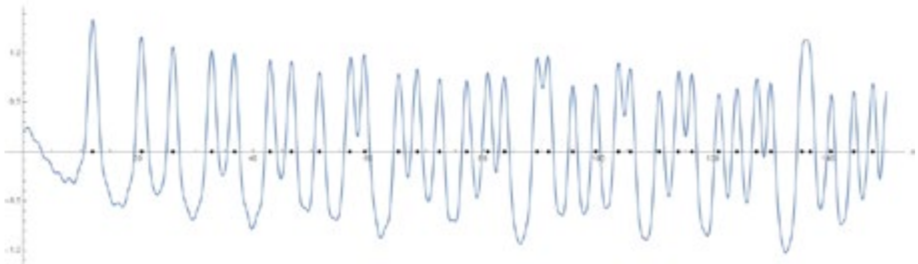
$$\hat{\omega}_{\mathrm{C}}(y,m,n) := -\sum_{k=1}^{m}(-1)^{n}\frac{\log(r_k)}{{r_k}^{n}}\cos\left(\frac{y}{2}\log({r_k}^{n})\right), \quad (\mathrm{C},m,n\in\mathbb{N},y\in\mathbb{R},{r_m}^{n}\le\mathrm{C}). \tag{16.95}$$

Then, calculate the function $\tilde{\Omega}_{1260}(y)-\hat{\Omega}_{1260}(y)$ using the following expression:

$$\begin{aligned}\tilde{\Omega}_{1260}(y)-\hat{\Omega}_{1260}(y) &= \tilde{\omega}_{1260}(y,101,1)+\tilde{\omega}_{1260}(y,4,2)+\sum_{n=3}^{4}\tilde{\omega}_{1260}(y,1,n)\\ &+\hat{\omega}_{1260}(y,103,1)+\hat{\omega}_{1260}(y,6,2)+\hat{\omega}_{1260}(y,2,3)+\sum_{n=4}^{6}\hat{\omega}_{1260}(y,1,n), \quad y\in\mathbb{R}.\end{aligned} \tag{16.96}$$

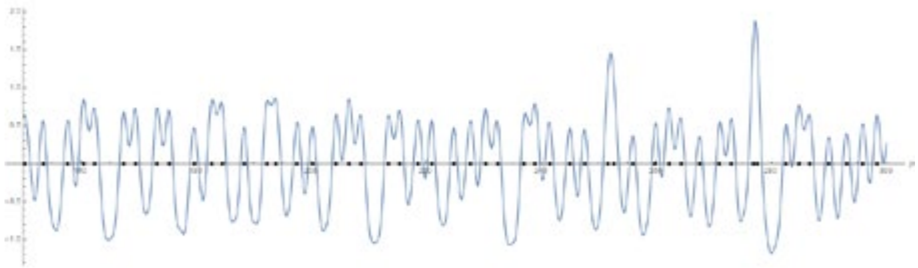
Here, the function $\tilde{\omega}_{\mathrm{C}}(y,m,n)$ is generated from a finite number of Pythagorean primes, and the function $\hat{\omega}_{\mathrm{C}}(y,m,n)$ is generated from a finite number of non-Pythagorean primes.
Subtracting the function $\hat{\Omega}_{\mathrm{C}}(y,m,n)$ from the function $\tilde{\Omega}_{\mathrm{C}}(y,m,n)$ can be said to eliminate the influence of the even prime 2.

Graph of $\tilde{\Omega}_{1260}(y)-\bar{\Omega}_{1260}(y)$, $y\in[0,150]$, $\bullet$: $\hat{\tau}_m$, $m\in\{1,2,3,\cdots,34\}$



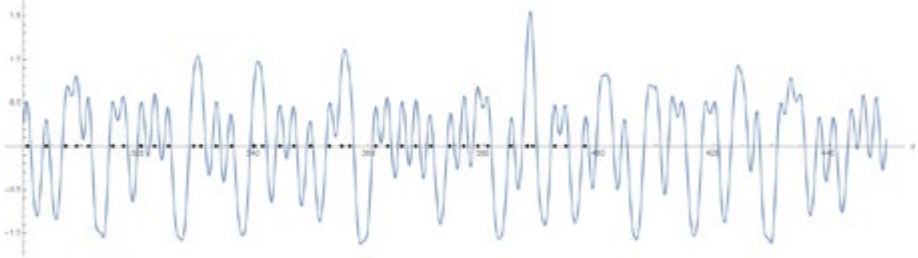
Graph. 16.29

Graph of $\tilde{\Omega}_{1260}(y)-\bar{\Omega}_{1260}(y)$, $y\in[150,300]$, $\bullet$: $\hat{\tau}_m$, $m\in\{=35,36,37,\cdots,85\}$



Graph. 16.30

Graph of $\tilde{\Omega}_{1260}(y)-\bar{\Omega}_{1260}(y)$, $y\in[300,450]$, $\bullet$: $\hat{\tau}_m$, $m\in\{86,87,88,\cdots,122\}$



Graph. 16.31

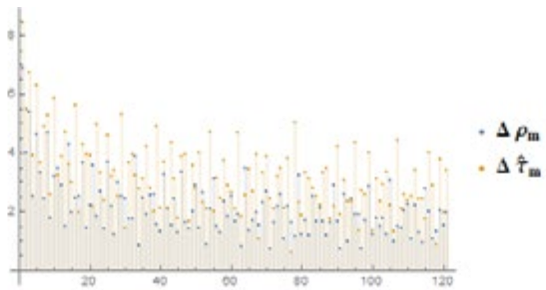
Very usefully, the above graph teaches us rough approximate values of 400 or more of the imaginary parts of conditional non-trivial zeros of the eta function.

16.7 Two Kinds of the Intervals of Adjacent Zeros Part2

The intervals of adjacent zeros $\rho_{m+1} - \rho_m$ on the critical line and the intervals of adjacent zeros of $\hat{\tau}_{m+1} - \hat{\tau}_m$ on the imaginary axis are shown on the same graph.

(16.97) $$\Delta\hat{\tau}_m := \hat{\tau}_{m+1} - \hat{\tau}_m, \quad (m \in \mathbb{N}, \hat{\tau}_m > 0, \eta(i\hat{\tau}_m) = 0, \zeta(1/2 + i\hat{\tau}_m/2) \neq 0).$$

Graph of $\Delta\rho_m$ and $\Delta\hat{\tau}_m$, $m \in \{1, 2, 3, \cdots, 121\}$

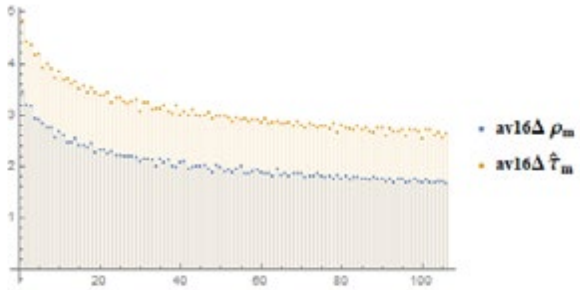


Graph. 16.32

Via taking the moving averages of 16 elements for the intervals of adjacent zeros,

(16.98) $$\text{av16}\Delta\hat{\tau}_m := \frac{1}{16}\sum_{k=m}^{m+15} \Delta\hat{\tau}_k, \quad m \in \mathbb{N}.$$

Graph of $\text{av16}\Delta\rho_m$ and $\text{av16}\Delta\hat{\tau}_m$, $m \in \{1, 2, 3, \cdots, 106\}$

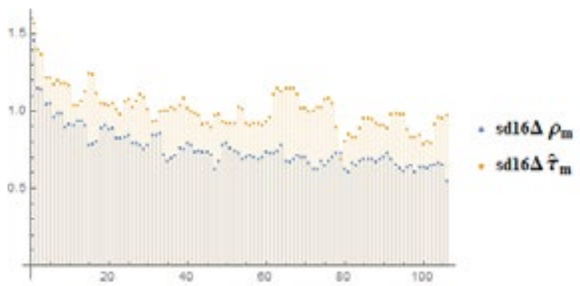


Graph. 16.33

Via taking the moving standard deviations of 16 elements for the intervals of adjacent zeros,

(16.99) $$\text{sd16}\Delta\hat{\tau}_m := \sqrt{\frac{1}{15}\sum_{k=m}^{m+15} (\Delta\hat{\tau}_k - \text{av16}\Delta\hat{\tau}_k)^2}, \quad m \in \mathbb{N}.$$

Graph of $\text{sd16}\Delta\rho_m$ and $\text{sd16}\Delta\hat{\tau}_m$, $m \in \{1, 2, 3, \cdots, 106\}$



Graph. 16.34

It is recognized that the moving standard deviations for the intervals of adjacent zeros $\Delta\rho_m$ are almost always smaller than those of $\Delta\hat{\tau}_m$ at each point in the observation range.

17 The Equivalent Function to the Riemann-Siegel $Z(t)$

The function $\tilde{\zeta}(1/2+it)$, which takes a real value on the critical line, is equivalent to the Riemann-Siegel $Z(t)$ function "[18]."

$$\tilde{\zeta}\left(\frac{1}{2}+it\right)=Z(t), \quad t\in\mathbb{R}. \tag{17.1}$$

Therefore, the new explicit formula for the Riemann-Siegel $Z(t)$ function is given as follows:

$$Z(t)=\left|\frac{2\pi^{\frac{1+2it}{4}}}{\left(\frac{1}{2}-it\right)\Gamma\left(\frac{5+2it}{4}\right)}\right|\sum_{p=1}^{\infty}\sigma_{-\frac{1}{2}-it}(p)\,p^{\frac{1+2it}{4}}\left(\begin{array}{r}\left(-t^2-\frac{1}{4}+\left(\frac{13}{2}-it\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2it}{4}}(2\pi p)\\-\left(\frac{1}{2}-it+2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{3-2it}{4}}(2\pi p)\end{array}\right), \quad t\in\mathbb{R}. \tag{17.2}$$

The rule in subsection 6-3 is applied to the above equation.

$$Z_\lambda(t)=\left|\frac{2\pi^{\frac{1+2it}{4}}}{\left(\frac{1}{2}-it\right)\Gamma\left(\frac{5+2it}{4}\right)}\right|\sum_{p=1}^{\lambda}\sigma_{-\frac{1}{2}-it}(p)\,p^{\frac{1+2it}{4}}\left(\begin{array}{r}\left(-t^2-\frac{1}{4}+\left(\frac{13}{2}-it\right)(2\pi p)^2\right)\mathrm{K}_{\frac{1+2it}{4}}(2\pi p)\\-\left(\frac{1}{2}-it+2(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{3-2it}{4}}(2\pi p)\end{array}\right), \quad (t\in\mathbb{R},\lambda\in\mathbb{R}). \tag{17.3}$$

Graph of $\mathrm{Re}(Z_{11}(t))$, $\mathrm{Im}(Z_{11}(t))$, and RiemannSiegelZ[t], $y\in[0,70]$

Graph. 17.1

To confirm the equivalence, the graph is superposed with Mathematica's built-in Riemann-Siegel $Z(t)$.

The following equation defines the Riemann-Siegel theta function $\theta(t)$:

$$Z(t)=: e^{i\,\theta(t)}\zeta\left(\frac{1}{2}+it\right)\in\mathbb{R}, \quad t\in\mathbb{R}. \tag{17.4}$$

On the critical line, the Riemann-Siegel theta function takes a real value. From the defining equation,

$$e^{i\,\theta(t)}=\frac{\left(\frac{1}{2}-it\right)\Gamma\left(\frac{5+2it}{4}\right)}{2\pi^{\frac{1+2it}{4}}}\left|\frac{2\pi^{\frac{1+2it}{4}}}{\left(\frac{1}{2}-it\right)\Gamma\left(\frac{5+2it}{4}\right)}\right|=\pi^{-\frac{it}{2}}\sqrt{\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}}, \quad t\in\mathbb{R}. \tag{17.5}$$

Taking a logarithm and dividing by i in both sides of the above equation,

$$\theta(t)+2k\pi=-\frac{\log(\pi)}{2}t-\frac{i}{2}\log\left(\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}\right), \quad (t\in\mathbb{R},k\in\mathbb{Z}). \tag{17.6}$$

Here, the integer k is chosen such that the Riemann-Siegel theta function is continuous.

Using the following proposition formula, the discontinued points of the second term on the right side of the above equation are obtained:

$$\text{Re}\left(\frac{\Gamma\left(\frac{1}{4}+\frac{if_m}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{if_m}{2}\right)}\right)+1=0 \implies \log\left(\frac{\Gamma\left(\frac{1}{4}+\frac{if_m}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{if_m}{2}\right)}\right)=\log\left(-1\right)=\pi i,\quad m\in\mathbb{Z}\backslash\{0\}. \tag{17.7}$$

Specifically, f_m can be determined as the zeros in the premise formula of the proposition.
The following are some f_m examples:

$$\{f_1,f_2,f_3,f_4,f_{35},f_{36}\}\simeq\{8.58494,12.3904,15.6226,18.5534,80.6619,82.3566\}. \tag{17.8}$$

$$f_{-m}=-f_m,\quad m\in\mathbb{N}. \tag{17.9}$$

For describing the partial Riemann-Siegel theta function, the following unit step function is proposed:

$$\text{U}\left(x\right):=\begin{cases}0 & x<0,\\ 1 & x\geq 0.\end{cases} \tag{17.10}$$

Hence, the partial Riemann-Siegel theta function is obtained as follows:

$$\begin{aligned}\theta_n\left(t\right)&=-\frac{\log\left(\pi\right)}{2}t-\frac{i}{2}\log\left(\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}\right)\\&+\pi\sum_{m=-n-1}^{n}\left(m+1-\text{U}\left(m\right)\right)\left(\text{U}\left(t-f_m\right)-\text{U}\left(t-f_{m+1}\right)\right),\quad\left(n\in\mathbb{N},t\in\mathbb{R},|t|<f_{n+1}\right).\end{aligned} \tag{17.11}$$

Here, f_0 is specially defined for simple expression.

$$f_0:=0. \tag{17.12}$$

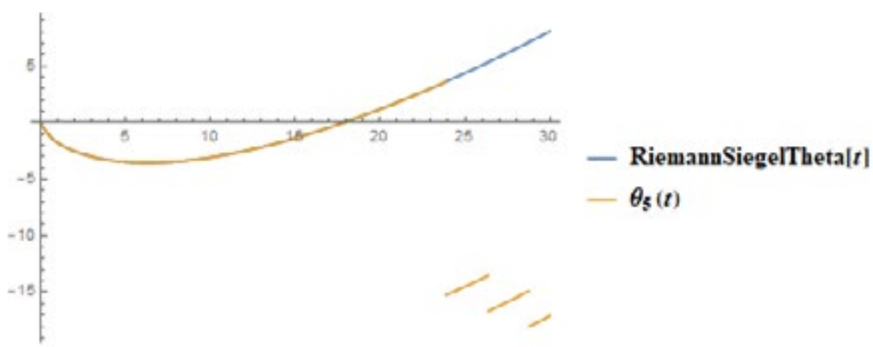
The partial Riemann-Siegel theta function is an odd function.

$$\theta_n\left(t\right)=-\theta_n\left(-t\right),\quad\left(n\in\mathbb{N},t\in\mathbb{R},|t|<f_{n+1}\right). \tag{17.13}$$

The Riemann-Siegel theta function is determined as the infinite limit of an integer n.

$$\theta\left(t\right)=\lim_{n\to\infty}\theta_n\left(t\right),\quad t\in\mathbb{R} \tag{17.14}$$

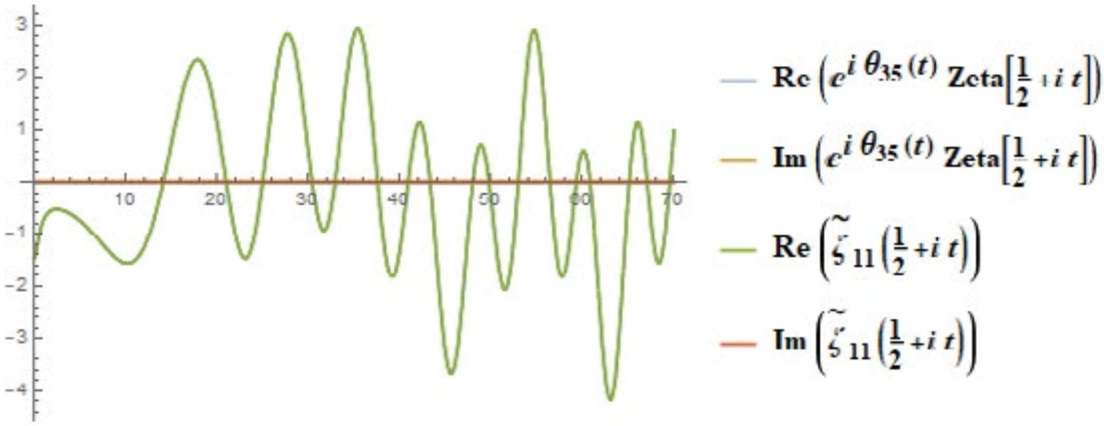
Graph of RiemannSiegelTheta[t] and $\theta_5\left(t\right)$, $t\in[0,30]$



Graph. 17.2

The graph is superposed with Mathematica's built-in Riemann-Siegel Theta function.
The built-in Riemann-Siegel Theta function is continuous on the whole interval, although the partial Riemann-Siegel theta function is discontinuous.

$$\text{Graph of Re}\left(e^{i\,\theta_{35}(t)}\,\text{Zeta}\left[\frac{1}{2}+i\,t\right]\right),\ \text{Im}\left(e^{i\,\theta_{35}(t)}\,\text{Zeta}\left[\frac{1}{2}+i\,t\right]\right),$$
$$\text{Re}\left(\tilde{\zeta}_{11}\left(\frac{1}{2}+i\,t\right)\right),\ \text{and Im}\left(\tilde{\zeta}_{11}\left(\frac{1}{2}+i\,t\right)\right),\ y\in[0,70]$$



Graph. 17.3

In the case of the zeta function, the essence of the theta function $\theta(t)$ is to convert the complex gamma factor into a real number on the clitical line.

The theta function for the eta function $\hat{\theta}(t)$ is defined by the following equation:

(17.15) $$e^{i\,\hat{\theta}(t)}\,\eta(it) := \tilde{\eta}(it) \in \mathbb{R},\quad t\in\mathbb{R}.$$

According to the defining equation,

(17.16) $$e^{i\,\hat{\theta}(t)} = \frac{(1-it)\,\Gamma\left(\frac{1+it}{2}\right)}{2\pi^{\frac{it}{2}}}\left|\frac{2\pi^{\frac{it}{2}}}{(1-it)\,\Gamma\left(\frac{1+it}{2}\right)}\right| = \pi^{-\frac{it}{2}}\frac{1-it}{\sqrt{1+t^2}}\sqrt{\frac{\Gamma\left(1+\frac{it}{2}\right)}{\Gamma\left(1-\frac{it}{2}\right)}},\quad t\in\mathbb{R}.$$

Taking a logarithm and dividing by i on both sides of the above equation,

(17.17) $$\hat{\theta}(t)+2k\pi = -\frac{\log(\pi)}{2}t - i\log\left(\frac{1-it}{\sqrt{1+t^2}}\right) - \frac{i}{2}\log\left(\frac{\Gamma\left(1+\frac{it}{2}\right)}{\Gamma\left(1-\frac{it}{2}\right)}\right),\quad (t\in\mathbb{R},k\in\mathbb{Z}).$$

Here, the integer k is chosen such that the theta-hat function $\hat{\theta}(t)$ is continuous.
Using the following proposition, the discontinued points of the third term on the right side of the above equation are obtained:

(17.18) $$\text{Re}\left(\frac{\Gamma\left(1+\frac{ig_m}{2}\right)}{\Gamma\left(1-\frac{ig_m}{2}\right)}\right)+1=0 \implies \log\left(\frac{\Gamma\left(1+\frac{ig_m}{2}\right)}{\Gamma\left(1-\frac{ig_m}{2}\right)}\right)=\log(-1)=\pi i,\quad m\in\mathbb{Z}\backslash\{0\}.$$

Specifically, g_m can be determined as the zeros in the premise formula of the proposition.
The following are such examples of g_m:

(17.19) $$\{g_1,g_2,g_3,g_4,g_{35},g_{36}\}\simeq\{6.87947,11.0772,14.4675,17.4912,80.0252,81.7234\}.$$

(17.20) $$g_{-m}=-g_m,\quad m\in\mathbb{N}.$$

The partial theta function for the eta function $\hat{\theta}_n(t)$ is obtained as follows:

(17.21) $$\hat{\theta}_n(t) = -\frac{\log(\pi)}{2}t - i\log\left(\frac{1-it}{\sqrt{1+t^2}}\right) - \frac{i}{2}\log\left(\frac{\Gamma\left(1+\frac{it}{2}\right)}{\Gamma\left(1-\frac{it}{2}\right)}\right)$$
$$+\pi\sum_{m=-n-1}^{n}(m+1-\text{U}(m))\left(\text{U}(t-g_m)-\text{U}(t-g_{m+1})\right),\quad (n\in\mathbb{N},t\in\mathbb{R},|t|<g_{n+1}).$$

Here, g_0 is specially defined for simple expression.

(17.22) $$g_0 := 0.$$

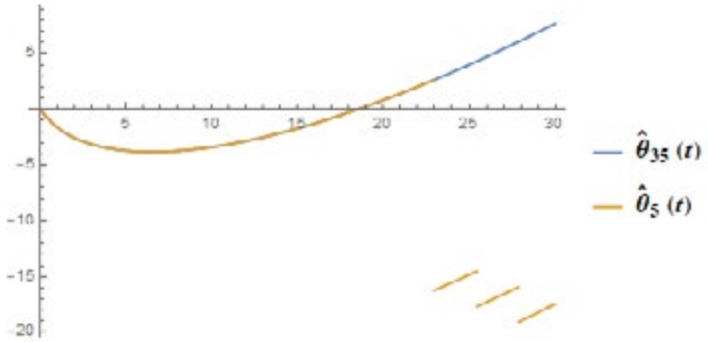
The partial theta function for the eta function $\hat{\theta}_n(t)$ is also an odd function.

(17.23) $$\hat{\theta}_n(t) = -\hat{\theta}_n(-t), \quad (n \in \mathbb{N}, t \in \mathbb{R}, |t| < g_{n+1}).$$

The theta function for the eta function $\hat{\theta}(t)$ is also determined as the infinite limit of an integer n.

(17.24) $$\hat{\theta}(t) = \lim_{n \to \infty} \hat{\theta}_n(t), \quad t \in \mathbb{R}.$$

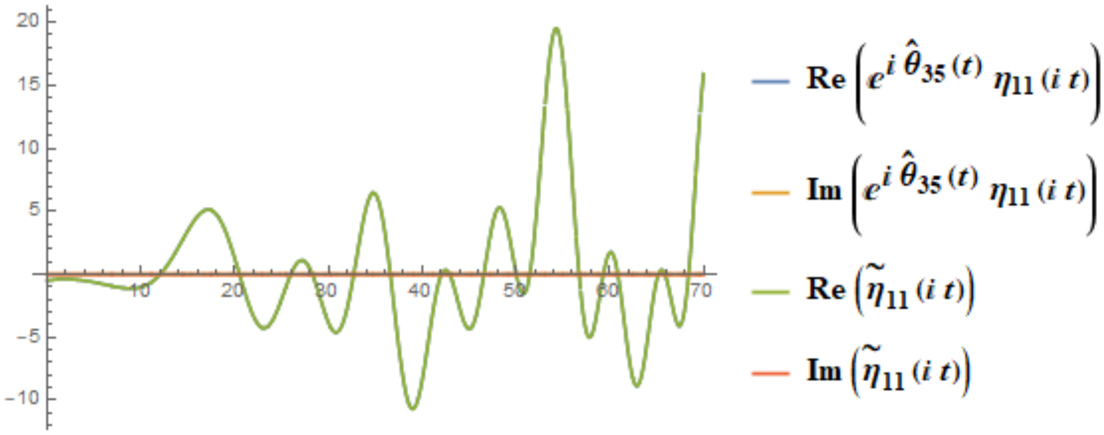
Graph of $\hat{\theta}_{35}(t)$ and $\hat{\theta}_5(t)$, $t \in [0, 30]$



Graph. 17.4

The function $\hat{\theta}_{35}(t)$ is continuous in the observational range, although the function $\hat{\theta}_5(t)$ is discontinuous.

Graph of $\mathrm{Re}\left(e^{i\,\hat{\theta}_{35}(t)}\,\eta_{11}(i\,t)\right)$, $\mathrm{Im}\left(e^{i\,\hat{\theta}_{35}(t)}\,\eta_{11}(i\,t)\right)$, $\mathrm{Re}\left(\tilde{\eta}_{11}(i\,t)\right)$, and $\mathrm{Im}\left(\tilde{\eta}_{11}(i\,t)\right)$, $y \in [0, 70]$



Graph. 17.5

In the case of the eta function as well, the essence of the theta-hat function $\hat{\theta}(t)$ is to convert the complex gamma factor into a real number on the imaginary axis.

The function $E(t)$ corresponding to the Riemann-Siegel $Z(t)$ function has already been found, and it is the eta-tilde function on the imaginary axis.

(17.25) $$E(t) = \tilde{\eta}(it) \in \mathbb{R}, \quad t \in \mathbb{R}.$$

Comparison table, $(S,\theta \in \mathbb{C}, m \in \mathbb{N}, t \in \mathbb{R})$

Classification	Zeta system (Known)	Eta system (New)
Basic function	The Riemann zeta function $\zeta(S)$	$\eta(\theta)$
Top function	The Riemann Xi function $\xi(S)$	$\chi(\theta)$
Relational expression	$\xi(S)=\frac{S(S-1)}{2}\pi^{-\frac{S}{2}}\Gamma\left(\frac{S}{2}\right)\zeta(S)$	$\chi(\theta)=\frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\eta(\theta)$
Symmetry	$\xi(S)=\xi(1-S)$	$\chi(\theta)=\chi(-\theta)$
Non-trivial zeros	$\zeta\left(\frac{1}{2}+i\rho_m\right)=0$	$\eta(i\tau_m)=0$
First special function	The Riemann-Siegel $Z(t)$ function	$E(t)$
Second special function	The Riemann-Siegel theta function $\theta(t)$	$\hat{\theta}(t)$

Table. 17.1

All functions in the table have their own explicit formulae derived from this study.

18 Conjectures

This section will go through five kinds of conjectures.

18.1 First Conjecture

Pythagorean primes, squares of non-Pythagorean primes, and even prime 2 would be generated as a main spectrum using information on the imaginary parts of the eta function's non-trivial zeros.

The following functions are developed to describe the next topic of the first conjecture:

(18.1)
$$\hat{\Omega}_{\mathrm{C}}(y):=\tilde{\Omega}_{\mathrm{C}}(y)-\bar{\Omega}_{\mathrm{C}}(y)=-\sum_{\substack{p^n\leq \mathrm{C}\\ p:\,\text{prime},\, p\equiv 1\,(\mathrm{mod}\,4),\, n\in\mathbb{N}}}\frac{\log(p)}{p^n}\cos\left(\frac{y}{2}\log(p^n)\right) -\sum_{\substack{p^n\leq \mathrm{C}\\ p:\,\text{prime},\, p\equiv 3\,(\mathrm{mod}\,4),\, n\in\mathbb{N}}}(-1)^n\frac{\log(p)}{p^n}\cos\left(\frac{y}{2}\log(p^n)\right),\quad (y\in\mathbb{R},\mathrm{C}\in\mathbb{N}).$$

The peaks of the functions $\tilde{\Omega}_{\mathrm{C}}(y)$ would be separated into two types.
Peaks of the first type are provided by the functions $\tilde{\Omega}_{\mathrm{C}}(2y)+\bar{\Omega}_{\mathrm{C}}(2y)$, and they would correspond to twice the imaginary parts of the zeta function's non-trivial zeros. Specifically, almost half of the imaginary parts of non-trivial zeros of the modified eta function $\eta(2\theta)$ would be generated as a spectrum using the information on all prime numbers.
Conversely, peaks of the second type are given by the functions $\hat{\Omega}_{\mathrm{C}}(y)$, and they do not correspond to twice the imaginary parts of the zeta function's non-trivial zeros. As a spectrum, almost half of the imaginary parts of non-trivial zeros of the eta function $\eta(\theta)$ would be generated using the information on all odd prime numbers.
It can be seen that the squares of non-Pythagorean primes play the same role as Pythagorean primes.
I believe the following facts are reasons for it: Over the field of complex integers, both Pythagorean primes and the squares of non-Pythagorean primes are not prime numbers, while non-Pythagorean primes are prime numbers.

Pythagorean primes are uniquely represented by the sum of two squares over the standard field of real integers, but the squares of non-Pythagorean primes are represented in at least one way by the sum of three squares. In contrast, non-Pythagorean primes are represented by the sum of three or four squares.

(18.2)
$$q_{\mathrm{i}}=k^2+l^2,\quad \left(k,l,\mathrm{i}\in\mathbb{N},{}^{\exists}k<{}^{\exists}l\right).$$

(18.3)
$$r_{\mathrm{i}}{}^2=k^2+l^2+m^2,\quad \left(k,l,m,\mathrm{i}\in\mathbb{N},{}^{\exists}k\leq{}^{\exists}l\leq{}^{\exists}m\right).$$

The following are examples of the squares of non-Pythagorean primes:

(18.4) $$\begin{aligned} {r_1}^2 &= 3^2 = 1^2+2^2+2^2,\\ {r_2}^2 &= 7^2 = 2^2+3^2+6^2,\\ {r_3}^2 &= 11^2 = 2^2+6^2+9^2 = 6^2+6^2+7^2,\\ {r_4}^2 &= 19^2 = 1^2+6^2+18^2 = 6^2+6^2+17^2 = 6^2+10^2+15^2,\\ {r_5}^2 &= 23^2 = 3^2+6^2+22^2 = 3^2+14^2+18^2 = 6^2+13^2+18^2,\\ {r_6}^2 &= 31^2 = 5^2+6^2+30^2 = 6^2+14^2+27^2 = 6^2+21^2+22^2 = 14^2+18^2+21^2. \end{aligned}$$

18.2 Second Conjecture

Twice the imaginary part of any non-trivial zero of the zeta function would belong to the subset of imaginary parts of non-trivial zeros of the eta function.

Specifically, the following proposition would hold:

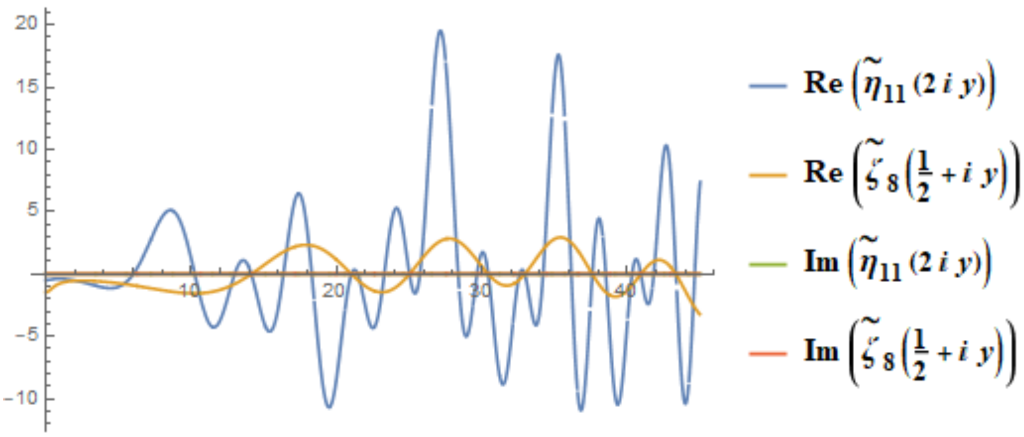
(18.5) $$\zeta\left(\frac{1}{2}+i\rho_m\right)=0 \implies \eta\left(2i\tau_m\right)=0, \quad m\in\mathbb{N}.$$

This conjecture supports the existence of an infinite number of non-trivial zeros of the eta function. The converse proposition would not be necessarily true.

(18.6)
$$\eta\left(i\tau_m\right)=0, \quad m\in\mathbb{N} \implies \zeta\left(\frac{1}{2}+i\frac{\tau_1}{2}\right)\neq 0,\ \zeta\left(\frac{1}{2}+i\frac{\tau_2}{2}\right)\neq 0,\ \zeta\left(\frac{1}{2}+i\frac{\tau_3}{2}\right)\neq 0,\ \zeta\left(\frac{1}{2}+i\frac{\tau_4}{2}\right)=0, \text{ etc.}$$

Of the 201 non-trivial zeros of the eta function whose values were examined in detail, a total of 79 are confirmed to satisfy the proposition.

Graph of $\mathrm{Re}\left(\tilde{\eta}_{11}\left(2\,i\,y\right)\right)$, $\mathrm{Re}\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\,y\right)\right)$, $\mathrm{Im}\left(\tilde{\eta}_{11}\left(2\,i\,y\right)\right)$, and $\mathrm{Im}\left(\tilde{\zeta}_8\left(\frac{1}{2}+i\,y\right)\right)$, $y\in[0,45]$



Graph. 18.1

A variable interval is constructed to describe the next topic of the second conjecture. On the imaginary axis, the first variable interval is defined as follows:

(18.7) $$\mathrm{V}_1\left(m\right) := \left\{iy : \tau_m>0, \eta\left(i\tau_m\right)=0, 0\leq y\leq \tau_m\right\}, \quad m\in\mathbb{N}.$$

From the observational results, the following equation would hold:

(18.8) $$\lim_{m\to\infty}\frac{\mathrm{M}_1\left(m\right)}{\mathrm{M}\left(m\right)}=\frac{1}{2}, \quad m\in\mathbb{N}.$$

Where

(18.9) $$\mathrm{M}\left(m\right) := \#\left\{\tau_n : n\in\mathbb{N}, \tau_n>0, \eta\left(i\tau_n\right)=0, i\tau_n\in \mathrm{V}_1\left(m\right)\right\}=m, \quad m\in\mathbb{N}.$$

(18.10) $$\mathrm{M}_1(m) := \#\{\rho_n : n\in\mathbb{N}, \rho_n>0, \zeta(1/2+i\rho_n)=0, 2i\rho_n\in \mathrm{V}_1(m)\},\quad m\in\mathbb{N}.$$

Here the symbol # express the number of elements of a set.
Assuming the second conjecture is correct, the eta function's non-trivial zeros can be classified into two groups:

(18.11) $$\begin{cases}\eta(i\hat{\tau}_k)=0,\ \zeta(1/2+i\hat{\tau}_k)\neq 0 & k\in\mathbb{N},\\ \eta(2i\rho_l)=\zeta(1/2+i\rho_l)=0 & l\in\mathbb{N}.\end{cases}$$

The Hadamard product representation of the Chi function is modified as follows:

(18.12) $$\chi(\theta)=\chi(0)\prod_{k=1}^{\infty}\left(1+\frac{\theta^2}{(\hat{\tau}_k)^2}\right)\prod_{l=1}^{\infty}\left(1+\frac{\theta^2}{(2\rho_l)^2}\right),\quad \theta\in\mathbb{C}.$$

Conversely, the Hadamard product representation of the Riemann Xi function is transformed as follows:

(18.13) $$\xi(\theta)=\xi(0)\prod_{m=1}^{\infty}\frac{(2\rho_m)^2+(2\theta-1)^2}{(2\rho_m)^2+1}=\frac{1}{2}\prod_{m=1}^{\infty}\frac{1+\frac{(2\theta-1)^2}{(2\rho_m)^2}}{1+\frac{1}{(2\rho_m)^2}},\quad \theta\in\mathbb{C}.$$

Therefore

(18.14) $$\prod_{m=1}^{\infty}\left(1+\frac{(2\theta-1)^2}{(2\rho_m)^2}\right)=2\xi(\theta)\prod_{m=1}^{\infty}\left(1+\frac{1}{(2\rho_m)^2}\right),\quad \theta\in\mathbb{C}.$$

Replace the complex variable θ with $(1+\theta)/2$.

(18.15) $$\prod_{m=1}^{\infty}\left(1+\frac{\theta^2}{(2\rho_m)^2}\right)=2\xi\left(\frac{1+\theta}{2}\right)\prod_{m=1}^{\infty}\left(1+\frac{1}{(2\rho_m)^2}\right),\quad \theta\in\mathbb{C}.$$

Using this result,

(18.16) $$\chi(\theta)=2\chi(0)\prod_{l=1}^{\infty}\left(1+\frac{1}{(2\rho_l)^2}\right)\prod_{k=1}^{\infty}\left(1+\frac{\theta^2}{(\hat{\tau}_k)^2}\right)\xi\left(\frac{1+\theta}{2}\right),\quad \theta\in\mathbb{C}.$$

Replace the complex variable θ with $-\theta$.

(18.17) $$\chi(-\theta)=2\chi(0)\prod_{l=1}^{\infty}\left(1+\frac{1}{(2\rho_l)^2}\right)\prod_{k=1}^{\infty}\left(1+\frac{\theta^2}{(\hat{\tau}_k)^2}\right)\xi\left(\frac{1-\theta}{2}\right),\quad \theta\in\mathbb{C}.$$

I define the Chi-hat function as follows:

(18.18) $$\hat{\chi}(\theta):=2\chi(\theta)\,\xi\left(\frac{1-\theta}{2}\right),\quad \theta\in\mathbb{C}.$$

Then the Chi-hat function satisfies the following functional equation:

(18.19) $$\hat{\chi}(\theta)=\hat{\chi}(-\theta),\quad \theta\in\mathbb{C}.$$

Proceed with the calculation according to the definition.

(18.20) $$\begin{aligned}\hat{\chi}(\theta)&=2\cdot\frac{\theta(\theta-1)}{2}\pi^{-\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}\right)\eta(\theta)\cdot\frac{1}{2}\left(\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}-1\right)\pi^{-\frac{1+\theta}{4}}\Gamma\left(\frac{1-\theta}{4}\right)\zeta\left(\frac{1-\theta}{2}\right)\\&=(1-\theta)(1+\theta)\pi^{-\frac{1+\theta}{4}}\Gamma\left(1+\frac{\theta}{2}\right)\Gamma\left(\frac{5-\theta}{4}\right)\zeta\left(\frac{1-\theta}{2}\right)\eta(\theta),\quad \theta\in\mathbb{C}.\end{aligned}$$

I define the eta-hat function as follows:

(18.21) $$\hat{\eta}(\theta):=-\frac{\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)}\hat{\chi}(\theta),\quad \theta\in\mathbb{C}\backslash\{1\}.$$

Therefore

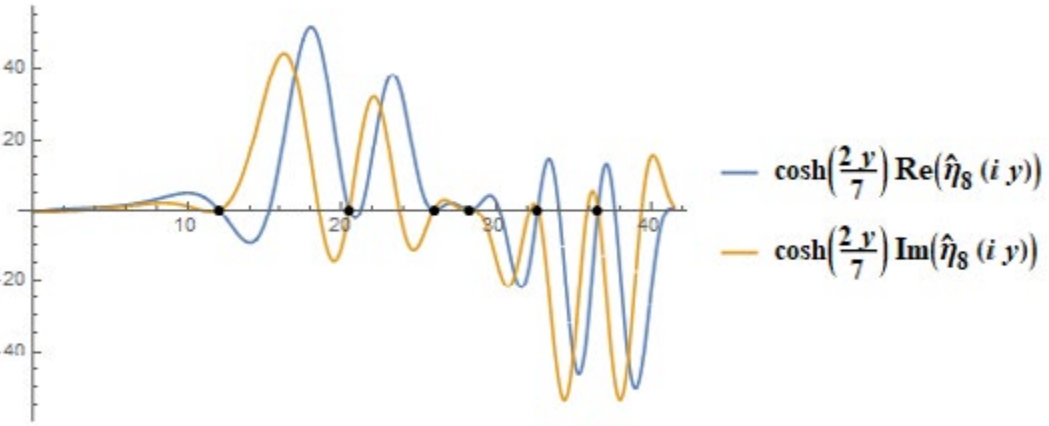
(18.22) $$\hat{\eta}(\theta)=-(1+\theta)\pi^{-\frac{1-\theta}{4}}\Gamma\left(\frac{5-\theta}{4}\right)\zeta\left(\frac{1-\theta}{2}\right)\eta(\theta),\quad \theta\in\mathbb{C}\backslash\{1\}.$$

In order to draw graphs, I modify the eta-hat function by two functions I have already obtained.

(18.23) $$\hat{\eta}_\lambda(\theta) = -(1+\theta)\,\pi^{-\frac{1-\theta}{4}}\,\Gamma\left(\frac{5-\theta}{4}\right)\zeta\left(\frac{1-\theta}{2}\right)\eta_\lambda(\theta), \quad \theta \in \mathbb{C}\backslash\{1\}.$$

(18.24) $$\bar{\eta}_\lambda(\theta) = -(1+\theta)\,\pi^{-\frac{1-\theta}{4}}\,\Gamma\left(\frac{5-\theta}{4}\right)\zeta\left(\frac{1-\theta}{2}\right)\tilde{\eta}_\lambda(\theta), \quad \theta \in \mathbb{C}\backslash\{1\}.$$

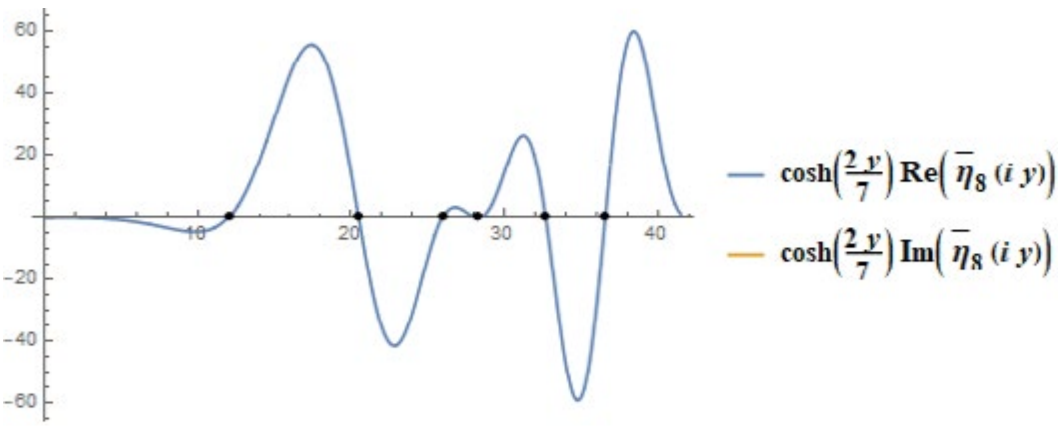
Graph of $\cosh\left(\frac{2y}{7}\right)\mathrm{Re}\left(\hat{\eta}_8(iy)\right)$ and $\cosh\left(\frac{2y}{7}\right)\mathrm{Im}\left(\hat{\eta}_8(iy)\right)$, $y \in [0, 41.5]$, $\bullet$: τ_m, $m \in \{1, 2, 3, \cdots, 6\}$



Graph. 18.2

Since the eta-hat function is a rapidly decreasing function on the imaginary axis, the displayed curves are the eta-hat function multiplied by the rapidly increasing function $\cosh(2y/7)$. The real part of the eta-hat function is positive without changing sign in its vicinity except for the fourth non-trivial zero of the eta function.

Graph of $\cosh\left(\frac{2y}{7}\right)\mathrm{Re}\left(\bar{\eta}_8(iy)\right)$ and $\cosh\left(\frac{2y}{7}\right)\mathrm{Im}\left(\bar{\eta}_8(iy)\right)$, $y \in [0, 41.5]$, $\bullet$: τ_m, $m \in \{1, 2, 3, \cdots, 6\}$



Graph. 18.3

Since the eta-dash function is also a rapidly decreasing function, the displayed curve is the eta-bar function multiplied by the rapidly increasing function $\cosh(2y/7)$. The function value is positive without changing sign in its vicinity except for the fourth non-trivial zero of the eta function.
Here are the details.

(18.25) $$\left\{\mathrm{Re}\left(\bar{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000} + 10^{-9}\right)\right)\right)\right\}$$
$$\simeq \{-4.82204 \times 10^{-10}, 2.09032 \times 10^{-10}\}.$$

$$\{\mathrm{Re}\left(\bar{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{2.22244\times 10^{-10}, -1.11178\times 10^{-10}\}. \tag{18.26}$$

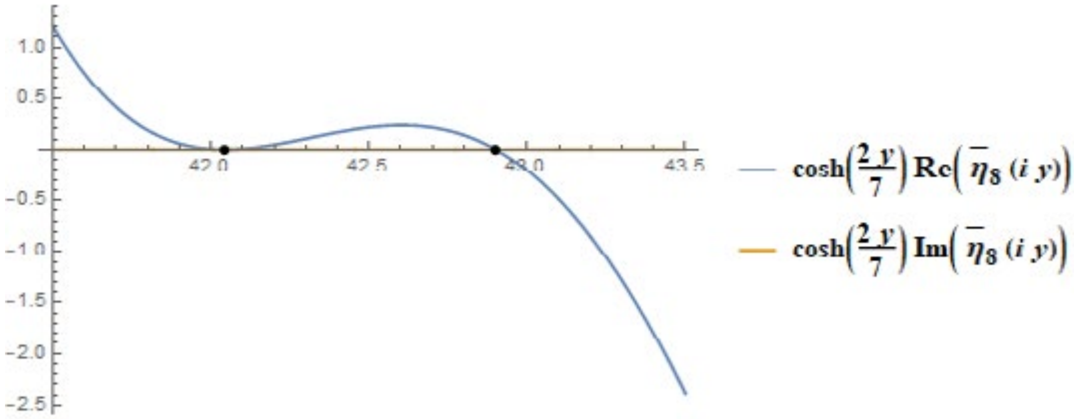
$$\{\mathrm{Re}\left(\bar{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{-6.34781\times 10^{-12}, 1.39702\times 10^{-11}\}. \tag{18.27}$$

$$\{\mathrm{Re}\left(\bar{\eta}_9\left(i\left(2\times\frac{141\,347\,251\,417}{10\,000\,000\,000}-10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_9\left(i\left(2\times\frac{141\,347\,251\,417}{10\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{2.96614\times 10^{-21}, 2.24625\times 10^{-21}\}. \tag{18.28}$$

$$\{\mathrm{Re}\left(\bar{\eta}_{10}\left(i\left(\frac{32\,685\,214\,209}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_{10}\left(i\left(\frac{32\,685\,214\,209}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{7.09716\times 10^{-12}, -4.98882\times 10^{-12}\}. \tag{18.29}$$

$$\{\mathrm{Re}\left(\bar{\eta}_{10}\left(i\left(\frac{36\,583\,986\,392}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_{10}\left(i\left(\frac{36\,583\,986\,392}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{-3.76781\times 10^{-12}, 2.27485\times 10^{-12}\}. \tag{18.30}$$

Graph of $\cosh\left(\frac{2y}{7}\right)\mathrm{Re}\left(\bar{\eta}_8(iy)\right)$ and $\cosh\left(\frac{2y}{7}\right)\mathrm{Im}\left(\bar{\eta}_8(iy)\right)$, $y\in[41.5, 43.5]$, $\bullet$: τ_m, $m\in\{7,8\}$



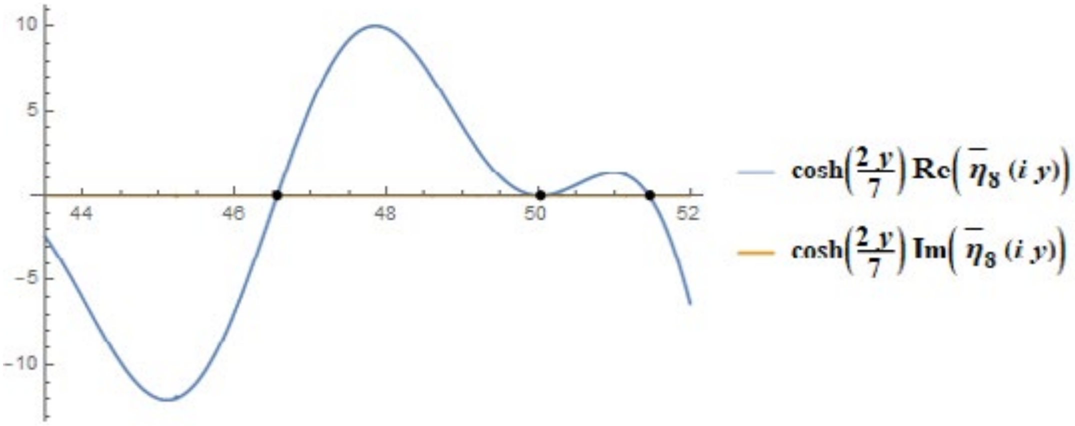
Graph. 18.4

The function value is also positive without changing sign in its vicinity except for the seventh non-trivial zero of the eta function. Here are the details.

$$\{\mathrm{Re}\left(\bar{\eta}_{11}\left(i\left(2\times\frac{210\,220\,396\,388}{10\,000\,000\,000}-10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_{11}\left(i\left(2\times\frac{210\,220\,396\,388}{10\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{2.59974\times 10^{-23}, 2.3.26487\times 10^{-23}\}. \tag{18.31}$$

$$\{\mathrm{Re}\left(\bar{\eta}_{11}\left(i\left(\frac{42\,901\,222\,688}{1\,000\,000\,000}-10^{-9}\right)\right)\right), \mathrm{Re}\left(\bar{\eta}_{11}\left(i\left(\frac{42\,901\,222\,688}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{1.58941\times 10^{-14}, -1.69816\times 10^{-14}\}. \tag{18.32}$$

Graph of $\cosh\left(\frac{2\,y}{7}\right)\operatorname{Re}\left(\overline{\eta}_8\left(i\,y\right)\right)$ and $\cosh\left(\frac{2\,y}{7}\right)\operatorname{Im}\left(\overline{\eta}_8\left(i\,y\right)\right),\; y\in[43.5,52]\,,\;\bullet\,:\,\tau_m,\; m\in\{9,10,11\}$



Graph. 18.5

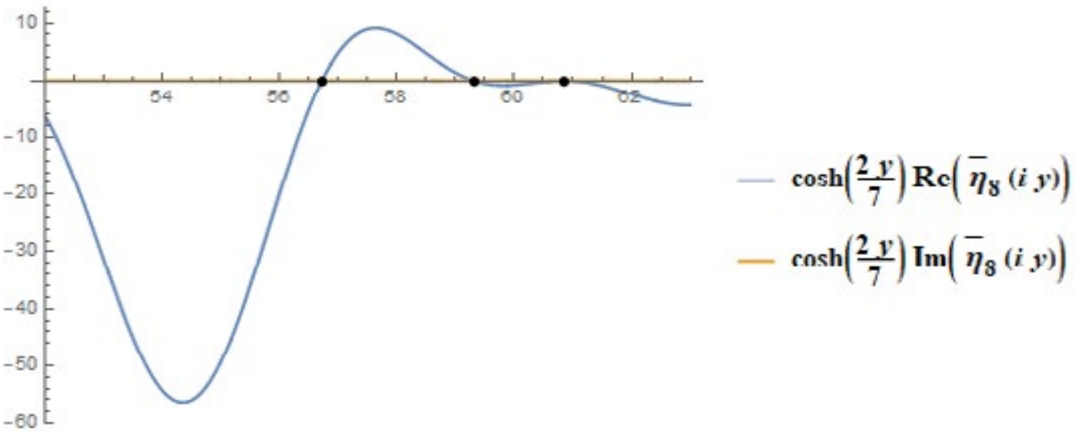
The function value is also positive without changing sign in its vicinity except for the tenth non-trivial zero of the eta function. Here are the details.

$$\begin{aligned}&\{\operatorname{Re}\left(\overline{\eta}_{11}\left(i\left(\frac{46\,556\,753\,041}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\operatorname{Re}\left(\overline{\eta}_{11}\left(i\left(\frac{46\,556\,753\,041}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\&\simeq\{-3.94975\times10^{-14},4.64542\times10^{-14}\}.\end{aligned}\tag{18.33}$$

$$\begin{aligned}&\{\operatorname{Re}\left(\overline{\eta}_{12}\left(i\left(2\times\frac{250\,108\,575\,801}{10\,000\,000\,000}-10^{-9}\right)\right)\right),\operatorname{Re}\left(\overline{\eta}_{12}\left(i\left(2\times\frac{250\,108\,575\,801}{10\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\&\simeq\{5.41705\times10^{-24},3.75474\times10^{-24}\}.\end{aligned}\tag{18.34}$$

$$\begin{aligned}&\{\operatorname{Re}\left(\overline{\eta}_{12}\left(i\left(\frac{51\,457\,512\,850}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\operatorname{Re}\left(\overline{\eta}_{12}\left(i\left(\frac{51\,457\,512\,850}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\&\simeq\{6.26619\times10^{-15},-4.37742\times10^{-15}\}.\end{aligned}\tag{18.35}$$

Graph of $\cosh\left(\frac{2\,y}{7}\right)\operatorname{Re}\left(\overline{\eta}_8\left(i\,y\right)\right)$ and $\cosh\left(\frac{2\,y}{7}\right)\operatorname{Im}\left(\overline{\eta}_8\left(i\,y\right)\right),\; y\in[52,63]\,,\;\bullet\,:\,\tau_m,\; m\in\{12,13,14\}$



Graph. 18.6

The function value is negative without changing sign in its vicinity except for the fourteenth non-trivial zero of the eta function. Here are the details.

$$\begin{aligned}&\{\operatorname{Re}\left(\overline{\eta}_{12}\left(i\left(\frac{56\,719\,268\,686}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\operatorname{Re}\left(\overline{\eta}_{12}\left(i\left(\frac{56\,719\,268\,686}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\}\\&\simeq\{-3.98984\times10^{-15},3.60511\times10^{-15}\}.\end{aligned}\tag{18.36}$$

$$\{\mathrm{Re}\left(\bar{\eta}_{12}\left(i\left(\frac{59\,312\,768\,029}{1\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\bar{\eta}_{12}\left(i\left(\frac{59\,312\,768\,029}{1\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{3.41677\times 10^{-16},-2.34359\times 10^{-16}\}. \tag{18.37}$$

$$\{\mathrm{Re}\left(\bar{\eta}_{13}\left(i\left(2\times\frac{304\,248\,761\,259}{10\,000\,000\,000}-10^{-9}\right)\right)\right),\mathrm{Re}\left(\bar{\eta}_{13}\left(i\left(2\times\frac{304\,248\,761\,259}{10\,000\,000\,000}+10^{-9}\right)\right)\right)\} \simeq \{-9.04002\times 10^{-26},-1.25067\times 10^{-25}\}. \tag{18.38}$$

The imaginary part of the eta-bar function's m-th contact zero with the imaginary axis would be twice the imaginary part of the zeta function's m-th non-trivial zero. This is a paraphrase of the second conjecture.

18.3 Third Conjecture

A variable interval is constructed to describe the first topic of the third conjecture. On the imaginary axis, the second variable interval is defined as follows:

$$\mathrm{V}_2\left(m\right) := \{iy : \rho_m > 0, \zeta\left(1/2+i\rho_m\right)=0, 0\leq y\leq \rho_m\},\quad m\in\mathbb{N}. \tag{18.39}$$

From the observational results, the following relations of inclusion would hold:

$$\mathrm{N}\left(m\right)\in\{m-1,m,m+1\},\quad m\in\mathbb{N}. \tag{18.40}$$

Where

$$\mathrm{N}\left(m\right) := \#\{\tau_n : n\in\mathbb{N}, \tau_n>0, \eta\left(i\tau_n\right)=0, \tau_n\in\mathrm{V}_2\left(m\right)\},\quad m\in\mathbb{N}. \tag{18.41}$$

The second topic: concerning the relation between the imaginary parts of non-trivial zeros of the zeta function and those of the eta function, I noticed that it is neither too close nor too far away in the observational range. The following relations between ρ_m and τ_m would be true in a narrow domain:

$$\tau_m<\rho_{m+1}<\tau_{m+3},\quad m\in\mathbb{N}. \tag{18.42}$$

$$\rho_m<\tau_{m+2}<\rho_{m+3},\quad m\in\mathbb{N}. \tag{18.43}$$

18.4 Fourth Conjecture

The following equation is defined to describe the fourth conjecture:

$$\lambda\left(\theta\right) := \zeta\left(\theta\right)+\eta\left(\theta\right),\quad \theta\in\mathbb{C}\backslash\{1\}. \tag{18.44}$$

And the habitable zone is defined.

$$\text{Habitable zone} := \{\theta\in\mathbb{C} : 0<\mathrm{Re}\left(\theta\right)<1/2\}. \tag{18.45}$$

The explicit formula for the lambda function is given as follows:

$$\lambda\left(\theta\right)=\frac{2\pi^{\frac{\theta}{2}}}{\left(1-\theta\right)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\infty}\sigma_{-\theta}\left(p\right)p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta\left(\theta-1\right)+2\left(7-\theta\right)\left(2\pi p\right)^2\right)\mathrm{K}_{\frac{\theta}{2}}\left(2\pi p\right)\\ +\left(\left(\theta-1\right)-4(2\pi p)^2\right)\left(2\pi p\right)\mathrm{K}_{\frac{2-\theta}{2}}\left(2\pi p\right)\end{pmatrix},\quad \theta\in\mathbb{C}\backslash\{1\}. \tag{18.46}$$

The rule in subsection 6-3 (using the character k instead of λ) is applied to the above equation.

(18.47)

$$\lambda_{\mathrm{k}}\left(\theta\right)=\frac{2\pi^{\frac{\theta}{2}}}{\left(1-\theta\right)\Gamma\left(1+\frac{\theta}{2}\right)}\sum_{p=1}^{\mathrm{k}}\sigma_{-\theta}\left(p\right)p^{\frac{\theta}{2}}\begin{pmatrix}\left(\theta\left(\theta-1\right)+2\left(7-\theta\right)\left(2\pi p\right)^2\right)\mathrm{K}_{\frac{\theta}{2}}\left(2\pi p\right)\\ +\left(\left(\theta-1\right)-4(2\pi p)^2\right)\left(2\pi p\right)\mathrm{K}_{\frac{2-\theta}{2}}\left(2\pi p\right)\end{pmatrix},\quad \left(\theta\in\mathbb{C}\backslash\{1\},\mathrm{k}\in\mathbb{N}\right).$$

I define that the positive imaginary part of the m-th non-trivial zero of the lambda function, which is the number allocated in the order that is closer to the real axis, is α_m. Under this assumption,

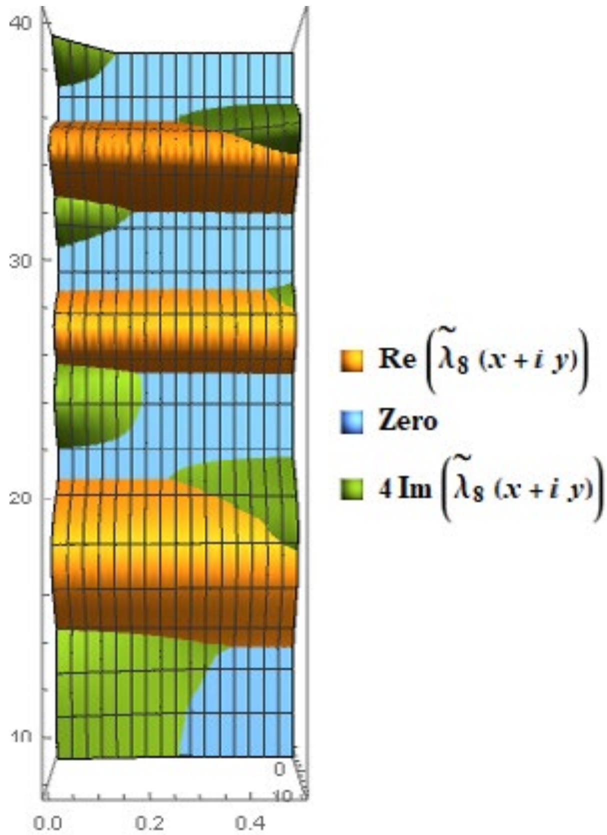
$$\lambda\left(u_m\pm i\alpha_m\right) := 0,\quad \left(u_m,\alpha_m\in\mathbb{R},m\in\mathbb{N}\right). \tag{18.48}$$

Here the real part of the m-th non-trivial zero of the lambda function is u_m. The fourth conjecture is that: all non-trivial zeros of the lambda function would be present in the habitable zone.

The following function is defined for drawing a graph that just shows the lambda function's non-trivial zeros:

$$\tilde{\lambda}_{\mathrm{k}}(\theta) := \left|\frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\,\Gamma\left(1+\frac{\theta}{2}\right)}\right| \sum_{p=1}^{\mathrm{k}} \sigma_{-\theta}(p)\, p^{\frac{\theta}{2}} \begin{pmatrix} \left(\theta(\theta-1)+2(7-\theta)(2\pi p)^2\right) \mathrm{K}_{\frac{\theta}{2}}(2\pi p) \\ +\left((\theta-1)-4(2\pi p)^2\right)(2\pi p)\,\mathrm{K}_{\frac{2-\theta}{2}}(2\pi p) \end{pmatrix}, \tag{18.49}$$
$$(\theta \in \mathbb{C}\backslash\{1\}, \mathrm{k} \in \mathbb{N}).$$

3D graph of $\mathrm{Re}\left(\tilde{\lambda}_8(x+i\,y)\right)$, Zero, and $4\,\mathrm{Im}\left(\tilde{\lambda}_8(x+i\,y)\right)$, $x \in [0, 1/2]$, $y \in [8, 40]$, Viewpoint $\to$ Above



Graph. 18.7

The non-trivial zeros of the lambda function are the cross points of three colors.
The coefficient number 4 is used in front of the imaginary part to emphasize the non-trivial zeros.
Here are the details.

$$\begin{aligned} u_1+i\alpha_1 &\simeq 0.36360066983+i12.970445119, \\ u_2+i\alpha_2 &\simeq 0.23314410196+i20.645405240, \\ u_3+i\alpha_3 &\simeq 0.16959589417+i25.432819891, \\ u_4+i\alpha_4 &\simeq 0.41836013776+i29.250622535, \\ u_5+i\alpha_5 &\simeq 0.17066664686+i32.737058741, \\ u_6+i\alpha_6 &\simeq 0.23291116250+i36.864192387. \end{aligned} \tag{18.50}$$

Calculation errors of non-trivial zeros in the lambda function are shown as follows:

$$\left|\lambda_8\left(\frac{36\,360\,066\,983}{100\,000\,000\,000}+i\frac{12\,970\,445\,119}{100\,000\,000\,000}\right)\right| \simeq 4.41687\times 10^{-10}. \tag{18.51}$$

$$\left|\lambda_8\left(\frac{23\,314\,410\,196}{100\,000\,000\,000}+i\frac{20\,645\,405\,240}{100\,000\,000\,000}\right)\right|\simeq 8.02663\times 10^{-7}. \tag{18.52}$$

$$\left|\lambda_9\left(\frac{16\,959\,589\,417}{100\,000\,000\,000}+i\frac{25\,432\,819\,891}{100\,000\,000\,000}\right)\right|\simeq 9.03159\times 10^{-9}. \tag{18.53}$$

$$\left|\lambda_9\left(\frac{41\,836\,013\,776}{100\,000\,000\,000}+i\frac{29\,250\,622\,535}{100\,000\,000\,000}\right)\right|\simeq 1.31080\times 10^{-8}. \tag{18.54}$$

$$\left|\lambda_9\left(\frac{17\,066\,664\,686}{100\,000\,000\,000}+i\frac{32\,737\,058\,741}{100\,000\,000\,000}\right)\right|\simeq 5.91107\times 10^{-8}. \tag{18.55}$$

$$\left|\lambda_{11}\left(\frac{23\,291\,116\,250}{100\,000\,000\,000}+i\frac{36\,864\,192\,387}{100\,000\,000\,000}\right)\right|\simeq 6.99155\times 10^{-10}. \tag{18.56}$$

18.5 Fifth Conjecture

This conjecture concerns the reciprocal sum of conditional Pythagorean primes and claims that it converges more slowly than the reciprocal sum of the primes. Consider adding a constraint that among three square numbers, the sum of the smallest square and the middle square is prime. The following simultaneous Diophantine equations describe this:

$$\begin{cases} {r_{\mathrm{i}}}^2 = x^2+y^2+z^2 & (x,y,z,\mathrm{i}\in\mathbb{N}, x\le y\le z), \\ \sigma_0\left(x^2+y^2\right)=2 & (x,y\in\mathbb{N}, x<y). \end{cases} \tag{18.57}$$

Some non-Pythagorean primes have unique solutions, and all verified solutions are of this form.

$${r_{\mathrm{i}}}^2 = {k_{\mathrm{i}}}^2+{l_{\mathrm{i}}}^2+(r_{\mathrm{i}}-1)^2, \quad \left({}^{\exists}k_{\mathrm{i}}, {}^{\exists}l_{\mathrm{i}}\in\mathbb{N}, \mathrm{i}=1,2,4,6,\cdots,174\,491\right). \tag{18.58}$$

Since primes except even prime 2, represented by the sum of two squares are limited to Pythagorean primes, let it be the j-th Pythagorean prime q_{j}. Furthermore, I assume that there are infinitely many solutions with this form.

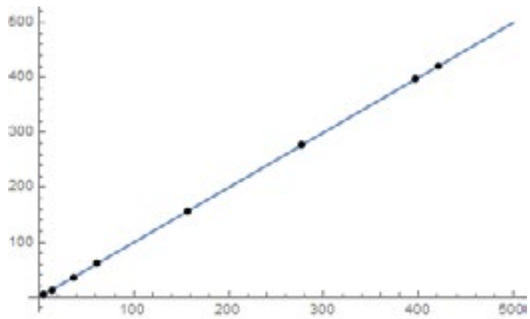
$${r_{\mathrm{i}}}^2 = q_{\mathrm{j}}+(r_{\mathrm{i}}-1)^2, \quad ((\mathrm{i},\mathrm{j})=(1,1),(2,2),(4,5),(6,8),\cdots). \tag{18.59}$$

Since the square of r_{i} is canceled on both sides,

$$2\,r_{\mathrm{i}}-1=q_{\mathrm{j}}, \quad ((\mathrm{i},\mathrm{j})=(1,1),(2,2),(4,5),(6,8),\cdots). \tag{18.60}$$

I obtain a rare relation between the non-Pythagorean primes and Pythagorean primes by adding only one condition. After this, the rare relationship is shown in two graphs.

Graph of the points $(2\,r_{\mathrm{i}}-1, q_{\mathrm{j}})$, with the line $y=x$, $x\in[0,500]$



Graph. 18.8

The dot on the graph indicates the point $(2\,r_{\mathrm{i}}-1, q_{\mathrm{j}})$ which is on the line $y=x$.
Here are the details.

(18.61)
$$\begin{aligned} q_1 &= 5 = 2\times 3-1 = 2\,r_1-1,\\ q_2 &= 13 = 2\times 7-1 = 2\,r_2-1,\\ q_5 &= 37 = 2\times 19-1 = 2\,r_4-1,\\ q_8 &= 61 = 2\times 31-1 = 2\,r_6-1,\\ q_{17} &= 157 = 2\times 79-1 = 2\,r_{12}-1,\\ q_{27} &= 277 = 2\times 139-1 = 2\,r_{18}-1,\\ q_{37} &= 397 = 2\times 199-1 = 2\,r_{24}-1,\\ q_{40} &= 421 = 2\times 211-1 = 2\,r_{25}-1. \end{aligned}$$

Graph of the points $(2\,r_{\mathrm{i}}-1, q_{\mathrm{j}})$, with the line $y=x$, $x\in[10\,000\,000, 10\,010\,000]$

Graph. 18.9

There is no regularity in distribution even for very high numbers.
Moreover, adjacent non-Pythagorean primes ($r_{174\,445}=5\,003\,659$ and $r_{174\,446}=5\,003\,671$) are found.
Here are the details.

(18.62)
$$\begin{aligned} q_{332\,228} &= 10\,001\,701 = 2\times 5\,000\,851-1 = 2\,r_{174\,346}-1,\\ q_{332\,245} &= 10\,002\,133 = 2\times 5\,001\,067-1 = 2\,r_{174\,354}-1,\\ q_{332\,293} &= 10\,003\,957 = 2\times 5\,001\,979-1 = 2\,r_{174\,390}-1,\\ q_{332\,319} &= 10\,005\,013 = 2\times 5\,002\,507-1 = 2\,r_{174\,403}-1,\\ q_{332\,327} &= 10\,005\,277 = 2\times 5\,002\,639-1 = 2\,r_{174\,407}-1,\\ q_{332\,372} &= 10\,006\,741 = 2\times 5\,003\,371-1 = 2\,r_{174\,434}-1,\\ q_{332\,384} &= 10\,007\,077 = 2\times 5\,003\,539-1 = 2\,r_{174\,442}-1,\\ q_{332\,392} &= 10\,007\,317 = 2\times 5\,003\,659-1 = 2\,r_{174\,445}-1,\\ q_{332\,394} &= 10\,007\,341 = 2\times 5\,003\,671-1 = 2\,r_{174\,446}-1,\\ q_{332\,402} &= 10\,007\,653 = 2\times 5\,003\,827-1 = 2\,r_{174\,453}-1,\\ q_{332\,451} &= 10\,009\,477 = 2\times 5\,004\,739-1 = 2\,r_{174\,481}-1,\\ q_{332\,462} &= 10\,009\,981 = 2\times 5\,004\,991-1 = 2\,r_{174\,491}-1. \end{aligned}$$

Up to 10 010 000, there are 28 202 conditional Pythagorean primes, but 56 089 Sophie Germain primes.
Here, Sophie Germain prime is a prime number that is prime again by adding 1 to it's double.
It is unknown whether there are infinitely many Sophie Germain primes "[19]."

Let now define the m-th conditional Pythagorean prime as twice the m-th conditional non-Pythagorean prime minus 1. They are denoted as q_m-hat and r_m-hat, respectively. Under the premise, there are infinitely many pairs of q_m-hat and q_m-hat.

(18.63)
$$\hat{q}_m = 2\,\hat{r}_m - 1, \quad m\in\mathbb{N}.$$

Let $\mathrm{S_q}\,(m)$ be the reciprocal sum up to the m-th conditional Pythagorean prime.

(18.64)
$$\mathrm{S_q}\,(m) := \sum_{\mathrm{i}=1}^{m} \hat{q}_{\mathrm{i}}^{-1} = \frac{1}{5}+\frac{1}{13}+\frac{1}{37}+\frac{1}{61}+\frac{1}{157}+\frac{1}{277}+\frac{1}{397}+\cdots+\hat{q}_m^{-1}, \quad m\in\mathbb{N}.$$

Similarly, let $\mathrm{S_r}(m)$ be the reciprocal sum up to the m-th conditional non-Pythagorean prime.

(18.65) $$\mathrm{S_r}(m) := \sum_{\mathrm{i}=1}^{m} \hat{r}_{\mathrm{i}}^{-1} = \frac{1}{3} + \frac{1}{7} + \frac{1}{19} + \frac{1}{31} + \frac{1}{79} + \frac{1}{139} + \frac{1}{199} + \cdots + \hat{r}_m^{-1}, \quad m \in \mathbb{N}.$$

The following inequality holds for any positive integer:

(18.66) $$\mathrm{S_r}(m) > \mathrm{S_q}(m), \quad m \in \mathbb{N}.$$

The following reciprocal series of the conditional Pythagorean primes would be a divergent series "[20]":

(18.67) $$\frac{1}{5} + \frac{1}{13} + \frac{1}{37} + \frac{1}{61} + \frac{1}{157} + \frac{1}{277} + \frac{1}{397} + \cdots + \hat{q}_m^{-1} + \cdots = +\infty.$$

The following reciprocal series of the conditional non-Pythagorean primes would also be a divergent series:

(18.68) $$\frac{1}{3} + \frac{1}{7} + \frac{1}{19} + \frac{1}{31} + \frac{1}{79} + \frac{1}{139} + \frac{1}{199} + \cdots + \hat{r}_m^{-1} + \cdots = +\infty.$$

19 Additional Knowledge

19.1 Proof of the Absolute Convergence of the Integral

$$\int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad (\alpha > 0, \theta \in \mathbb{C})$$

At first, the function of three real variables is defined as follows:

(19.1) $$u(x,t,\alpha) := x^{t-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}, \quad (x, \alpha > 0, t \in \mathbb{R}).$$

The variable x integrates the above function into the open interval $(0, \infty)$.

(19.2) $$\int_0^\infty u(x,t,\alpha)\, dx = \int_0^\infty x^{t-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad (\alpha > 0, t \in \mathbb{R}).$$

The left-sided limit of the integrand is zero as follows:

(19.3) $$\lim_{x\to+0} u(x,t,\alpha) = \lim_{x\to+0} e^{-x^2} \cdot \lim_{x\to+0} x^{t-1} e^{-\frac{\alpha^2}{x^2}} = \lim_{y\to\infty} y^{1-t} e^{-\alpha^2 y^2} = 0, \quad (\alpha > 0, t \in \mathbb{R}).$$

The right-sided limit of the integrand is also zero, as follows:

(19.4) $$\lim_{x\to\infty} u(x,t,\alpha) = \lim_{x\to\infty} e^{-\frac{\alpha^2}{x^2}} \cdot \lim_{x\to\infty} x^{t-1} e^{-x^2} = \lim_{y\to+0} e^{-\alpha^2 y^2} \cdot \lim_{x\to\infty} x^{t-1} e^{-x^2} = 0, \quad (\alpha > 0, t \in \mathbb{R}).$$

The integrand is partially differentiated by the variable x.

(19.5) $$\begin{aligned} \frac{\partial}{\partial x} u(x,t,\alpha) &= \left((t-1)\, x^{t-2} - 2x^t + 2\alpha^2 x^{t-4}\right) e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} \\ &= -\left(2x^4 + (1-t)\, x^2 - 2\alpha^2\right) x^{t-4} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}, \quad (x, \alpha > 0, t \in \mathbb{R}). \end{aligned}$$

And I assume that the integrand's differentiation is zero at $x = \beta$ $(\beta > 0)$.

(19.6) $$\left.\frac{\partial}{\partial x} u(x,t,\alpha)\right|_{x=\beta} = 0, \quad (x, \alpha, \beta > 0, t \in \mathbb{R}).$$

The dependent variable β is then determined as the solution of the following quartic equation:

(19.7) $$2\beta^4 + (1-t)\,\beta^2 - 2\alpha^2 = 0, \quad (\alpha, \beta > 0, t \in \mathbb{R}).$$

The quartic equation is defined as the quadratic equation of the variable β^2. Based on the formula for solving the quadratic equations,

(19.8) $$\beta^2 = \frac{-(1-t) + \sqrt{(1-t)^2 + 16\alpha^2}}{4}, \quad (\alpha, \beta > 0, t \in \mathbb{R}).$$

The quartic equation has the unique solution β in the open interval $(0,\infty)$.

$$\beta = \frac{1}{2}\sqrt{\sqrt{(1-t)^2+16\alpha^2}-(1-t)}, \quad (\alpha,\beta>0, t\in\mathbb{R})\,. \tag{19.9}$$

Table of the increase and decrease for the integrand

x	$+0$	$\cdots$	β	$\cdots$	∞
$\frac{\partial}{\partial x}u(x,t,\alpha)$		$+$	0	$-$	
$u(x,t,\alpha)$	0	$\nearrow$	maximum	$\searrow$	0

Table. 19.1

The open interval of the improper integral is divided into two semi-open intervals on the right side of equation (19.2) as follows:

$$\int_0^\infty u(x,t,\alpha)\,dx = \int_0^1 x^{t-1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx + \int_1^\infty x^{t-1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx, \quad (\alpha>0, t\in\mathbb{R})\,. \tag{19.10}$$

For the second term of the right side, I perform the variable transformation $x=y^{-1}$.

$$\begin{aligned}\int_1^\infty x^{t-1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx &= \int_1^0 \left(\frac{1}{y}\right)^{t-1} e^{-\left(\frac{1}{y^2}+\alpha^2y^2\right)}\left(-y^2\right)dy \\ &= \int_0^1 y^{-1-t}e^{-\left(\frac{1}{y^2}+\alpha^2y^2\right)}dy, \quad (\alpha>0, t\in\mathbb{R})\,.\end{aligned} \tag{19.11}$$

The following inequality is obtained by applying the Schwarz's inequality to the equation (19.10):

$$\int_0^\infty |u(x,t,\alpha)|\,dx \le \int_0^1 x^{t-1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx + \int_0^1 y^{-1-t}e^{-\left(\frac{1}{y^2}+\alpha^2y^2\right)}dy, \quad (\alpha>0, t\in\mathbb{R})\,. \tag{19.12}$$

The function of three real variables $v(x,t,\alpha)$ is defined as the integrand of the result of equation (19.11) for convenience.

$$v(x,t,\alpha) := x^{-1-t}e^{-\left(\frac{1}{x^2}+\alpha^2x^2\right)}, \quad (x,\alpha>0, t\in\mathbb{R})\,. \tag{19.13}$$

Hereafter, I will find the upper limit of the improper integral of the function $|u(x,t,\alpha)|$. I begin by considering the two-dimensional $\alpha-t$ phase space. The boundary curve divides the space's right-half-plane into two regions. To generate the curve's representation, 1 is substituted for β on the left side of equation (19.8).
The result is

$$t = 3-2\alpha^2, \quad \alpha>0. \tag{19.14}$$

Of course, the value of β is equal to 1 throughout the boundary curve.
Under the condition of $0<\beta\leqq 1$,

$$t \le 3-2\alpha^2, \quad \alpha>0. \tag{19.15}$$

The improper integral's upper limit is calculated as the sum of two square areas.

$$\begin{aligned}\int_0^\infty |u(x,t,\alpha)|\,dx &\le 1\cdot u(\beta,t,\alpha) + 1\cdot v(1,t,\alpha) \\ &= \beta^{t-1}e^{-\left(\beta^2+\frac{\alpha^2}{\beta^2}\right)} + e^{-\left(\alpha^2+1\right)}, \quad \left(\alpha>0, t\le 3-2\alpha^2\right)\,.\end{aligned} \tag{19.16}$$

Under the condition of $\beta>1$ i.e.,

$$t > 3-2\alpha^2, \quad \alpha>0. \tag{19.17}$$

The upper limit of the improper integral is also calculated as the sum of two square areas.

$$\begin{aligned}\int_0^\infty |u(x,t,\alpha)|\,dx \leq & 1\cdot u(1,t,\alpha) + 1\cdot v\left(\frac{1}{\beta},t,\alpha\right) \\ &= e^{-\left(\alpha^2+1\right)} + \left(\frac{1}{\beta}\right)^{-1-t} e^{-\left(\beta^2+\frac{\alpha^2}{\beta^2}\right)} \\ &= \beta^{t+1} e^{-\left(\beta^2+\frac{\alpha^2}{\beta^2}\right)} + e^{-\left(\alpha^2+1\right)}, \quad \left(\alpha>0, t>3-2\alpha^2\right).\end{aligned} \tag{19.18}$$

The results of equations (19.16) and (19.18) show that the improper integral absolutely converges.
Finally, I expand the domain of definition of the variable t from a real number to a complex number. To clearly express this, I change the character from t to θ. Herein, I take the absolute value of the improper integral.

$$\begin{aligned}\left|\int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx\right| &\leq \int_0^\infty \left|x^{\theta-1}\right| e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx \\ &= \int_0^\infty x^{-1+\mathrm{Re}(\theta)} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad \left(\alpha>0, \theta\in\mathbb{C}\right).\end{aligned} \tag{19.19}$$

The right side in inequality (19.19) is the same as substituting the real value $\mathrm{Re}(\theta)$ for t on the right side of equation (19.2).
Therefore, the proof of the assertion in the title of this subsection is complete.

19.2 Proof of the Recurrence Formula for the Integral

$$\int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad \left(\alpha>0, \theta\in\mathbb{C}\right)$$

I apply the integration by parts to the integral excluding the case when $\theta=0$ as follows:

$$\begin{aligned}&\int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx \\ &= \int_0^\infty \left(\frac{x^\theta}{\theta}\right)' e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx \\ &= \left[\frac{x^\theta}{\theta} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}\right]_0^\infty - \int_0^\infty \frac{x^\theta}{\theta}\left(-2x+\frac{2\alpha^2}{x^3}\right) e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx \\ &= \frac{2}{\theta}\int_0^\infty x^{\theta+1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx - \frac{2\alpha^2}{\theta}\int_0^\infty x^{\theta-3} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad \left(\alpha>0, \theta\in\mathbb{C}\backslash\{0\}\right).\end{aligned} \tag{19.20}$$

Thus, the integral of the second term on the right side of the result is written as follows:

$$\begin{aligned}&\int_0^\infty x^{\theta-3} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx \\ &= \frac{1}{\alpha^2}\left(\int_0^\infty x^{\theta+1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx - \frac{\theta}{2}\int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx\right), \quad \left(\alpha>0, \theta\in\mathbb{C}\backslash\{0\}\right).\end{aligned} \tag{19.21}$$

In case of $\theta=0$, I perform the variable transformation $x=y^{-1}$ for the left side of equation (19.21).

$$\int_0^\infty x^{-3} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \int_\infty^0 \left(\frac{1}{y}\right)^{-3} e^{-\left(\frac{1}{y^2}+\alpha^2 y^2\right)}\left(-y^{-2}\right) dy = \int_0^\infty y e^{-\left(\frac{1}{y^2}+\alpha^2 y^2\right)} dy, \quad \alpha>0. \tag{19.22}$$

I perform the variable transformation $y=x/\alpha$, one more time.

$$\int_0^\infty y e^{-\left(\frac{1}{y^2}+\alpha^2 y^2\right)} dy = \int_0^\infty \frac{x}{\alpha} e^{-\left(\frac{\alpha^2}{x^2}+x^2\right)} \frac{1}{\alpha} dx = \frac{1}{\alpha^2}\int_0^\infty x e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad \alpha>0. \tag{19.23}$$

For the right side of equation (19.21), 0 is substituted directly for θ and the result is the same as the above result.
Thus, I can include the condition $\theta=0$ for the equation (19.21).

$$\int_0^\infty x^{\theta-3} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{1}{\alpha^2}\left(\int_0^\infty x^{\theta+1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx - \frac{\theta}{2}\int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx\right), \quad \left(\alpha>0, \theta\in\mathbb{C}\right). \tag{19.24}$$

The integral of the first term on the right side of equation (19.24) is written as follows:

$$\int_0^\infty x^{\theta+1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx=\alpha^2\int_0^\infty x^{\theta-3}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx+\frac{\theta}{2}\int_0^\infty x^{\theta-1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx,\quad (\alpha>0,\theta\in\mathbb{C}).\tag{19.25}$$

For equation (19.25), 0 is substituted for α. And for the integral of the left side, I perform the variable transformation $x=y^{1/2}$.

$$\begin{aligned}&\int_0^\infty x^{\theta+1}e^{-x^2}dx=\int_0^\infty y^{\frac{\theta+1}{2}}e^{-y}\frac{1}{2}y^{-\frac{1}{2}}dy=\frac{1}{2}\int_0^\infty y^{1+\frac{\theta}{2}-1}e^{-y}dy\\&=\frac{1}{2}\Gamma\left(1+\frac{\theta}{2}\right),\quad \theta\in\mathbb{C}.\end{aligned}\tag{19.26}$$

For equation (19.25), 0 is substituted for α again. And for the integral of the right side, I perform the variable transformation $x=y^{1/2}$.

$$\begin{aligned}&\frac{\theta}{2}\int_0^\infty x^{\theta-1}e^{-x^2}dx=\frac{\theta}{2}\int_0^\infty y^{\frac{\theta-1}{2}}e^{-y}\frac{1}{2}y^{-\frac{1}{2}}dy=\frac{\theta}{4}\int_0^\infty y^{\frac{\theta}{2}-1}e^{-y}dy=\frac{\theta}{4}\Gamma\left(\frac{\theta}{2}\right)\\&=\frac{1}{2}\Gamma\left(1+\frac{\theta}{2}\right),\quad \theta\in\mathbb{C}.\end{aligned}\tag{19.27}$$

Both sides of the equation (19.25) are the same when $\alpha=0$ and the results have same poles at even numbers of -2 or less. As a result, the integral's recurrence formula is as follows:

$$\begin{aligned}&\int_0^\infty x^{\theta+1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx=\alpha^2\int_0^\infty x^{\theta-3}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx+\frac{\theta}{2}\int_0^\infty x^{\theta-1}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx,\\&(\alpha=0,\theta\in\mathbb{C}\backslash\{-2n:n\in\mathbb{N}\})\vee(\alpha>0,\theta\in\mathbb{C}).\end{aligned}\tag{19.28}$$

19.3 Representations for the Integrals

$$\int_0^\infty x^{2p}e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx,\quad (\alpha>0,p=-1)\vee(\alpha\ge 0,p=0,1,2,\cdots)$$

When $p=0$,
I consider the integral

$$\int_0^\infty x^0 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx,\quad \alpha\ge 0.$$

At first, I show the defining equation of the function $J(\alpha)$ for preparation "[21]."

$$J(\alpha):=\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2}dx,\quad \alpha\in\mathbb{R}.\tag{19.29}$$

The function $J(\alpha)$ is also written as follows:

$$J(\alpha)=e^{2\alpha}\int_0^\infty e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx>0,\quad \alpha\in\mathbb{R}.\tag{19.30}$$

The value of $J(0)$ is given as follows:

$$J(0)=\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.\quad \text{(Gaussian integral)}\tag{19.31}$$

The value of left-sided limit of $J(\alpha)$ is given as follows:

$$\lim_{\alpha\to-\infty}J(\alpha)=\lim_{\alpha\to-\infty}e^{2\alpha}\cdot\lim_{\alpha\to-\infty}\int_0^\infty e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx\le\lim_{\alpha\to-\infty}e^{2\alpha}\cdot\int_0^\infty e^{-x^2}dx=0.\tag{19.32}$$

Assuming that the function $J(\alpha)$ has uniform convergence, the differentiation of $J(\alpha)$ is carried out for the integrand as partial differentiation..

$$\frac{d}{d\alpha}J(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha} e^{-\left(x-\frac{\alpha}{x}\right)^2} dx = \int_0^\infty -2\left(x-\frac{\alpha}{x}\right)\left(-\frac{1}{x}\right) e^{-\left(x-\frac{\alpha}{x}\right)^2} dx$$

(19.33)
$$= 2\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx - 2\alpha\int_0^\infty x^{-2} e^{-\left(x-\frac{\alpha}{x}\right)^2} dx$$

For the second integral, the variable transformation $x = y^{-1}$ is perfomed.

$$= 2\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx - 2\alpha\int_\infty^0 y^2 e^{-\left(\frac{1}{y}-\alpha y\right)^2}\left(-y^{-2}\right) dy, \quad \alpha \in \mathbb{R}.$$

Therefore,

(19.34)
$$\frac{d}{d\alpha}J(\alpha) = 2\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx - 2\alpha\int_0^\infty e^{-\left(\frac{1}{y}-\alpha y\right)^2} dy, \quad \alpha \in \mathbb{R}.$$

For the second integral on the right side of equation (19.34), the variable transformation $y = -x/\alpha$ is perfomed. When $\alpha < 0$, the differentiation of $J(\alpha)$ takes positive value as follows:

(19.35)
$$\frac{d}{d\alpha}J(\alpha) = 2\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx - 2\alpha\int_0^\infty e^{-\left(-\frac{\alpha}{x}+x\right)^2}\left(-\frac{1}{\alpha}\right) dx = 4\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx > 0, \quad \alpha < 0.$$

The right-sided limit of the differentiation of $J(\alpha)$ in the neighborhood of $\alpha = 0$ takes the fixed value as follows:

(19.36)
$$\lim_{\alpha\to-0}\frac{d}{d\alpha}J(\alpha) = 4\int_0^\infty e^{-x^2} dx = 4\cdot\frac{\sqrt{\pi}}{2} = 2\sqrt{\pi}.$$

For the second integral on the right side of equation (19.34), the variable transformation $y = x/\alpha$ is perfomed. When $\alpha > 0$, the differentiation of $J(\alpha)$ takes zero as follows:

(19.37)
$$\frac{d}{d\alpha}J(\alpha) = 2\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx - 2\alpha\int_0^\infty e^{-\left(\frac{\alpha}{x}-x\right)^2}\left(\frac{1}{\alpha}\right) dx = 0, \quad \alpha > 0.$$

The left-sided limit of the differentiation of $J(\alpha)$ in the neighborhood of $\alpha = 0$ takes zero as follows:

(19.38)
$$\lim_{\alpha\to+0}\frac{d}{d\alpha}J(\alpha) = 0.$$

Therefore, when $\alpha \geqq 0$, the function $J(\alpha)$ takes the constant as follows:

(19.39)
$$J(\alpha) = \text{CONSTANT} = J(0) = \frac{\sqrt{\pi}}{2}, \quad \alpha \geq 0.$$

Table shows the increase and decrease in the function $J(\alpha)$

α	$-\infty$	$\cdots$	0	$\cdots$	∞
$\frac{d}{d\alpha}J(\alpha)$		$+$	discontinuity	0	0
$J(\alpha)$	0	$\nearrow$	$\frac{\sqrt{\pi}}{2}$	$\rightarrow$	$\frac{\sqrt{\pi}}{2}$

Table. 19.2

By referring to the equation (19.30),

(19.40)
$$J(\alpha) = e^{2\alpha}\int_0^\infty e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2}, \quad \alpha \geq 0.$$

Both the central and right sides are multiplied by $e^{-2\alpha}$ to obtain the formula in the case of $p = 0$.

(19.41)
$$\int_0^\infty x^0 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2} e^{-2\alpha}, \quad \alpha \geq 0.$$

I show another proof of the above formula that makes no assumptions. If $\alpha > 0$,
For the right side of equation (19.30), I perform the variable transformation $x = (\alpha y)^{1/2}$.

(19.42)
$$J(\alpha) = e^{2\alpha} \int_0^\infty e^{-\alpha\left(y+\frac{1}{y}\right)} \frac{\sqrt{\alpha}}{2} y^{-\frac{1}{2}} dx = \sqrt{\alpha}\, e^{2\alpha} \cdot \frac{1}{2} \int_0^\infty y^{\frac{1}{2}-1} e^{-\frac{2\alpha}{2}\left(y+\frac{1}{y}\right)} dy$$
$$= \sqrt{\alpha}\, e^{2\alpha}\, \mathrm{K}_{\frac{1}{2}}(2\alpha), \quad \alpha > 0.$$

Now, recall the formula for the modified Bessel function of the second kind at the index $\nu = \pm 1/2$.

(19.43)
$$\mathrm{K}_{\pm\frac{1}{2}}(Z) = \sqrt{\frac{\pi}{2Z}}\, e^{-Z}, \quad Z \in \mathbb{C}\backslash\{0\}.$$

The formula (19.43) is introduced into the result of equation (19.42) to obtain the result as follows:

(19.44)
$$J(\alpha) = \sqrt{\alpha}\, e^{2\alpha} \cdot \sqrt{\frac{\pi}{2(2\alpha)}}\, e^{-2\alpha} = \frac{\sqrt{\pi}}{2}, \quad \alpha > 0.$$

In case of $\alpha < 0$
For the right side of equation (19.30), I perform the variable transformation $x = (-\alpha y)^{1/2}$.

(19.45)
$$J(\alpha) = e^{2\alpha} \int_0^\infty e^{\alpha\left(y+\frac{1}{y}\right)} \frac{\sqrt{-\alpha}}{2} y^{-\frac{1}{2}} dx = \sqrt{-\alpha}\, e^{2\alpha} \cdot \frac{1}{2} \int_0^\infty y^{\frac{1}{2}-1} e^{-\frac{(-2\alpha)}{2}\left(y+\frac{1}{y}\right)} dy$$
$$= \sqrt{-\alpha}\, e^{2\alpha}\, \mathrm{K}_{\frac{1}{2}}(-2\alpha), \quad \alpha < 0.$$

The formula (19.43) is introduced into the result of equation (19.45) to obtain the result as follows:

(19.46)
$$J(\alpha) = \sqrt{-\alpha}\, e^{2\alpha} \cdot \sqrt{\frac{\pi}{2(-2\alpha)}}\, e^{2\alpha} = \frac{\sqrt{\pi}}{2}\, e^{4\alpha}, \quad \alpha < 0.$$

Because equations (19.44) and (19.46) show that the function $J(\alpha)$ is continuous in the neighborhood of zero,

(19.47)
$$J(\alpha) = \frac{\sqrt{\pi}}{2}, \quad \alpha \geq 0.$$

The above result is the same as that of the equation (19.40). Therefore, another proof of the formula (19.41) is completed.
When $p = -1$,
I consider the integral
$$\int_0^\infty x^{-2} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad \alpha > 0.$$
For this integral, I perform the variable transformation $x = \alpha/y$.

(19.48)
$$\int_0^\infty x^{-2} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \int_\infty^0 \left(\frac{\alpha}{y}\right)^{-2} e^{-\left(\frac{\alpha^2}{y^2}+y^2\right)} \left(-\alpha y^{-2}\right) dy = \frac{1}{\alpha} \int_0^\infty e^{-\left(y^2+\frac{\alpha^2}{y^2}\right)} dy, \quad \alpha > 0.$$

Therefore

(19.49)
$$\int_0^\infty x^{-2} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2\alpha}\, e^{-2\alpha}, \quad \alpha > 0.$$

When $p = 1$,
For the recurrence formula (19.28), 1 is substituted for θ.

(19.50)
$$\int_0^\infty x^2 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \alpha^2 \int_0^\infty x^{-2} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx + \frac{1}{2} \int_0^\infty x^0 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx$$
$$= \frac{(2\alpha)^2}{4} \cdot \frac{\sqrt{\pi}}{2\alpha}\, e^{-2\alpha} + \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2}\, e^{-2\alpha}$$
$$= \frac{1}{4}\left((2\alpha) + 1\right) \sqrt{\pi}\, e^{-2\alpha}, \quad \alpha \geq 0.$$

When $p = 2$,
For the recurrence formula (19.28), 3 is substituted for θ.

$$\begin{aligned}\int_0^\infty x^4 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx &= \alpha^2 \int_0^\infty x^0 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx + \frac{3}{2}\int_0^\infty x^2 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx \\ &= \frac{(2\alpha)^2}{4}\cdot\frac{\sqrt{\pi}}{2}e^{-2\alpha} + \frac{3}{2}\cdot\frac{1}{4}\left((2\alpha)+1\right)\sqrt{\pi}\,e^{-2\alpha} \\ &= \frac{1}{8}\left((2\alpha)^2+3\,(2\alpha)+3\right)\sqrt{\pi}\,e^{-2\alpha}, \quad \alpha \geq 0. \end{aligned} \tag{19.51}$$

19.4 General Representation for the Integrals

$$\int_0^\infty x^{2k} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx, \quad (\alpha \geq 0, k \in \mathbb{N})$$

Hereafter, I will use mathematical induction to find the general representation for the integrals. First, in order to prove, I prepared the following equation that I had found in advance:

$$\int_0^\infty x^{2k} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx = \frac{1}{2^{k+1}}\sum_{\mu=0}^{k} a_{k,\mu}(2\alpha)^\mu \sqrt{\pi}\,e^{-2\alpha}, \quad (\alpha \geq 0, k \in \mathbb{N})\,. \tag{19.52}$$

Where

$$a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}, \quad (k \in \mathbb{N}, \mu = \{0,1,2,\cdots,k\})\,. \tag{19.53}$$

When $k = 1$,
For equation (19.52), 1 is substituted for k.

$$\begin{aligned}\int_0^\infty x^2 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx &= \frac{1}{2^2}\sum_{\mu=0}^{1}\frac{(2-\mu)!}{2^{1-\mu}\mu!\,(1-\mu)!}(2\alpha)^\mu\sqrt{\pi}\,e^{-2\alpha} \\ &= \frac{1}{4}\left((2\alpha)+1\right)\sqrt{\pi}\,e^{-2\alpha}, \quad \alpha \geq 0. \end{aligned} \tag{19.54}$$

The results of equations (19.50) and (19.54) are the same, so in the case of $k = 1$, equation (19.52) is correct.
When $k = 2$,
For equation (19.52), 2 is substituted for k.

$$\begin{aligned}\int_0^\infty x^4 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx &= \frac{1}{2^3}\sum_{\mu=0}^{2}\frac{(4-\mu)!}{2^{2-\mu}\mu!\,(2-\mu)!}(2\alpha)^\mu\sqrt{\pi}\,e^{-2\alpha} \\ &= \frac{1}{8}\left((2\alpha)^2+3\,(2\alpha)+3\right)\sqrt{\pi}\,e^{-2\alpha}, \quad \alpha \geq 0. \end{aligned} \tag{19.55}$$

The results of equations (19.51) and (19.55) are the same, so in the case of $k = 2$, equation (19.52) is correct.
When $k = m$, I assume that equation (19.52) is correct. The following equation holds under the assumption:

$$\int_0^\infty x^{2m} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx = \frac{1}{2^{m+1}}\sum_{\mu=0}^{m}\frac{(2m-\mu)!}{2^{m-\mu}\mu!\,(m-\mu)!}(2\alpha)^\mu\sqrt{\pi}\,e^{-2\alpha}, \quad (\alpha \geq 0, m \in \mathbb{N})\,. \tag{19.56}$$

When $k = m + 1$, I assume that equation (19.52) is also correct. The following equation holds under the assumption:

$$\int_0^\infty x^{2(m+1)} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}dx = \frac{1}{2^{m+2}}\sum_{\mu=0}^{m+1}\frac{(2m+2-\mu)!}{2^{m+1-\mu}\mu!\,(m+1-\mu)!}(2\alpha)^\mu\sqrt{\pi}\,e^{-2\alpha}, \quad (\alpha \geq 0, m \in \mathbb{N})\,. \tag{19.57}$$

When $k = m+2$, I apply the recurrence formula (19.28) to the integral.

$$\begin{aligned}\int_0^\infty x^{2(m+2)} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx &= \alpha^2 \int_0^\infty x^{2m} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx + \frac{2m+3}{2} \int_0^\infty x^{2(m+1)} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx,\\ &= \frac{(2\alpha)^2}{4} \cdot \frac{1}{2^{m+1}} \sum_{\mu=0}^{m} \frac{(2m-\mu)!}{2^{m-\mu}\mu!\,(m-\mu)!} (2\alpha)^\mu \sqrt{\pi}\, e^{-2\alpha}\\ &\quad + \frac{2m+3}{2} \cdot \frac{1}{2^{m+2}} \sum_{\mu=0}^{m+1} \frac{(2m+2-\mu)!}{2^{m+1-\mu}\mu!\,(m+1-\mu)!} (2\alpha)^\mu \sqrt{\pi}\, e^{-2\alpha}\\ &= \frac{1}{2^{m+3}} \left(\begin{array}{c} \displaystyle\sum_{\mu=2}^{m+2} \frac{(2m+2-\mu)!}{2^{m+2-\mu}\,(\mu-2)!\,(m+2-\mu)!} (2\alpha)^\mu \\ \displaystyle + (2m+3) \sum_{\mu=0}^{m+1} \frac{(2m+2-\mu)!}{2^{m+1-\mu}\mu!\,(m+1-\mu)!} (2\alpha)^\mu \end{array} \right) \sqrt{\pi}\, e^{-2\alpha}, \quad (\alpha \geq 0, m \in \mathbb{N}).\end{aligned} \tag{19.58}$$

The coefficient of the highest degree of the polynomial of the variable (2α) in the sum is simplified as follows:

$$\left.\frac{(2m+2-\mu)!}{2^{m+2-\mu}\,(\mu-2)!\,(m+2-\mu)!}\right|_{\mu=m+2} = \frac{m!}{2^0\, m!\, 0!} = 1, \quad m \in \mathbb{N}. \tag{19.59}$$

This polynomial is called a monic polynomial since the coefficient of the highest degree is unity.
The coefficient of the first degree of the polynomial is simplified as follows:

$$(2m+3) \left.\frac{(2m+2-\mu)!}{2^{m+1-\mu}\mu!\,(m+1-\mu)!}\right|_{\mu=1} = \frac{(2m+3)\,(2m+1)!}{2^m\, 1!\, m!} = (2m+3)!!, \quad m \in \mathbb{N}. \tag{19.60}$$

The constant term of the polynomial is simplified as follows:

$$(2m+3) \left.\frac{(2m+2-\mu)!}{2^{m+1-\mu}\mu!\,(m+1-\mu)!}\right|_{\mu=0} = \frac{(2m+3)\,(2m+2)!}{2^{m+1}\, 0!\, (m+1)!} = (2m+3)!!, \quad m \in \mathbb{N}. \tag{19.61}$$

Thus, the first-degree coefficient and the constant term are the same.
The above results are introduced into the result of equation (19.58) to obtain the general term.

$$\begin{aligned}\int_0^\infty x^{2(m+2)} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx &= \frac{1}{2^{m+3}} \left(\begin{array}{c} \displaystyle (2\alpha)^{m+2} + \sum_{\mu=2}^{m+1} \frac{(2m+2-\mu)!}{2^{m+2-\mu}\,(\mu-2)!\,(m+2-\mu)!} (2\alpha)^\mu \\ \displaystyle + (2m+3) \sum_{\mu=2}^{m+1} \frac{(2m+2-\mu)!}{2^{m+1-\mu}\mu!\,(m+1-\mu)!} (2\alpha)^\mu \\ + (2m+3)!!\,(2\alpha) + (2m+3)!! \end{array} \right) \sqrt{\pi}\, e^{-2\alpha}\\ &= \frac{1}{2^{m+3}} \left(\begin{array}{c} (2\alpha)^{m+2} \\ \displaystyle + \sum_{\mu=2}^{m+1} \left(\frac{\mu\,(\mu-1)}{2^{m+2-\mu}\mu!\,(m+2-\mu)!} + \frac{2\,(m+2-\mu)\,(2m+3)}{2^{m+2-\mu}\mu!\,(m+2-\mu)!} \right) (2m+2-\mu)!(2\alpha)^\mu \\ + (2m+3)!!\,(2\alpha) + (2m+3)!! \end{array} \right) \sqrt{\pi}\, e^{-2\alpha}\\ &= \frac{1}{2^{m+3}} \left(\begin{array}{c} \displaystyle (2\alpha)^{m+2} + \sum_{\mu=2}^{m+1} \frac{(2m+4-\mu)!}{2^{m+2-\mu}\mu!\,(m+2-\mu)!} (2\alpha)^\mu \\ + (2m+3)!!\,(2\alpha) + (2m+3)!! \end{array} \right) \sqrt{\pi}\, e^{-2\alpha}, \quad (\alpha \geq 0, m \in \mathbb{N}).\end{aligned} \tag{19.62}$$

Therefore

$$\begin{aligned}&\int_0^\infty x^{2(m+2)} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx\\ &= \frac{1}{2^{m+3}} \left((2\alpha)^{m+2} + \sum_{\mu=2}^{m+1} a_{m+2,\mu} (2\alpha)^\mu + (2m+3)!!\,(2\alpha) + (2m+3)!! \right) \sqrt{\pi}\, e^{-2\alpha}, \quad (\alpha \geq 0, m \in \mathbb{N}).\end{aligned} \tag{19.63}$$

For the coefficient of the general term of the polynomial, $m+2$ is substituted for μ.

$$a_{m+2,m+2} = \left.\frac{(2m+4-\mu)!}{2^{m+2-\mu}\mu!\,(m+2-\mu)!}\right|_{\mu=m+2} = \frac{(m+2)!}{2^0\,(m+2)!\,0!} = 1, \quad m \in \mathbb{N}. \tag{19.64}$$

Also, 1 is instituted for μ.

$$a_{m+2,1} = \left.\frac{(2m+4-\mu)!}{2^{m+2-\mu}\mu!\,(m+2-\mu)!}\right|_{\mu=1} = \frac{(2m+3)!}{2^{m+1}\,1!\,(m+1)!} = (2m+3)!!, \quad m \in \mathbb{N}. \tag{19.65}$$

And also, 0 is instituted for μ.

$$a_{m+2,0} = \left.\frac{(2m+4-\mu)!}{2^{m+2-\mu}\mu!\,(m+2-\mu)!}\right|_{\mu=0} = \frac{(2m+4)!}{2^{m+2}\,0!\,(m+2)!} = (2m+3)!!, \quad m \in \mathbb{N}. \tag{19.66}$$

The results of equations (19.64), (19.65), and (19.66) are the same as that of equation (19.63), so in the case of $k = m+2$, equation (19.52) is also correct.

$$\int_0^\infty x^{2(m+2)} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{m+3}} \sum_{\mu=0}^{m+2} a_{m+2,\mu} (2\alpha)^\mu \sqrt{\pi}\, e^{-2\alpha}, \quad (\alpha \geq 0, m \in \mathbb{N})\,. \tag{19.67}$$

Where

$$a_{m+2,\mu} = \frac{(2m+4-\mu)!}{2^{m+2-\mu}\mu!\,(m+2-\mu)!}, \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \cdots, m+2)\,. \tag{19.68}$$

I can obtain the general representation for the integrals as a result of mathematical induction. At this stage, the proof is complete. Therefore, the following equation holds true:

$$\int_0^\infty x^{2k} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^{k} \frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!} (2\alpha)^\mu \sqrt{\pi}\, e^{-2\alpha}, \quad (\alpha \geq 0, k \in \mathbb{N})\,. \tag{19.69}$$

Finally, the general representation for the integrals is given by using the coefficients.

$$\int_0^\infty x^{2k} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^{k} a_{k,\mu} (2\alpha)^\mu \sqrt{\pi}\, e^{-2\alpha}, \quad (\alpha \geq 0, k \in \mathbb{N})\,. \tag{19.70}$$

Where

$$a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu}\mu!\,(k-\mu)!}, \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \cdots, k)\,. \tag{19.71}$$

And the coefficients deserving special mention are shown as follows:

$$a_{k,k} = 1, \quad k \in \mathbb{N}. \tag{19.72}$$

$$a_{k,1} = a_{k,0} = (2k-1)!!, \quad k \in \mathbb{N}. \tag{19.73}$$

Table of the coefficients $a_{k,\mu}$

k	$a_{k,10}$	$a_{k,9}$	$a_{k,8}$	$a_{k,7}$	$a_{k,6}$	$a_{k,5}$	$a_{k,4}$	$a_{k,3}$	$a_{k,2}$	$a_{k,1}$	$a_{k,0}$
1										1	1
2									1	3	3
3								1	6	15	15
4							1	10	45	105	105
5						1	15	105	420	945	945
6					1	21	210	1260	4725	10395	10395
7				1	28	378	3150	17325	62370	135135	135135
8			1	36	630	6930	51975	270270	945945	2027025	2027025
9		1	45	990	13860	135135	945945	4729725	16216200	34459425	34459425
10	1	55	1485	25740	315315	2837835	18918900	91891800	310134825	654729075	654729075

Table. 19.3

$$\sum_{n=1}^{\infty} n^k e^{-2nx}, \quad (x > 0, k \in \mathbb{N})$$

The hyperbolic cotangent of the real variable x is a convergent function in the intervals of $0 < |x| < \pi$. And it has the Laurent series expansion described by the Bernoulli numbers "[22]."

$$\coth(x) = \frac{1}{x} + \sum_{\mu=1}^{\infty} \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-1}, \quad 0 < |x| < \pi. \tag{19.74}$$

The infinite series,

$$\sum_{n=1}^{\infty} e^{-2nx}, \quad x > 0$$

can be written using the hyperbolic cotangent of the positive variable x as follows:

$$\sum_{n=1}^{\infty} e^{-2nx} = \frac{e^{-2x}}{1 - e^{-2x}} = \frac{e^{-x}}{e^{x} - e^{-x}} = \frac{\cosh(x)}{2\sinh(x)} = \frac{1}{2}(\coth(x) - 1), \quad x > 0. \tag{19.75}$$

By combining the above two equations,

$$\sum_{n=1}^{\infty} e^{-2nx} = \frac{1}{2}\left(\frac{1}{x} - 1 + \sum_{\mu=1}^{\infty} \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-1}\right), \quad 0 < x < \pi. \tag{19.76}$$

For the right side of equation (19.76), I show several terms of the series in order.

$$\sum_{n=1}^{\infty} e^{-2nx} = \frac{1}{2}\left(x^{-1} - 1 + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \frac{2x^9}{93555} + \sum_{\mu=6}^{\infty} \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-1}\right), \quad 0 < x < \pi. \tag{19.77}$$

The infinite series,

$$\sum_{n=1}^{\infty} n^k e^{-2nx}, \quad (x > 0, k \in \mathbb{N})$$

has a recurrent formula based on differentiation.

$$\sum_{n=1}^{\infty} n^k e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx} \sum_{n=1}^{\infty} n^{k-1} e^{-2nx}, \quad (x > 0, k \in \mathbb{N}). \tag{19.78}$$

When $k = 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n e^{-2nx} &= -\frac{1}{2} \cdot \frac{d}{dx} \sum_{n=1}^{\infty} e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx}\left(\frac{1}{2}\left(\frac{1}{x} - 1 + \sum_{\mu=1}^{\infty} \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-1}\right)\right) \\
&= -\frac{1}{4}\left(-\frac{1}{x^2} + \sum_{\mu=1}^{\infty} (2\mu - 1) \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-2}\right) \\
&= \frac{1}{4}\left(\frac{1}{x^2} - (2\mu - 1) \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-2}\Big|_{\mu=1} - \sum_{\mu=2}^{\infty} (2\mu - 1) \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-2}\right) \\
&= \frac{1}{4}\left(\frac{1}{x^2} - \frac{1}{3} - \sum_{\mu=1}^{\infty} (2\mu + 1) \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu}\right), \quad 0 < x < \pi.
\end{aligned} \tag{19.79}$$

Therefore

$$\sum_{n=1}^{\infty} n e^{-2nx} = \frac{1}{4}\left(x^{-2} - \frac{1}{3} + \frac{x^2}{15} - \frac{2x^4}{189} + \frac{x^6}{675} - \frac{2x^8}{10395} - \sum_{\mu=5}^{\infty} (2\mu + 1) \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu}\right), \quad 0 < x < \pi. \tag{19.80}$$

The following Pochhammer symbol will be used for convenience hereafter "[3]":

$$(a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{N}. \tag{19.81}$$

When $k = 2$,

(19.82)
$$\sum_{n=1}^{\infty} n^2 e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx} \sum_{n=1}^{\infty} n e^{-2nx} = -\frac{1}{8}\left(-\frac{2}{x^3} - \sum_{\mu=1}^{\infty} (2\mu)(2\mu+1) \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-1}\right)$$
$$= \frac{1}{4}\left(\frac{1}{x^3} + \frac{1}{2} \sum_{\mu=1}^{\infty} (2\mu)_2 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-1}\right), \quad 0 < x < \pi.$$

Therefore

(19.83)
$$\sum_{n=1}^{\infty} n^2 e^{-2nx}$$
$$= \frac{1}{4}\left(x^{-3} + 0 - \frac{x}{15} + \frac{4x^3}{189} - \frac{x^5}{225} + \frac{8x^7}{10395} + \frac{1}{2} \sum_{\mu=5}^{\infty} (2\mu)_2 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-1}\right), \quad 0 < x < \pi.$$

When $k = 3$,

(19.84)
$$\sum_{n=1}^{\infty} n^3 e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx} \sum_{n=1}^{\infty} n^2 e^{-2nx} = -\frac{1}{8}\left(-\frac{3}{x^4} + \frac{1}{2} \sum_{\mu=1}^{\infty} (2\mu-1)_3 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-2}\right)$$
$$= \frac{3}{8}\left(\frac{1}{x^4} - \frac{1}{3!}(2\mu-1)_3 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-2}\Big|_{\mu=1} - \frac{1}{3!} \sum_{\mu=2}^{\infty} (2\mu-1)_3 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-2}\right)$$
$$= \frac{3}{8}\left(\frac{1}{x^4} + \frac{1}{45} - \frac{1}{3!} \sum_{\mu=1}^{\infty} (2\mu+1)_3 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu}\right), \quad 0 < x < \pi.$$

Therefore

(19.85)
$$\sum_{n=1}^{\infty} n^3 e^{-2nx} = \frac{3}{8}\left(x^{-4} + \frac{1}{45} - \frac{4x^2}{189} + \frac{x^6}{135} - \frac{8x^8}{4455} - \frac{1}{3!} \sum_{\mu=4}^{\infty} (2\mu+1)_3 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu}\right), \quad 0 < x < \pi.$$

When $k = 4$,

(19.86)
$$\sum_{n=1}^{\infty} n^4 e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx} \sum_{n=1}^{\infty} n^3 e^{-2nx} = -\frac{3}{16}\left(-\frac{4}{x^5} - \frac{1}{3!} \sum_{\mu=1}^{\infty} (2\mu)_4 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-1}\right)$$
$$= \frac{3}{4}\left(\frac{1}{x^5} + \frac{1}{4!} \sum_{\mu=1}^{\infty} (2\mu)_4 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-1}\right), \quad 0 < x < \pi.$$

Therefore

(19.87)
$$\sum_{n=1}^{\infty} n^4 e^{-2nx} = \frac{3}{4}\left(x^{-5} + 0 + \frac{2x}{189} - \frac{x^3}{135} + \frac{4x^5}{1485} + \frac{1}{4!} \sum_{\mu=1}^{\infty} (2\mu)_4 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-1}\right), \quad 0 < x < \pi.$$

When $k = 5$,

(19.88)
$$\sum_{n=1}^{\infty} n^5 e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx} \sum_{n=1}^{\infty} n^4 e^{-2nx} = -\frac{3}{8}\left(-\frac{5}{x^6} + \frac{1}{4!} \sum_{\mu=1}^{\infty} (2\mu-1)_5 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-2}\right)$$
$$= \frac{15}{8}\left(\frac{1}{x^6} - \frac{1}{5!}(2\mu-1)_5 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-2}\Big|_{\mu=1} - \frac{1}{5!} \sum_{\mu=2}^{\infty} (2\mu-1)_5 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-2}\right)$$
$$= \frac{15}{8}\left(\frac{1}{x^6} - \frac{2}{945} - \frac{1}{5!} \sum_{\mu=1}^{\infty} (2\mu+1)_5 \frac{2^{2\mu+6} B_{2\mu+6}}{(2\mu+6)!} x^{2\mu}\right), \quad 0 < x < \pi.$$

Therefore

(19.89)
$$\sum_{n=1}^{\infty} n^5 e^{-2nx} = \frac{15}{8}\left(x^{-6} - \frac{2}{945} + \frac{x^2}{225} - \frac{4x^4}{1485} - \frac{1}{5!} \sum_{\mu=3}^{\infty} (2\mu+1)_5 \frac{2^{2\mu+6} B_{2\mu+6}}{(2\mu+6)!} x^{2\mu}\right), \quad 0 < x < \pi.$$

19.6 Formulae Using the Hyperbolic Functions for the Infinite Series

$$\sum_{n=1}^{\infty} n^k e^{-2nx}, \quad (x > 0, k \in \mathbb{N})$$

When $k = 1$, taking derivative and multiplying by $-1/2$ for the left and result of equation (19.75),

(19.90) $$\sum_{n=1}^{\infty} n e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx}\left(\frac{\cosh(x)}{2\sinh(x)} - \frac{1}{2}\right) = -\frac{1}{4} \cdot \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = \frac{1}{4\sinh^2(x)}, \quad x > 0.$$

When $k = 2$, taking derivative and multiplying by $-1/2$ for the left and result of equation (19.90),

(19.91) $$\sum_{n=1}^{\infty} n^2 e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx}\left(\frac{1}{4\sinh^2(x)}\right) = -\frac{1}{8} \cdot \frac{-2\sinh(x)\cosh(x)}{\sinh^4(x)} = \frac{\coth(x)}{4\sinh^2(x)}, \quad x > 0.$$

When $k = 3$,

(19.92)
$$\sum_{n=1}^{\infty} n^3 e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx}\left(\frac{\cosh(x)}{4\sinh^3(x)}\right) = -\frac{1}{8} \cdot \frac{\sinh^4(x) - 3\cosh^2(x)\sinh^2(x)}{\sinh^6(x)}$$
$$= \frac{1}{8} \cdot \frac{3\cosh^2(x) - \sinh^2(x)}{\sinh^4(x)} = \frac{1}{8} \cdot \frac{3\left(\sinh^2(x) + 1\right) - \sinh^2(x)}{\sinh^4(x)} = \frac{1}{8} \cdot \frac{2\sinh^2(x) + 3}{\sinh^4(x)}, \quad x > 0.$$

Therefore

(19.93) $$\sum_{n=1}^{\infty} n^3 e^{-2nx} = \frac{3}{8}\left(\frac{2}{3} + \frac{1}{\sinh^2(x)}\right)\frac{1}{\sinh^2(x)}, \quad x > 0.$$

When $k = 4$, I apply the same operations to the result of equation (19.92),

(19.94)
$$\sum_{n=1}^{\infty} n^4 e^{-2nx} = -\frac{1}{2} \cdot \frac{d}{dx}\left(\frac{1}{8} \cdot \frac{2\sinh^2(x) + 3}{\sinh^4(x)}\right)$$
$$= -\frac{1}{16} \cdot \frac{4\sinh^5(x)\cosh(x) - 4\left(2\sinh^2(x) + 3\right)\sinh^3(x)\cosh(x)}{\sinh^8(x)}$$
$$= \frac{1}{4} \cdot \frac{\left(2\sinh^2(x) + 3\right)\cosh(x) - \sinh^2(x)\cosh(x)}{\sinh^5(x)} = \frac{1}{4} \cdot \frac{\left(\sinh^2(x) + 3\right)\cosh(x)}{\sinh^5(x)}, \quad x > 0.$$

Therefore

(19.95) $$\sum_{n=1}^{\infty} n^4 e^{-2nx} = \frac{3}{4}\left(\frac{1}{3} + \frac{1}{\sinh^2(x)}\right)\frac{\coth(x)}{\sinh^2(x)}, \quad x > 0.$$

Hereafter, I show some results in order.

(19.96) $$\sum_{n=1}^{\infty} n^5 e^{-2nx} = \frac{15}{8}\left(\frac{2}{15} + \frac{1}{\sinh^2(x)} + \frac{1}{\sinh^4(x)}\right)\frac{1}{\sinh^2(x)}, \quad x > 0.$$

(19.97) $$\sum_{n=1}^{\infty} n^6 e^{-2nx} = \frac{45}{8}\left(\frac{2}{45} + \frac{2}{3\sinh^2(x)} + \frac{1}{\sinh^4(x)}\right)\frac{\coth(x)}{\sinh^2(x)}, \quad x > 0.$$

(19.98) $$\sum_{n=1}^{\infty} n^7 e^{-2nx} = \frac{315}{16}\left(\frac{4}{315} + \frac{2}{5\sinh^2(x)} + \frac{4}{3\sinh^4(x)} + \frac{1}{\sinh^6(x)}\right)\frac{1}{\sinh^2(x)}, \quad x > 0.$$

(19.99) $$\sum_{n=1}^{\infty} n^8 e^{-2nx} = \frac{315}{4}\left(\frac{1}{315} + \frac{1}{5\sinh^2(x)} + \frac{1}{\sinh^4(x)} + \frac{1}{\sinh^6(x)}\right)\frac{\coth(x)}{\sinh^2(x)}, \quad x > 0.$$

(19.100)
$$\sum_{n=1}^{\infty} n^9 e^{-2nx}$$
$$= \frac{2835}{8}\left(\frac{2}{2835} + \frac{17}{189\sinh^2(x)} + \frac{7}{9\sinh^4(x)} + \frac{5}{3\sinh^6(x)} + \frac{1}{\sinh^8(x)}\right)\frac{1}{\sinh^2(x)}, \quad x > 0.$$

19.7 Relation between the Modified Bessel Function of the Second Kind and the Integral

$$\int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad (\alpha > 0, \theta \in \mathbb{C})$$

For the integral in the subtitle, the variable transformation

$$x = (\alpha y)^{\frac{1}{2}}, \quad (\alpha, y > 0)$$

is performed.

$$(19.101) \qquad \int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \int_0^\infty (\alpha y)^{\frac{\theta-1}{2}} e^{-\left(\alpha y+\frac{\alpha}{y}\right)} \frac{\alpha^{\frac{1}{2}}}{2} y^{-\frac{1}{2}} dy = \alpha^{\frac{\theta}{2}} \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta}{2}-1} e^{-\frac{2\alpha}{2}\left(y+\frac{1}{y}\right)} dy$$
$$= \alpha^{\frac{\theta}{2}} \mathrm{K}_{\frac{\theta}{2}}(2\alpha), \quad (\alpha > 0, \theta \in \mathbb{C}).$$

Therefore

$$(19.102) \qquad \mathrm{K}_{\frac{\theta}{2}}(2\alpha) = \alpha^{-\frac{\theta}{2}} \int_0^\infty x^{\theta-1} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx, \quad (\alpha > 0, \theta \in \mathbb{C}).$$

Thus, when α is a positive real number, the above modified Bessel function of the second kind absolutely converges.

20. Conclusion

I obtained three types of general representations for the zeta function for any odd number of either 3 or 7, or more; one has the leading term, while the others do not. Unfortunately, because the general representation lacks essential information on the zeta function at these points, finding exact values (closed representations) such as $\zeta(2) = \pi^2/6$ would be extremely difficult. Because the explicit formulae for the zeta, eta, and null functions are obtained using four arithmetic operations ((+,−,×, and÷) from the complex functional equations, it can be said that these explicit formulae are in the same class as the complex functional equations.

I discovered that some infinite series represented using the divisor sigma function give transcendental numbers with fixed values. Two proofs of the Riemann hypothesis for the zeta function were obtained, one using the reduction to absurdity and the other using the deductive method. And the generalized Riemann hypothesis for the eta function was also proven using the deductive method. I believe that the Riemann hypothesis is a symmetry problem unrelated to prime numbers.

From the relationship between the conditional non-trivial zeros of the eta function and odd prime numbers, the implication of prime numbers divided into three subsets: even prime 2, Pythagorean primes, and non- Pythagorean primes was demonstrated. And It was shown that odd and even powers of non-Pythagorean primes have important meaning. In particular, under certain conditions, the square of a non-Pythagorean prime acts as well as a Pythagorean prime "[23]". Proving the conjecture that the almost half of imaginary parts of non-trivial zeros of the eta function would be twice the imaginary parts of non-trivial zeros of the zeta function will be an important problem in the future. The exploration of the Dirichlet series and Euler product representations for the eta function has not been started.

When interpreting that the functional equation (2.2) stipulates the relation between whole positive real number and all squares of natural numbers, I must conclude that the essence of the numbers is condensed here.

By the way, security of the current information society depends on the difficulty of factorizing the product of two huge prime numbers. In contrast, we know some people believe that when the Riemann hypothesis is solved, safety will be violated. Because the explicit formula for the zeta function has the same difficulty, I believe the safety will be maintained in the future as well.

Thanks

Letter to Professor Hardy "[11, 12, 23]"

Dear Professor G. H. Hardy,
I'm delighted to inform you that you and I have proved the Riemann hypothesis, which was proposed by
Bernhard Riemann himself in 1859. You demonstrated (together with Professor J. E. Littlewood) in 1914 that
there are an infinite number of non-trivial zeros of the Riemann zeta function $\zeta(S)$ on the critical line.
Recently, I was able to demonstrate that the non-trivial zeros of the zeta function don't lie off the critical line.
Combining these two results, we arrived at the conclusion that all non-trivial zeros of the zeta function lie on
the critical line.
Problems with symmetry in general, not just prime numbers, were raised by the Riemann hypothesis.
I want to sincerely commend you for your great achievement.
From future Japan 106 years after your time, on April 30, 2020.
Hideharu Maki · · · was born on June 30, 1959.

To those who supported and encouraged me

Many people gave me words of support and encouragement as I was writing this article. I would like to express
my gratitude. further, please allow me to introduce the following people:
Kenta Iijima, Shigeko Iijima, Yoichi Okuno, Yukiharu Okumura, Hiroshi Kamide, Kunihiko Saito, Norio Sasaki,
Yukiko Shimoda, Toshiyuki Sugio, Erdenebat Bayarbat, Toshi Furuta (Yu-Min), Kyoko Matsumoto, Nobuko
Watanuki, and Noriyoshi Yamazaki (in the order of the Kana syllabary, without title of honor) and I express
my thank you to all for giving me your words of support. Above all, I want to express my heartfelt appreciation
to my parents. Moreover, I want to express my gratitude, especially for giving me the opportunity to attend
university for four years.

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Profile

Full Name: Hideharu Maki, Date and place of birth: June 30, 1959, Saiki, Japan
Languages: English(poor), Japanese
E-mail: sequoiacypressantares [@] jcom.zaq.ne.jp (When sending an email, Please remove "[ ].")

Education (from high school):
1. Oita Prefectural Saiki Kakujo High School (Saiki, Japan), April 1975–March 1978
2. Department of Electrical and Electronic Engineering, Faculty of Engineering, Kumamoto University
(Kumamoto, Japan), April 1978–March 1982

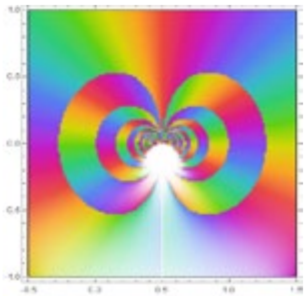
Employment history:
1. MOCHIDA PHARMACEUTICAL CO.,LTD. (Tokyo, Japan)
2. Transfer, Mochida Siemens Medical Systems Co.,Ltd. (Tokyo, Japan)
3. Transfer, Siemens Healthcare Diagnostics K.K. (Tokyo, Japan), April 1982–December 31, 2011 (1-3)
Engaged in related work such as research, development, and maintenance of medical equipment
(laser surgical equipment, ultrasound diagnostic equipment, etc.).
4. SOFTRONICS CO.,LTD. (Saitama, Japan), April 2012–June 2013
Skills: Electronic circuit design and fabrication.
Improved the production yield and quality of electronic circuits by analyzing and correcting design.

Research history:
1. Start research on the Riemann zeta function $\zeta(2k+1)$, around October 2013
2. Purchase one desktop computer for calculations and one laptop computer for communications, March 2020
3. Purchase Mathematica, March 20, 2020
For further research, I felt the need for powerful calculation tool. (2-3)
4. Complete the unpublished, Japanese, handwritten paper,
"The proof of the Riemann hypothesis (abbreviated title) ", April 30, 2020
5. Submit the article" Beyond the Riemann hypothesis "(Version 1) on viXra.org, July 25, 2023

I believe that future research will require higher performance computers and ample funding.

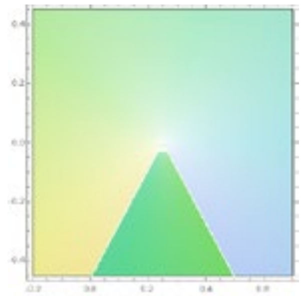
Art Gallery (Beyond the limits of logical expression)

The formulae for the two works below will not be made public. There isn't enough space to write them down.



Work No.1: "Yoake Mae (*Before the sun rises*)"
Production date: December 24, 2020.

This work depicts an image of understanding the Riemann hypothesis before it was proven.
The half-line in the work indicates the critical line (Re(S)=1/2) of the Riemann zeta function $\zeta(S)$.
The first 10 trillion non-trivial zeros of the Riemann zeta function have been confirmed.
On the other hand, no one could deny the possibility that a non-trivial zero lies off the critical line.
Beyond infinity (mathematically vague expression) was shrouded in fog.



Work No.2: "Shin-Sekai (*A new world*)"
Production date: June 30, 2023.
Dedicated to my deceased parents.

This work depicts an image of a new world in which the Riemann hypothesis has been solved and the fog
has cleared. The half-line on the left represent the imaginary axis, and the half-line on the right represent
the critical line. The non-trivial zeros of the eta function lie on the imaginary axis, those of the Riemann
zeta function lie on the critical line.
The area between the half-lines is called the Habitable zone (colored green).
It is believed that the non-trivial zeros of the function ζ (θ) + η (θ) would present in irregular positions
hear. History tends to repeat itself, and new mysteries were born as well

Refernces

1. Norio Adachi, Mitsuo Sugiura, and Ryosuke Nagaoka. *Riemann Ronbun-shu*, pages 155–185. Asakura, 2004.
2. B. Riemann. ¨ *Ueber die Anzahl der Primzahlen unter einer gegebenenen Gr¨osse,* pages 671–680. Monatsberichte Preuss. Akad. Wissens, 1859/1860. (Werke, 145–155).
3. Yoshihiko Ootsuki and Yoshiaki Muroya (2). *Shin Sugaku Koshiki-shu II Tokushu Kansu*, pages 719–730. Maruzen, 1992.
4. Shin Hitotsumatsu. *Sugaku Teisu Jiten (Steven R. Finch, Mathematical Constants, Cambridge University Press, 2003), pages* 39–44. Asakura, 2010.
5. B. C. Berndt. *Ramanujan's Notebook's:* Part II. Springer-Verlag, 1989. pp. 153–155, 275–276, 293; MR 90b:01039.
6. Nihon Sugakkai, editor. *Sugaku Jiten*, pages 1351–1353. Iwanami, fourth edition, 2007.
7. Kiyoshi Yomogida. *Self-study guide to special functions and integral transformations in practice*, page 106. Kyoritsu, 2007.
8. MATHEMATICA 12.0J (Japanese version), Wolfram Research, Inc.
9. Nobushige Kurokawa. *Riemann Yosou no 150 nen*, pages 130–131. Iwanami, 2009.
10. E. C. Titchmarsh. *The theory of the Riemann zeta-function*. Clarendon Press, Oxford, second edition,1951. reviced by D. R. Herth-Brown, 1986.
11. G. H. Hardy. Sur les Zeros de la Function ζ (s) de Riemann. *C R. Acad. Sci, Paris,* 158:1012–1014, 1914.
12. G. H. Hardy and J. E. Littlewood. The Zeros of Riemann's zeta-function on the critical line. *Math. Zeit*, 10:283–317, 1921.
13. Yoshihiko Ootsuki and Yoshiaki Muroya (1). *Shin Sugaku Koshiki-shu* I Shoto Kansu, pages 722,725. Maruzen, 1991.
14. Kumiko Nishioka. *Choetsusu toha nanika*, pages 10–30. Number B-1911 in Blue Backs. Kodansha, 2015.
15. Takashi Kiuchi. *A Visual Guide to the Riemann Hypothesis*, pages 150–184. Gijutsuhyoronsha, 2020.
16. Chebyshev function-Wikipedia, the free encyclopedia.
17. Akira Nakamura. *Riemann Yoso toha nanika*, pages 18–28. Number B-1828 in Blue Backs. Kodansha, 2015.
18. Koji Matsumoto. Riemann no Zeta kansu, pages 124–128. Asakura, 2005.
19. Hideo Wada, Masanari Kida, Kazuto Matsuo, Iwao Kimura, Atsushi Sato, and Yuji Hasegawa. *Sosu Zensho (Richard Carandall, Carl Pomerance, Prime Numbers, A Computational Perspective, second Edition, Springer-Verlag, 2005),* page 227. Asakura, 2010.
20. Shin-ya Kiyama. *Primes, zeta functions and arithmetic quantum chaos*, pages 2–20. Nihonhyoronsha, 2010.
21. Teiji Takagi. *Teihon Kaiseki Gairon,* page 214. Iwanami, 2010.
22. Shoyu Nagaoka. *Suronteki Koten-kaiseki (Max. Koecher, Klassische elementare Analysis, Birkh¨auser Basel- Boston, 1987),* pages 196–197. Maruzen, 2012.
23. TV program (DVD), *Riemann Y osou, Tensai-tachi no 150 nen no tatakai, The Cosmi Code Breakers,* NHK BS-hi, 21 Nov, 2009 on air.

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