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Analogue of Koëbe's and Bieberbach's Theorems in a Class of Locally Quasi-Conformal Maps

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Abstract
In this work, an analogue of Köebe and Bieberbach's theorems has been proposed for a class of locally quasi-conformal maps defined on the open unit disc $\Delta = \{z \in C : |z| < 1\}$.

Keywords: Quasi-Conformal Map, Modulus of a Double-Connected Domain, First Lavrentiev Characteristic.

1. Introduction

In this paper, an analogue of Köebe's and Bieberbach's theorems has been proposed in a class of locally quasi-conformal maps, defined on the open unit disc, by applying the notions of extremal length of a family of curves and of the modulus of a double-connected domain which made their appearance in the first part of the 20th century, in the work of G. GRECH. and L. AHLFORS. The methods used by the latter proved effective in solving many problems in the geometric theory of functions with complex variables and the theory of Riemann surfaces. The different aspects of using these methods are significantly mentioned in a large number of works (for example [2-3]). It's worth noting that the interest in this theme continues to this day. Recall that, the modulus of a double-connected D, domain in the complex plane C is called the number

$$M(D) = \frac{1}{\lambda(G)},$$

where $\lambda(G)$ is the extremal length in of the family G of curves connecting the boundaries of D. In particular, for the crown $K(r,R) = \{\mathbf{z} \in C : r < |z| < R, 0 < r < R < \infty\}$, we have:

$$M(r,R) = \frac{1}{2\pi}\ln\left(\frac{R}{r}\right).$$

The monotonicity property of the modulus plays a key role in this work.

Recall that if $D_1 \subset \mathbb{C}$ and $D_2 \subset \mathbb{C}$, double-connectors such that, $D_1 \subset D_2$, alors $M(D_1) \le M(D_2)$.
Let us also note that by writing a ring K as a union of a finite number of concentric crowns $D_k, k = 1,\dots,n$, two by two disjoint, bounded by curves γ_k, $k = 1,\dots,n-1$ we will have the following relation

$$M(K) \ge \sum_{k=1}^{n} M(D_k), \qquad (1)$$

and the equality is reached that, when the curves γ_k are circles $\{z : |z| = r_k\}, r_k \in (r,R)$, $k = 1,\dots,n-1$.
It is important to note that the modulus admits the property of invariance by conformal maps and quasi-invariance by quasi-conformal maps.

Thus, we can state the theorem of L. Ahlfors:

2. Theorem 1

Let D be a double-connected domain in the complex plane $\mathbb{C}$. Let f be a K_f-quasi-conformal map. Then we have the Inequality

$$\frac{1}{\mathbf{K}_f} M(D) \leq M(f(D)) \leq \mathbf{K}_f M(D) \qquad (2)$$

From the inequalities (2), we obtain equalities if, and only if, f is an affine map. This does not justify the correctness of these inequalities in many current problems. It appears obvious that the inequalities (2) are not applicable for moderately quasi-conformal or locally quasi-conformal maps. The following theorem allows us to overcome some difficulties.

3. Theorem 2 ([2])

Let f be a crown map $K(r,R) = \{z : r < |z| < R, 0 < r < R < \infty\}$ on the locally quasi-conformal double-connected domain $D \subset C$. Let

$$p_f(z) = \frac{|f_z(z)| + |f_{\bar{z}}(z)|}{|f_z(z)| - |f_{\bar{z}}(z)|}$$

the first Lavrentiev characteristic of the map f:

Let's put $P_f(r) = essup_{|z|}=r\, p_f(z)$:

If the integrals $\int_r^R P_f^{-1}(t)\frac{dt}{t}$ *and* $\int_r^R P_f(t)\frac{dt}{t}$ *are finite, then the modulus of the domain verifies the relation*

$$\frac{1}{2\pi}\int_r^R P_f^{-1}(t)\frac{dt}{t} \leqslant M(D) \leqslant \frac{1}{2\pi}\int_r^R Pf(t)\frac{dt}{t} \qquad (3)$$

For $P_f(r) = \mathbf{K}_f$, we find the increase (2).

An example of the use of the theorem 2 is expressed in the solution of certain majorization problems in a class of locally quasi-conformal maps in the open unit disc:

4. Theorem 3 (Analog of Köebe Theorem in a Class of Locally Quasi-Conformal Maps).

Let f be an automorphism of Δ, locally quasi-conformal such that, $f(0) = 0$, $f_z(0) = 1$ and if $|\mu_f(z_0)| \leq |z_0|$, then for all $z_0 = re^{i\theta} \in \Delta$, we have :

$$\frac{1}{4}\frac{1}{1-r^2} \leq \left|f_z(z_0)\right| \leq \frac{1}{1-r^2}$$

<u>**Démonstration.**</u>

Let's consider the map $h(z) = \dfrac{f(z)}{(1-|z_0|^2)f_z(z_0)}$.

h is an automorphism of Δ, locally quasi-conformal such that $h(0) = 0$ and satisfying the inequality $\left|\mu_{h(z_0)}\right| \leq |z_0|$

We know that

$$P_{h(z_0)} = \frac{|h_z(z_0)| + |h_{\bar{z}}(z_0)|}{|h_z(z_0)| - |h_{\bar{z}}(z_0)|}.$$

$$P_{h(z_0)} = \frac{1 + |\mu_{h(z_0)}|}{1 - |\mu_{h(z_0)}|}.$$

With $P_{h(z_0)}$ the first characteristic of Lavrentiev

Let's put

$$P_{h(r)} = essup_{|z_0|=r} P_{h(z_0)}.$$

Therefore

$$P_{h(r)} \leq \frac{1+r}{1-r} \text{ et } P^{-1}_{h(r)} \geq \frac{1-r}{1+r}$$

If $\varepsilon > 0$, then the image of the ring $\Delta_\varepsilon = \left\{z : \varepsilon < |z| < 1\right\}$ by h is a doubly connected domain to within $o(\varepsilon)$ merging with the ring $\Delta'_\varepsilon = \left\{z : \varepsilon\left|h_z(0)\right| < |z| < 1\right\}$.

Thus

$$\frac{1}{2\pi}\int_\varepsilon^1 P^{-1}_{h(r)}\frac{dr}{r} \leq M(\Delta'_\varepsilon) \tag{4}$$

Or

$$M(\Delta'_\varepsilon) = \frac{1}{2\pi}\ln\left(\frac{1}{\varepsilon|h_z(0)|}\right) \quad et \quad P^{-1}_{h(r)} \geq \frac{1-r}{1+r}$$

The relation (4) becomes :

$$\frac{1}{2\pi}\int_\varepsilon^1 \frac{1-r}{1+r}\frac{dr}{r} \leq \frac{1}{2\pi}\ln\left(\frac{1}{\varepsilon|h_z(0)|}\right)$$

$$\int_\varepsilon^1 \left(\frac{1}{r} - \frac{2}{1+r}\right)dr \leq -\ln(\varepsilon) - \ln(|h_z(0)|)$$

$$-2\ln 2 - \ln(\varepsilon) + 2\ln(1+\varepsilon) \leq -\ln(\varepsilon) - \ln(|h_z(0)|)$$

$$-\ln 4 + 2\ln(1+\varepsilon) \leq -\ln(|h_z(0)|)$$

By stretching ε to 0, we find :

$$\ln(|h_z(0)|) \leq \ln 4$$

Taking exponential on both sides, we find :

$$\left|h_z(0)\right| \leq 4 \tag{5}$$

Let's consider the crown $\Delta_0 = \left\{z : \varepsilon < |z| < |h_z(0)|\right\}$. Constructively $\Delta'_\varepsilon \subset \Delta_0$

Due to the monotonicity of the module, we have : $M(\Delta'_\varepsilon) \leq M(\Delta_0)$

$$\frac{1}{2\pi}\ln\left(\frac{1}{\varepsilon|h_z(0)|}\right) \leq \frac{1}{2\pi}\ln\left(\frac{|h_z(0)|}{\varepsilon}\right)$$

$$-\ln(\varepsilon) - \ln(|h_z(0)|) \leq \ln(|h_z(0)|) - \ln(\varepsilon)$$

$$\ln(|h_z(0)|) \geq 0$$

Taking exponential on both sides, we find :

$$1 \leq |h_z(0)| \tag{6}$$

The relations (5) et (6) give :

$$1 \leq |h_z(0)| \leq 4 \tag{7}$$

$$h_z(z) = \frac{f_z(z)}{(1-|z_0|^2)f_z(z_0)}$$

Therefore

$$h_z(0) = \frac{1}{(1-|z_0|^2)f_z(z_0)} \tag{8}$$

By replacing relation (8) in relation (7), we get the results of our theorem, i.e. :

$$\boxed{\frac{1}{4}\frac{1}{1-r^2} \leq \left|f_z(z_0)\right| \leq \frac{1}{1-r^2}} \tag{9}$$

5. Theorem 4 (analog of Bieberbach's theorem in a class of locally quasi-conformal maps)

Let f be an automorphism of Δ, locally quasi-conformal such that, $f(0) = 0$, $f_z(0) = 1$ and if $|\mu_{f(z_0)}| \leq |z_0|$, then for all $z_0 = re^{i\theta} \in \Delta$, we have :

$$\frac{1}{8}\ln\left(\frac{1+r}{1-r}\right) \leq \left|f(z_0)\right| \leq \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)$$

Démonstration.

We know that :

$$f(z_0) = \int_0^{z_0} f_z(\rho)d\rho, \quad avec \;\; \rho = re^{i\theta}$$

Therefore

$$f(z_0) = \int_0^{r} f_z(te^{i\theta})e^{i\theta}dt$$

Taking the modulus on both sides, we have :

$$|f(z_0)| \leq \int_0^{r} |f_z(te^{i\theta})|dt \tag{10}$$

According to relation (9), $\left|f_z(z_0)\right| \leq \dfrac{1}{1-r^2}$

Then

$$|f(z_0)| \leq \int_0^{r} \frac{1}{1-t^2}dt$$

$$|f(z_0)| \leq \frac{1}{2}\int_0^{r} \left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt$$

$$|f(z_0)| \leq \frac{1}{2}\Big[\ln(1+r) - \ln(1-r)\Big]$$

Therefore :

$$|f(z_0)| \leq \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) \tag{11}$$

Let γ be a map of class $\mathscr{C}^1$ per piece of $[0,1]$ in $\mathbb{C}$ such that $\gamma(0) = 0$ and $\gamma(1) = z_0$. Consider the path $([0,1];\gamma)$.

$$|f(z_0)| = \int_0^{1} |f'(\gamma(t))||\gamma'(t)|dt \tag{12}$$

Let's put : $\gamma(t) = x(t) + iy(t)$, alors $|\gamma(t)| = \sqrt{x^2(t) + y^2(t)}$

$$\frac{d|\gamma(t)|}{dt} = \frac{x'(t).x(t) + y'(t).y(t)}{\sqrt{x^2(t) + y^2(t)}} \tag{13}$$

According to the Cauchy-Schwarz inequality we have :

$$x'(t).x(t) + y'(t).y(t) \leq \sqrt{x'^2(t) + y'^2(t)}.\sqrt{x^2(t) + y^2(t)}$$

Then the relation (13) becomes :

$$\frac{d|\gamma(t)|}{dt} \leq \sqrt{x'^2(t) + y'^2(t)}$$

$$\frac{d|\gamma(t)|}{dt} \leq |\gamma'(t)|$$

$$|f'(\gamma(t))|.\frac{d|\gamma(t)|}{dt} \leq |f'(\gamma(t))|.|\gamma'(t)|$$

Therefore

$$\int_0^{1} |f'(\gamma(t))|d|\gamma(t)| \leq \int_0^{1} |f'(\gamma(t)|.|\gamma'(t)|dt \tag{14}$$

Replacing the relation (12) in the relation (14), we have :

$$\int_0^1 |f'(\gamma(t))| d|\gamma(t)| \leq |f(z_0)| \tag{15}$$

According to the relation (9), we have :

$$\frac{1}{4}\frac{1}{1-|\gamma(t)|^2} \leq |f'(\gamma(t)|$$

Then

$$\frac{1}{4}\int_0^1 \frac{1}{1-|\gamma(t)|^2} d|\gamma(t)| \leq |f(z_0)|$$

$$\frac{1}{8}\left[\ln\left(\frac{1+|\gamma(t)|}{1-|\gamma(t)|}\right)\right]_0^1 \leq |f(z_0)|$$

$$\frac{1}{8}\left[\ln\left(\frac{1+|\gamma(1)|}{1-|\gamma(1)|}\right) - \ln\left(\frac{1+|\gamma(0)|}{1-|\gamma(0)|}\right)\right] \leq |f(z_0)|$$

Thus :

$$\frac{1}{8}\ln\left(\frac{1+|z_0|}{1-|z_0|}\right) \leq |f(z_0)|$$

Since $|z_0| = r$, then :

$$\frac{1}{8}\ln\left(\frac{1+r}{1-r}\right) \leq |f(z_0)| \tag{16}$$

Consequently, relations (11) and (16) give:

$$\boxed{\frac{1}{8}\ln\left(\frac{1+r}{1-r}\right) \leq \left|f(z_0)\right| \leq \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)} \tag{17}$$

6. Conclusion

In this paper, we have found an analogue of Köebe theorem and an analogue of Bieberbach's theorem in a particular class of quasi-conformal maps, based on the properties of the modulus of a double-connected domain, the first quaracteristic of Lavrentev.

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