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# A New Zerbet's Statistic for Detecting Outliers in Exponential Distribution

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**Abstract**
*The problem of multiple upper outlier detection in a sample from an exponential distribution is considered in this paper. A new test statistic for an exponential sample is thus proposed. Distribution of the test based on this new statistic under slippage alternative also under the null hypothesis is obtained. The critical values of the new statistic for various n (size of the sample) are tabulated. An extensive Monte Carlo simulation study is conducted for comparing the performance of the new test with other available tests. The proposed test has the highest outlier identification power.*

**Keywords:** Critical Values, Outliers, Slippage Alternative

## 1. Introduction

The problem of detecting outliers is universal. In the presence of good theoretical tools, an analysis based on erroneous data often leads to false conclusions, hence the importance of developing statistical tests to detect anomalies and ensure the quality of the data is warranted. A so-called outlier value does not imply that this value is necessarily not useful and interesting, but it presents the extreme case of a particularly important phenomenon. This explains why we do not seek to remove outliers but to minimize their undesirable effects when using them. The objective of this paper is to focus on alternative models, namely slippage alternatives in exponential distribution [1]. This distribution has several applications in various fields, such as queueing systems, reliability engineering, survival analysis, etc. It is specifically used to model processes or events that occur with a constant rate (or intensity) $\lambda$. The rate $\lambda$ determines the average number of events or occurrences per unit of time. An exponential random variable can be regarded as the time between consecutive events in a Poisson process with a constant rate $\lambda$. For this distribution, a number of discordancy tests for multiple upper outliers have been proposed by various authors; for example, see [2-7].

Let $X_1, X_2, ..., X_n$ be arbitrary independent random variables. In this paper, we want to test the hypothesis $H_0$: $X_1, X_2,....., X_n$ have an exponential distribution, denoted by $E(\frac{1}{\sigma})$, with the density function given by

$$f(x,\sigma) = \frac{1}{\sigma}e^{-x/\sigma} \qquad (x>0),$$

where the parameter scale, or inverse rate $\sigma > 0$, is unknown. But, under the slippage alternative $H_r$ of existence of $r$ upper outliers, we have:

$$X_{(1)}, X_{(2)}, ..., X_{(n-r)} \text{ derive from } E(\frac{1}{\sigma});$$

$$X_{(n-r+1)}, X_{(n-r+2)}, ..., X_{(n)} \text{ derive from } E(\frac{1}{b\sigma}), b \geq 1, b \text{ is unknown.}$$

To test $H_0$ against $H_r$, proposed the test statistic $ZN(r,n)$ which is given by

$$ZN(r,n) = \frac{X_{(n-r)} - X_{(1)}}{\sum_{j=n-r+1}^{n} (X_{(j)} - X_{(1)})}.$$

A smaller value of $ZN(r;n)$ indicates the presence of r upper outliers in the sample. The null and slippage alternative distribution of this statistic was derived [7]. They also compared this statistic with Dixon's statistic $D_1(r;n)$ which is given by

$$D_1(r,n) = \frac{X_{(n)} - X_{(n-r)}}{X_{(n)}}.$$

Through a power comparison, they showed that the $ZN(r, n)$ statistic is more powerful than the Dixon's statistic [2].

For testing upper outliers with a slippage alternative in an exponential sample, proposed the test statistic $B(r, n)$ which is given by

$$BG(r,n) = \frac{\sum_{j=1}^{r}(X_{(n-r+j)} - X_{(n-r)})}{\sum_{j=2}^{n}(X_{(j)} - X_{(1)})}$$

The hypothesis $H_0$ is rejected if the test statistic $BG$(r,n) takes large values [5].

Another Dixon type statistic, $D_2$, for detecting outliers from an exponential distribution is proposed in [4]. It is based on the idea that the dispersion of the suspect observations accounts for a large proportion of the sample dispersion:

$$D_2(r,n) = \frac{X_{(n)} - X_{(n-r)}}{X_{(n)} - X_{(1)}}$$

These statistics will have large values if upper outliers are present in the data and declare them as discordant if the value of this statistic is greater than the corresponding critical value.

The Maximum Likelihood Ratio test statistic $MLR$ is based on the ratio of the sum of observations suspected to be outliers to the sum of all observations in the sample:

$$MLR(r,n) = \frac{\sum_{j=1}^{r} X_{(n-r+j)}}{\sum_{j=1}^{n} X_{(j)}}$$

The null distribution of this statistic was derived for an exponential sample. A larger value of $MLR(r,n)$ above a specified level indicates the presence of $r$ upper outliers [3,6,8]

## 2. The Proposed New Test Statistic

In this paper, to test $H_0$ against $H_r$, we propose the new following statistic:

$$Z(r,n)=\frac{(n-r)\sum_{j=n-r+1}^{n}X_{(j)}-r\sum_{i=1}^{n-r}X_{(i)}}{\sum_{i=2}^{n-r}(n-i+1)(X_{(i)}-X_{(i-1)})},$$

Here r denotes the maximum number of suspected upper outliers.

Note that under the null hypothesis $H_0,\ X_{(i)}=\sigma X'_{(i)}$, where $X'_{(i)}$ are the order statistics from the standard exponential distribution. So that under $H_0$, the distribution of the statistic Z (r, n) is parameter-free.

Under the alternative hypothesis $H_r$, the statistic $Z(r, n)$ have a tendency to take greater values than under the null hypothesis $H_0$. Thus, the hypothesis $H_0$ should be rejected if the value of the test statistic $Z(r, n)$ is greater than the corresponding critical value.

## 3. The Probability Density Function of The New Statistic

The main objective in this section is to obtain the probability density function of the statistic $Z(r, n)$, with $r \geq 1$ in the explicit form, under the slippage alternative Hr and also under null hypothesis.

**Theorem 3.1.** *The probability density function of the statistic Z(r,n) under Hr is given by the formula*:

$$f_{Z(r,n)}(x)=$$
$$=\frac{(r/b+n-r-1)!}{(r/b)!\,(n-r)\;b\,(r+1)^{r-1}}\sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}\,j^{r-1}\sum_{k=2}^{n-r}\frac{(-1)^{n-r-k+1}}{(k-2)!\,(n-k-r)!\,(n-k+1)}\left[\left(x+\frac{(k-1)r}{n-k+1}\right)\frac{(r-j+1)}{(n-r)jb}-\frac{(\frac{r}{b}+n-r-k+1)}{(n-k+1)}\right]^{-2}$$

$$if\ \left[\left(x+\frac{(k-1)r}{n-k+1}\right)\frac{(r-j+1)}{(n-r)jb}-\frac{(\frac{r}{b}+n-r-k+1)}{(n-k+1)}\right]>0.$$

*Proof.* We intend, here, to give the distribution of the statistic $Z(r, n)$ under the alternative hypothesis $H_r$.

Let $Y_i = X_{(i)} - X_{(i-1)}$, we remark that:

$$V=\sum_{j=1}^{r}\sum_{k=2}^{n-r+1}\sum_{i=k}^{n-r+j}Y_i=(n-r)\sum_{j=n-r+1}^{n}X_{(j)}-r\sum_{i=1}^{n-r}X_{(i)}$$

and

$$W=\sum_{i=2}^{n-r}(n-i+1)Y_i$$

then

$$Z(r,n)=\frac{\sum_{j=1}^{r}\sum_{k=2}^{n-r+1}\sum_{i=k}^{n-r+j}Y_i}{\sum_{i=2}^{n-r}(n-i+1)Y_i}=\frac{V}{W},\qquad r\geq 1.$$

The caracteristic function of $(V,W)$ is

$$\varphi_{(V,W)}(t,z) = E\left(e^{i(Vt+Wz)}\right)$$

$$= E\left( e^{i\left(\sum_{j=1}^{r}\sum_{k=2}^{n-r+1}\sum_{l=k}^{n-r+j} Y_l\ t + \sum_{j=2}^{n-r}(n-j+1)Y_j\ z\right)} \right)$$

$$= \int_{\mathbb{R}^{n-1}} e^{i\left(\sum_{j=1}^{r}\sum_{k=2}^{n-r+1}\sum_{l=k}^{n-r+j} Y_l\ t + \sum_{j=2}^{n-r}(n-j+1)Y_j\ z\right)} f_{(Y_2,\ldots,Y_n)}(y_2,\ldots,y_n)dy_2\ldots dy_n$$

Knowing that $Y_j$, $j = 2,\ldots, n - r$ follows the exponential law of parameters 0 and $\sigma\,(r / b + n - r - j + 1)^{-1}$, as well as $Y_{n-r+j}$, $j = 2,\ldots, r$, but of parametres 0 and $\sigma \times b\,(r - j + 1)^{-1}$, we can write the caracteristic function $\varphi(V,W)$ as [3]

$$\varphi_{(V,W)}(t,z) = \int_{-\infty}^{+\infty} \left[ e^{it\sum_{j=1}^{r}\sum_{k=2}^{n-r+1}\sum_{l=k}^{n-r+j} y_l} \right] \left[ e^{iz\sum_{j=2}^{n-r}(n-j+1)y_j} \right] \left[\prod_{k=2}^{n-r}\frac{1}{a_k}e^{-y_k/a_k}\right]\left[\prod_{k=1}^{r}\frac{1}{b_k}e^{-y_{n-r+k}/b_k}\right] dy_2\ldots dy_n$$

$$= \int_{-\infty}^{+\infty} \left[\prod_{j=1}^{r}\prod_{k=2}^{n-r+1}\prod_{l=k}^{n-r+j} e^{ity_l}\right] \times \left[\prod_{j=2}^{n-r} e^{iz(n-j+1)y_j}\right]\left[\prod_{k=2}^{n-r}\frac{1}{a_k}e^{-y_k/a_k}\right]\left[\prod_{k=1}^{r}\frac{1}{b_k}e^{-y_{n-r+k}/b_k}\right] dy_2\ldots dy_n$$

$$= \left[\prod_{k=2}^{n-r}\frac{1}{a_k}\left(1/a_k - i(k-1)rt - i(n-k+1)z\right)^{-1}\right] \times \left[\prod_{j=1}^{r}\frac{1}{b_j}\left(1/b_j - it(n-r)j\right)^{-1}\right]$$

With $a_k = \sigma(r/b + n - r - k + 1)^{-1}$ and $b_j = \sigma \times b(r - j + 1)^{-1}$.

Based on the inversion formula, the joint density function of (V, W) can be obtained as follows:

$$f_{(V,W)}(v,w) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi_{((V,W)}(t,z)e^{-i(tv+wz)}dtdz$$

$$= \tfrac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\prod_{k=2}^{n-r}\frac{1}{a_k}\left(1/a_k - i(k-1)rt - i(n-k+1)z\right)^{-1}\right]\left[\prod_{j=1}^{r}\frac{1}{b_j}\left(1/b_j - it(n-r)j\right)^{-1}\right] e^{-itv}e^{-iwz}dtdz$$

$$= \tfrac{1}{(2\pi)^2}\left[\prod_{k=2}^{n-r}\frac{1}{a_k}\right]\left[\prod_{j=1}^{r}\frac{1}{b_j}\right]\int_{-\infty}^{+\infty}\left\{\prod_{j=1}^{r}\frac{1}{1/b_j - it(n-r)j}e^{-itv}\left[\int_{-\infty}^{+\infty}\prod_{k=2}^{n-r}\frac{1}{1/a_k - i(k-1)rt - i(n-k+1)z}e^{-iwz}dz\right]\right\}dt \quad (1)$$

To find the joint density function of (V,W), we first calculate the following products:

$$\left[\prod_{j=1}^{r}\frac{1}{b_j}\right]=\prod_{j=1}^{r}\frac{(r-j+1)}{b\times\sigma}=\frac{r!}{b^r\,\sigma^r}, \tag{2}$$

$$\left[\prod_{k=2}^{n-r}\frac{1}{a_k}\right]=\prod_{k=2}^{n-r}\frac{1}{\sigma}(r/b+n-r-k+1)=\frac{1}{\sigma^{n-r-1}}\frac{(r/b+n-r-1)!}{(r/b)!}, \tag{3}$$

$$\prod_{j=1}^{r}\frac{1}{1/b_j-it(n-r)j}=\frac{1}{(n-r)^r\,r!}\times\sum_{j=1}^{r}\frac{(-1)^r}{(it-\frac{1}{(n-r)jb_j})\prod_{k=1,k\neq j}^{r}(\frac{1}{(n-r)jb_j}-\frac{1}{(n-r)kb_k})}$$

$$=\frac{\sigma^r\,b^r}{(r+1)^{r-1}}\times\sum_{j=1}^{r}\frac{(-1)^{r-j}\,j^{r-1}}{(j-1)!\,(r-j)!\,(r-j+1-i(n-r)j\sigma bt)}, \tag{4}$$

and

$$\prod_{k=2}^{n-r}\frac{1}{1/a_k-i(k-1)rt-i(n-k+1)z}=\prod_{k=2}^{n-r}\frac{1}{1/a_k-i[(k-1)rt-(n-k+1)z]}$$

$$=\prod_{k=2}^{n-r}\frac{1}{\frac{1}{a_k}-iq_{k,t,z}}$$

$$=(-1)^{n-r-1}\sum_{k=2}^{n-r}\frac{1}{(iq_{k,t,z}-\frac{1}{a_k})\prod_{j=2,j\neq k}^{n-r}(\frac{1}{a_k}-\frac{1}{a_j})}$$

$$=(-1)^{n-r-1}\sum_{k=2}^{n-r}\frac{1}{(iq_{k,t,z}-\frac{1}{a_k})\prod_{j=2,j\neq k}^{n-r}\frac{j-k}{\sigma}}$$

$$=(-1)^{n-r-1}\sum_{k=2}^{n-r}\frac{1}{(iq_{k,t,z}-\frac{1}{a_k})\frac{(-1)^{k-2}(k-2)!(n-k-r)!}{\sigma^{n-r-2}}}$$

$$=\sum_{k=2}^{n-r}\frac{(-1)^{n-r-k+1}\;\sigma^{n-r-1}}{(k-2)!\,(n-k-r)!\,[i(k-1)rt\sigma-i(n-k+1)\sigma z-\frac{r}{b}-n+r+k-1]} \tag{5}$$

with $q_{k,t,z}=(k-1)rt-(n-k+1)z$.

By substituting Equations (2) to (5) into Equation (1), the joint probability density function (pdf) of (V,W) will be as follows:

$$f_{(V,W)}(v,w) = a \times \int_{-\infty}^{+\infty} \left\{ \sum_{j=1}^{r} \frac{c_j}{(r-j+1-i(n-r)j\sigma bt)} e^{-itv} \left[ \sum_{k=2}^{n-r} d_k \times \int_{-\infty}^{+\infty} \frac{1}{[i(k-1)rt\sigma - i(n-k+1)\sigma z - \frac{r}{b} - n + r + k - 1]} e^{-iwz} dz \right] \right\} dt$$

with

$$a = \frac{1}{(2\pi)^2} \left[ \frac{1}{\sigma^{n-r-1}} \frac{(r/b+n-r-1)!}{(r/b)!} \right] \left[ \frac{r!}{(r+1)^{r-1}} \right], \quad c_j = \frac{(-1)^{r-j}\, j^{r-1}}{(j-1)!\,(r-j)!} \quad \text{and} \quad d_k = \frac{(-1)^{n-r-k+1}\, \sigma^{n-r-1}}{(k-2)!\,(n-k-r)!}$$

knowing that

$$\int_{-\infty}^{+\infty} \frac{e^{-iwz}}{i(k-1)rt\sigma - i(n-k+1)\sigma z - \frac{r}{b} - n + r + k - 1} dz = \frac{2\pi}{(n-k+1)\sigma} \times e^{\frac{w(\frac{r}{b}+n-r-k+1)}{(n-k+1)\sigma}} \times e^{\frac{-i(k-1)r\sigma wt}{(n-k+1)\sigma}}$$

and

$$\int_{-\infty}^{+\infty} \frac{e^{-itv}\, e^{\frac{-i(k-1)rwt}{(n-k+1)}}}{(r-j+1-i(n-r)j\sigma bt)} dt = \int_{-\infty}^{+\infty} \frac{e^{-it(v+\frac{(k-1)rw}{n-k+1})}}{r-j+1-i(n-r)j\sigma bt} dt = \frac{2\pi}{(n-r)j\sigma b} e^{-(v+\frac{(k-1)rw}{n-k+1})\times\frac{(r-j+1)}{(n-r)j\sigma b}}$$

As a conclusion, the joint pdf of (V,W) is

$$f_{(V,W)}(v,w) = a \times \sum_{j=1}^{r} c_j \times \frac{2\pi}{(n-r)j\sigma b} \sum_{k=2}^{n-r} d_k \times \frac{2\pi}{(n-k+1)\sigma} \times e^{\frac{w(\frac{r}{b}+n-r-k+1)}{(n-k+1)\sigma}} \times e^{-(v+\frac{(k-1)rw}{n-k+1})\times\frac{(r-j+1)}{(n-r)j\sigma b}}$$

Then the pdf of *Z*(*r*, *n*) is

$$f_{Z(r,n)}(x) = \int_{-\infty}^{+\infty} |t|\, f_{(V,W)}(tx,\, t)\, dt = \int_{0}^{+\infty} t\, f_{(V,W)}(tx,\, t)\, dt$$

$$= \int_{0}^{+\infty} t\, a \times \sum_{j=1}^{r} c_j \times \frac{2\pi}{(n-r)j\sigma b} \sum_{k=2}^{n-r} d_k \times \frac{2\pi}{(n-k+1)\sigma} \times e^{\frac{t(\frac{r}{b}+n-r-k+1)}{(n-k+1)\sigma}} \times e^{-(tx+\frac{(k-1)rt}{n-k+1})\times\frac{(r-j+1)}{(n-r)j\sigma b}}\, dt$$

$$= \frac{a(2\pi)^2}{(n-r)\sigma b} \times \sum_{j=1}^{r} \frac{c_j}{j} \sum_{k=2}^{n-r} \frac{d_k}{(n-k+1)\sigma} \int_{0}^{+\infty} t \times exp\left\{ -t \left[ \left( x + \frac{(k-1)r}{n-k+1} \right) \frac{(r-j+1)}{(n-r)j\sigma b} - \frac{(\frac{r}{b}+n-r-k+1)}{(n-k+1)\sigma} \right] \right\} dt$$

$$= \frac{a(2\pi)^2}{(n-r)\sigma b} \times \sum_{j=1}^{r} \frac{c_j}{j} \sum_{k=2}^{n-r} \frac{d_k}{(n-k+1)\sigma} \left[ \left( x + \frac{(k-1)r}{n-k+1} \right) \frac{(r-j+1)}{(n-r)j\sigma b} - \frac{(\frac{r}{b}+n-r-k+1)}{(n-k+1)\sigma} \right]^{-2}$$

$$= \frac{(r/b+n-r-1)!}{(r/b)!\,(n-r)\; b\,(r+1)^{r-1}} \sum_{j=1}^{r} \binom{r}{j} (-1)^{r-j} j^{r-1} \sum_{k=2}^{n-r} \frac{(-1)^{n-r-k+1}}{(k-2)!\,(n-k-r)!\,(n-k+1)} \left[ \left( x + \frac{(k-1)r}{n-k+1} \right) \frac{(r-j+1)}{(n-r)jb} - \frac{(\frac{r}{b}+n-r-k+1)}{(n-k+1)} \right]^{-2}$$

$$\text{if } \left[ \left( x + \frac{(k-1)r}{n-k+1} \right) \frac{(r-j+1)}{(n-r)jb} - \frac{(\frac{r}{b}+n-r-k+1)}{(n-k+1)} \right] > 0.$$

**Corollary 1.** *Under $H_0$, the probability density function of statistic Z(r, n) is obtained from the theorem taking b* = 1:

$$f_{Z(r,n)}(x) = \frac{(n-1)!}{(r)!\,(n-r)\,(r+1)^{r-1}} \sum_{j=1}^{r} \binom{r}{j} (-1)^{r-j}\, j^{r-1} \sum_{k=2}^{n-r} \frac{(-1)^{n-r-k+1}}{(k-2)!\,(n-k-r)!\,(n-k+1)} \left[ \left( x + \frac{(k-1)r}{n-k+1} \right) \frac{(r-j+1)}{(n-r)j} - 1 \right]^{-2}$$

$$if \left( x + \frac{(k-1)r}{n-k+1} \right) > \frac{(n-r)j}{(r-j+1)}.$$

## 4. Critical Values

The main objective in this section is to give the critical values of the six tests *ZN*, $D_1$, $D_2$, *BG, MLR* and *Z* computed by simulation using the Monte Carlo method (100 000 replications) at the 0.2,0.15,0.1,0.05,0.02 and 0.01 levels of significance, for $r = 2$, $n = 20, 40,...,100$, the results are summarized in the tables 1, 2, ..., 6 below.

| | $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| n | 0.2 | 0.15 | 0.1 | 0.05 | 0.02 | 0.01 |
| 20 | 0.2789907490 | 0.2622229563 | 0.2420281459 | 0.2130299881 | 0.1833568717 | 0.1653573709 |
| 40 | 0.3184780706 | 0.3033189728 | 0.2845119152 | 0.2576372619 | 0.2284969742 | 0.2106925453 |
| 60 | 0.3347496611 | 0.3209348712 | 0.3030583169 | 0.2770341195 | 0.2484796888 | 0.2311081133 |
| 80 | 0.3448380825 | 0.3310295177 | 0.3141725953 | 0.2894883351 | 0.2624875821 | 0.2444915947 |
| 100 | 0.3524744817 | 0.3389949382 | 0.3226244831 | 0.2976247145 | 0.2709360256 | 0.2533216621 |

**Table 1: Critical Values of ZN (r, n)**

| | $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| n | 0.2 | 0.15 | 0.1 | 0.05 | 0.02 | 0.01 |
| 20 | 0.5366332766 | 0.5717351358 | 0.6135360008 | 0.6677462395 | 0.7230180034 | 0.7549959956 |
| 40 | 0.4591325789 | 0.4925572636 | 0.5336366767 | 0.5902478814 | .6487897017 | .6826191715 |
| 60 | 0.4243911524 | 0.4568284064 | 0.4964347621 | 0.5535345990 | 0.6122159986 | 0.6489579830 |
| 80 | 0.4024231929 | 0.4339175664 | 0.4737655105 | 0.5292754163 | 0.5884519846 | 0.6239754161 |
| 100 | 0.3854495408 | 0.4169851661 | 0.4562241160 | 0.5114120109 | 0.5704714061 | 0.6048463751 |

**Table 2: Critical Values of *D*1 (*r, n*)**

| | $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| n | 0.2 | 0.15 | 0.1 | 0.05 | 0.02 | 0.01 |
| 20 | 0.5435416023 | 0.5788448481 | 0.6201434408 | 0.6741228822 | 0.7294705736 | 0.7612886833 |
| 40 | 0.4613822194 | 0.4950108894 | 0.5360956633 | 0.5928503233 | 0.6510465201 | 0.6857106433 |
| 60 | 0.4258321748 | 0.4580530589 | 0.4978573802 | 0.5553083101 | 0.6136920167 | 0.6502607214 |
| 80 | 0.4039927529 | 0.4353260279 | 0.4743377233 | 0.5297943698 | 0.5867500569 | 0.6238195209 |
| 100 | 0.3879954632 | 0.4195087340 | 0.4584946718 | 0.5141856814 | 0.5729155826 | 0.6075978162 |

**Table 3: Critical Values of *D*2 (*r, n*)**

| | $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| n | 0.2 | 0.15 | 0.1 | 0.05 | 0.02 | 0.01 |
| 20 | 0.1578541140 | 0.1755476292 | 0.2002714346 | 0.2380827778 | 0.2849957482 | 0.3183391455 |
| 40 | 0.07689115966 | 0.08612584760 | 0.09848174005 | 0.1188600106 | 0.1435782317 | 0.1619150420 |
| 60 | 0.05100723546 | 0.05717833138 | 0.06568560262 | 0.07968509288 | 0.09704236238 | 0.1089062728 |
| 80 | 0.03823125597 | 0.04286788346 | 0.04910540500 | 0.05935684604 | 0.07248847822 | 0.08189230625 |
| 100 | 0.03051307959 | 0.03421594578 | 0.03932448970 | 0.04771625447 | 0.05834402307 | 0.06611964785 |

**Table 4: Critical Values of *BG* (*r, n*)**

| | $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| n | 0.2 | 0.15 | 0.1 | 0.05 | 0.02 | 0.01 |
| 20 | 0.6075978162 | 0.3711432211 | 0.3898142590 | 0.4189874853 | 0.4539074318 | 0.4766014397 |
| 40 | 0.2165331220 | 0.2249014707 | 0.2359697559 | 0.2534304009 | 0.2748360360 | 0.2899472084 |
| 60 | 0.1591537972 | 0.1652772390 | 0.1734840997 | 0.1860174168 | 0.2013230575 | 0.2131105495 |
| 80 | 0.1267968052 | 0.1315948453 | 0.1379400928 | 0.1480592629 | 0.1604662982 | 0.1693900790 |
| 100 | 0.1063535799 | 0.1102501239 | 0.1153743605 | 0.1234616653 | 0.1337864073 | 0.1407900456 |

**Table 5: Critical Values of *MLR* (*r, n*)**

| | $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| n | 0.2 | 0.15 | 0.1 | 0.05 | 0.02 | 0.01 |
| 20 | 7.088943034 | 7.543187738 | 8.161117281 | 9.257344726 | 10.71168608 | 11.91616070 |
| 40 | 7.778242199 | 8.185593375 | 8.742833066 | 9.650735410 | 10.86851785 | 11.75959513 |
| 60 | 8.378671308 | 8.787563347 | 9.340021233 | 10.23050304 | 11.36799856 | 12.26065940 |
| 80 | 8.830780726 | 9.236434072 | 9.781161280 | 10.65751959 | 11.84119820 | 12.65697846 |
| 100 | 9.195387291 | 9.604545024 | 10.16699683 | 11.05116620 | 12.20186147 | 13.05011057 |

**Table 6: Critical Values of *Z* (*r, n*)**

According to the above tables, we can see the critical values of $D_1$, $D_2$, *BG* and *MLR* decrease when n is increased. But the critical values of *ZN* and *Z* increase when n is increased.

## 5. Simulation Experiments

Under the null hypothesis, the distribution of any of the considered test statistic is parameterfree; hence, investigating by simulation the power of the tests, the value $\sigma = 1$ could be taken when simulating the data under the null hypothesis, and different values of $b$ could be considered for fixed $\sigma = 1$ when simulating the data under the alternative hypothesis $H_r$.

Assume that $r = 2$. For performance comparison of the six tests *ZN*, $D_1$, $D_2$, *BG, MLR* and *Z*, we generated N = 100 000 samples of size n = 20 from the alternative distributions considered $H_r$. We computed then the power of the six tests by simulation for various values of $b$. The results are presented in Table 7. A good test should have a high power.

| | Tests | | | | | |
|---|---|---|---|---|---|---|
| **b** | **Dixon-1** | **Dixon-2** | **MLR** | **Balasooriya and Gadag** | **Zerbet** | **Zerbet and Nikulin** |
| 1 | 0.05 | 0.05 | 0.05 | 0.05 | 0.049930 | 0.04999 |
| 2 | 0.047630 | 0.047850 | 0.046610 | 0.048210 | 0.53175 | 0.04763 |
| 3 | 0.067850 | 0.066860 | 0.081710 | 0.074020 | 0.91151 | 0.06744 |
| 4 | 0.10840 | 0.10715 | 0.16043 | 0.13382 | 0.99333 | 0.11401 |
| 5 | 0.17326 | 0.17080 | 0.28963 | 0.23375 | 0.99974 | 0.19275 |
| 6 | 0.25113 | 0.24710 | 0.44261 | 0.35786 | 1 | 0.29262 |
| 7 | 0.34747 | 0.34244 | 0.60295 | 0.49830 | 1 | 0.40951 |
| 8 | 0.44985 | 0.44316 | 0.74042 | 0.63089 | 1 | 0.52783 |
| 9 | 0.54799 | 0.54098 | 0.84488 | 0.74355 | 1 | 0.63606 |
| 10 | 0.63762 | 0.62976 | 0.91360 | 0.82904 | 1 | 0.72731 |
| 11 | 0.71496 | 0.70743 | 0.95502 | 0.89045 | 1 | 0.80013 |
| 12 | 0.77548 | 0.76852 | 0.97793 | 0.93044 | 1 | 0.85267 |
| 13 | 0.82803 | 0.82154 | 0.98976 | 0.95824 | 1 | 0.8961 |
| 14 | 0.86969 | 0.86414 | 0.99572 | 0.97537 | 1 | 0.92705 |

| b | Tests | | | | | |
|---|---|---|---|---|---|---|
| | **Dixon-1** | **Dixon-2** | **MLR** | **Balasooriya and Gadag** | **Zerbet** | **Zerbet and Nikulin** |
| 15 | 0.90414 | 0.89976 | 0.99845 | 0.98572 | 1 | 0.94954 |
| 16 | 0.92758 | 0.92364 | 0.99931 | 0.99157 | 1 | 0.96484 |
| 17 | 0.94658 | 0.94318 | 0.99973 | 0.99534 | 1 | 0.97567 |
| 18 | 0.96032 | 0.95769 | 0.99986 | 0.99715 | 1 | 0.98318 |
| 19 | 0.97078 | 0.96858 | 0.99997 | 0.99836 | 1 | 0.98843 |
| 20 | 0.97856 | 0.97673 | 1 | 0.99911 | 1 | 0.99209 |
| 21 | 0.98442 | 0.98297 | 1 | 0.99945 | 1 | 0.99459 |
| 22 | 0.98842 | 0.98741 | 1 | 0.99970 | 1 | 0.99609 |
| 23 | 0.99141 | 0.99054 | 1 | 0.99988 | 1 | 0.99735 |
| 24 | 0.99416 | 0.99365 | 1 | 0.99990 | 1 | 0.9984 |
| 25 | 0.99591 | 0.99545 | 1 | 0.99995 | 1 | 0.99884 |
| 26 | 0.99654 | 0.99615 | 1 | 0.99997 | 1 | 0.99908 |
| 27 | 0.99778 | 0.99748 | 1 | 0.99999 | 1 | 0.99951 |
| 28 | 0.99840 | 0.99815 | 1 | 0.99999 | 1 | 0.99968 |
| 29 | 0.99891 | 0.99874 | 1 | 0.99999 | 1 | 0.99976 |
| 30 | 0.99902 | 0.99886 | 1 | 1 | 1 | 0.99986 |
| 31 | 0.99940 | 0.99929 | 1 | 1 | 1 | 0.99986 |
| 32 | 0.99956 | 0.99948 | 1 | 1 | 1 | 0.99993 |
| 33 | 0.99959 | 0.99949 | 1 | 1 | 1 | 0.99991 |
| 34 | 0.99966 | 0.99962 | 1 | 1 | 1 | 0.99995 |
| 35 | 0.99978 | 0.99973 | 1 | 1 | 1 | 0.99998 |
| 36 | 0.99983 | 0.99978 | 1 | 1 | 1 | 0.99999 |
| 37 | 0.99988 | 0.99986 | 1 | 1 | 1 | 0.99999 |
| 38 | 0.99992 | 0.99989 | 1 | 1 | 1 | 1 |
| 39 | 0.99995 | 0.99992 | 1 | 1 | 1 | 1 |
| 40 | 0.99997 | 0.99997 | 1 | 1 | 1 | 1 |
| 41 | 0.99997 | 0.99997 | 1 | 1 | 1 | 1 |
| 42 | 0.99998 | 0.99997 | 1 | 1 | 1 | 1 |
| 43 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 7: Empirical Power for n=20, r=2 and α = 0.05**

From Table 7, we observed that:

The powers of the six tests $ZN$, $D_1$, $D_2$, $BG$, $MLR$ and $Z$ increase to 1 when b increases;

The $D_1$ and $D_2$ tests are equivalent;

For $b \geq 3$, the $ZN$ test has more power than the $D_1$ and $D_2$ tests;

The test based on the $BG$ test is more powerful than the $D_1$, $D_2$ and $ZN$ tests;

The power of $MLR$ test increases to 1 rapidly than the $D_1$, $D_2$, $ZN$ and $BG$ tests;

The power of the new $Z$ test increases to 1 very rapidly than the other tests $D_1$, $D_2$, $BG$, $MLR$ and $ZN$.

The curves of the empirical power of different test statistics at $n = 20$, $r = 2$ and $\alpha = 0.05$ are depicted in the following figure:

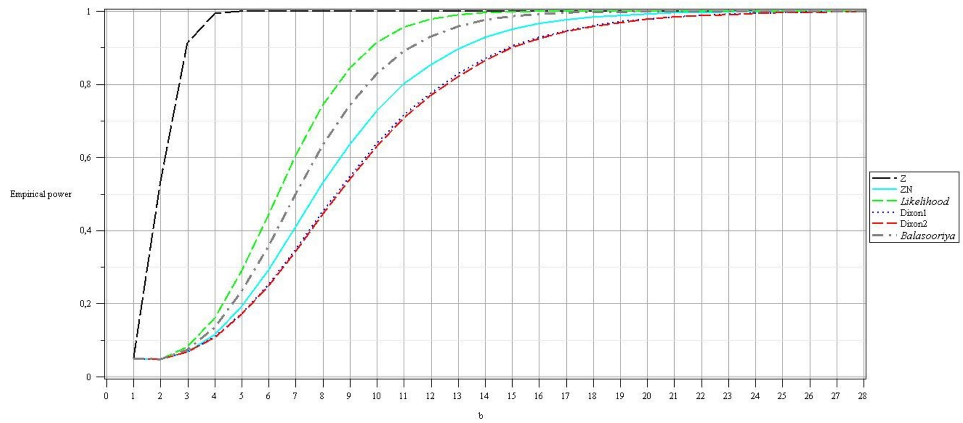


**Figure 1:** Empirical Power of Tests

## 6. Conclusion

The power analysis shows that, for exponential distribution, the $Dixon_1$, $Dixon_2$, Zerbet and Nikulin, and Balasooriya and Gadag tests have considerably smaller power than the new Zerbet and Maximum Likelihood Ratio tests. The power of the new Zerbet test increases to 1 very rapidly than the Maximum Likelihood Ratio test. We conclude that, for exponential distribution, the new Zerbet test has a higher probability of identifying the contaminant observations as outliers than the $Dixon_1$, $Dixon_2$, Zerbet and Nikulin, Balasooriya and Gadag, and Maximum Likelihood Ratio tests for all values of $b > 1$ [2-8].


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